K. C. NAG
GHOSH & BASU

HIGHER SECONDARY MATHEMATICS

CALCUTTA BOOK HOUSE

Written in accordance with the revised Syllabus for Classes XI & XII of West Bengal Council of Higher Secondary Education.

HIGHER SECONDARY MATHEMATICS

[SECOND PAPER]

[● Differential Calculus ● Integral Calculus & Differential Equations ● Dynamics ● Statics]

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INTRODUCTION

Our book "Higher Secondary Mathematics: Paper I" was published in October, 1976. Having appreciated the special features of that book some student and respected teachers have recently been enquiring whether Paper II of the book has been published. We are, however, extremely sorry to say that due to frequent loadshedding the publication of our book "Higher Secondary Mathematics: Paper II" has been long delayed. But for the untiring efforts of my son Sri Debiprasad Nag and the reputed publisher Sri Paresh Chandra Bhowal and the devoted co-operation of the workers of Calcutta Book House and the printing concerns the publication would have been further delayed.

This book (Paper II) has been prepared with my co-authors Prof. Keshab Basu (my beloved ex-student) and Prof. Sri Anandamohan Ghosh in active collaboratian with Dr. Rajkumar Roychoudhury (ex-professor of mathematics, Maulana Azad College, Calcutta). In this book also all the special features of my previous books on Mathematics, such as copious worked-out examples, have been maintained.

I hope this book also will be as cordially received by the learned teachers and professors as my previous works and our Book Paper I. Any suggestion from them towards the improvement of this book will be gratefully considered.

15.8.77
Gurup

Keshab Chandra Nag

PREFACE TO THE THIRD EDITION

In this edition all the errors and misprints of the previous edition have been corrected. Many new sums have been worked out and also set as exercises. Addition of *short answer type questions* on both the branches of Mechanics is a special feature of this edition.

Calcutta
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Authors

REVISED SYLLABUS FOR MATHEMATICS

FULL MARKS—200

Paper II : [Marks 100]

Elements of Calculus and Mechanics

Elements of Differential Calculus : 30 Marks

Rational Numbers. Real Numbers and their geometrical representation. Functions of a single variable including trigonometrical functions, Limit and continuity (geometrical and intuitive approach). Inverse functions. Standard Limits such as

$$\underset{x\to 0}{Lt}\ \frac{\sin x}{x},\ \underset{x\to 0}{Lt}\ \frac{e^x-1}{x}\ \underset{x\to 0}{Lt}\ \frac{\log(1+x)}{x}$$

(Proofs not required)

Fundamental limit theorems (statement only). Differential coefficient of a function and its significances. Rules of differentiation of sum, product and quotient of two functions. Rule of differentiation of function of a function (Proof not required). Differential coefficients of x^n (n rational), $\sin x$, $\cos x$, e^x, $\log x$ from first principles. Second order derivative of a function.

Integral Calculus and Differential Equations : 30 Marks

Indefinite Integrals. Standard forms.

Integration by simple substitution. Integral by parts in simple cases. Definite Integral as the limit of a sum and as an area. Fundamental theorem connecting primitive with definite integral (Proof not required). Determination of areas in simple cases.

Solution of first order differential equations by separation Method.

Elements of Mechanics : 40 Marks

Resultant of any two velocities and of any two accelerations of a particle. Relative Velocity :

Expressions for velocity and acceleration of a particle in cartesian co-ordinates. Newton's laws of motion ; Equation of

motion. Motion of a particle in a straight line. Simple Harmonic motion. Vertical motion under gravity, Motion of a projectile under gravity, Motion of a projectile under constant gravity neglecting air resistance.

Resultant of two concurrent forces, two like parallel forces, two unlike parallel forces; Couples. Centre of gravity of a uniform plane Lamina : simple cases.

Resultant of three or more non-concurrent forces in a plane, conditions of equilibrium—Triangle of forces.

List of Greek Letters used in Mathematics

α (alpha)	ϕ (phi)
β (beta)	Σ, σ (sigma)
γ (gamma)	ψ (psi)
Δ, δ (delta)	ω (omega)
θ (theta)	ρ (rho)
λ (lambda)	ξ (xi)
μ (mu)	η (eta)
π (pi)	

CONTENTS

Differential Calculus

Integral Calculus and Differential Equations

Dynamics

Statics

"Give me a place to stand on and I will move the earth."
—*Archimedes (287-212 B. C.)*

"Eupper si muove"
(And, yet it moves)
—*Galileo Galilie* (1564-1642)

"If I have seen **a little furthor than** others, it is bec**ause I have stood on** the shoulder of g **ants**."
—*Isaac Newton* (*1642-1721*)

"I have so many ideas that may perhaps be of some use in time if others more penetrating than I, go deeply into them some day and join the beauty of their minds to the labour of mine."—*G. W. Leibniz* (*1646-1716*)

DIFFERENTIAL CALCULUS

CHAPTER ONE
REAL NUMBERS

§ 1·1. Introduction. What is Calculus ? It is not possible to answer this question correctly at the beginning. In fact it is very difficult to define any subject. Indeed, one cannot understand the exact nature of a subject unless he acquires some knowledge about it. On the other hand, some elementary idea about a subject makes the students interested in learning the subject. So, in this article we are briefly discussing the content of calculus.

In Algebra or Geometry, up to this stage, you have used four operations, viz, addition and multiplication and their inverse operations, subtraction and division. In calculus a fifth operation, viz, *limit operation* is defined.

The difference between elementary algebra and calculus is, in the main, due to the use of this limit operation. In Trigonometry in proving that the ratio of the lengths of the circumference of a circle and its diameter is constant, you have unknowingly used this fifth operation. In algebra while evaluating a recurring decimal or in geometry, while finding the tangent to a curve at a point you have made use of this operation, though not in a formal way.

In calculus, the process of finding the limiting value of a function has two-fold applications. In the first case, by the limit operation the instantaneous rate of change or the derivative of a function at a point is determined. The second application is concerned with the determination of the area bounded by a function or the definite integral of the function. The first application is the subject matter of the branch of calculus known as the *Differential Calculus.* The other branch of calculus, viz, the *Integral Calculus* is concerned with the second application.

§ 1·2. Historical Notes. In Latin, the word calculus is the name of a stone. These stones were used by the Romans for calculations. But now the term Calculus means the most

important branch of Mathematics. In modern days, in every field of learning, where there is use of mathematics, there is extensive use of Calculus.

In ancient times, the process which the greek mathematicians adopted for determining the area of a circle, the length of a portion of a parabola, the volumes of regular solids such as parallelopipeds, cones and spheres, was nothing but finding the limit of sum. There is evident similarity between this process and the branch of Calculus, known as Integral Culculus.

In the writings of *Zeno* (495—435 B. C.) and *Eudoxus* (408 B. C. —355 B. C.) there are evidences of the concepts of *the infinite* and *continuity*. The methods used by *Archimedes* (287—212 B. C.) for determination of the area of a circle and tracing of the curve known as *Archimedian spiral* were nothing but elementary forms of the Differential Calculus and Integral Calculus, the two branches of Calculus.

In the seventeenth century *Descartes* (1595—1650) introduced the concept of variable quantity in Geometry. The introduction of this new concept was epoch-making in the history of mathematics and led the mathematicians of the west to use this concept in the discussion about drawing of tangents to curves, determination of extreme values, determination of areas and volumes, etc. *Fermate* (1608—1665), *Pascal* (1623—1662), *Kepler* (1571—1630), *Roberval* (1602—1675), *Haygens* (1629—1695), *Barrow* (1630—1677) and many other mathematicians discovered methods for drawing tangents of particular curves and determination of areas bounded by particular curves. They discussed every problem independently and could not discover any general rule. But in each case there were evidences of use of the concepts of Derivative of a function or the Definite integral of a function. Amongst the contemporary mathematicians of the orient, there are evidences of the concept of calculus in the writings of *Sekikowa* (1642—1708) of Japan.

In the later half of the seventeenth century these unorganised concepts led the English Mathematician *Newton* (1642—1727)

and the German Mathematician *Leibnitz* (1651—1708) to define, independent of each other, Derivatives and Integrals of functions. Leibnitz used the symbol $\frac{dy}{dx}$ for differential coefficient and the symbol of integral now in use. Both, Newton and Leibnitz are known as discoverers of calculus.

After the discovery of the elementary rules of calculus by Newton and Leibnitz, mathematicians of different European countries used calculus in solving different problems of mathematics and different branches of science. Amongst these mathematicians names of Euler (1707—1783) D' Alembert (1717—1783), Lagrange (1736—1813), Gauss (1777—1855), Laplace (1749—1827), Cauchy (1789—1857), Abel (1802—1829), Weirstrass (1815—1897), Reimann (1826—1866) are worth mentioning.

It may be said that the improvement of modern physics and chemistry have been possible due to the application of calculus in those subjects. Indeed, in the realm of knowledge, where there is use of mathematics, there we find use of calculas.

§ 1·3. Calculus is based on two concepts. The first one is the *concept of function* and the second one is the *concept of limit*. Again both the concepts are based on the notion of numbers. Hence before discussing the concepts of functions and their limits we must discuss about numbers. In this chapter we shall discuss about numbers. The subject matter of the next two chapters are functions and the concept of limits respectively.

§ 1·4. **Rational number :**

The numbers 1, 2, 3,...... are called *Natural numbers* or *Counting numbers* as they are used for counting. The sum of two natural numbers is the sum total of the two quantities. Multiplication of natural numbers is defined as contracted addition, as for example $4\times3=4+4+4=12$; subtraction and division are respectively inverse operations of addition and multiplication. For example, as $4+3=7$, we say, $7-3=4$ and as $4\times3=12$, so $12\div3=4$.

The natural numbers are *closed* with respect to addition and multiplication. i.e., the sum and product of two natural numbers

are natural numbers. But the natural numbers are not closed with respect to subtraction and division. For example $2-7$ or $5 \div 3$ are not natural numbers. For making the subtraction and division possible, new numbers are created.

With the discovery of the number *zero* and negative integers, subtraction of a natural number from another natural number becomes always possible. This means that the totality of natural numbers (positve integers), zero and negative integers are closed with respect to subtraction over above addition and multiplication. The totality of the positive integers, zero and the negative integers constitute what are known as *integers*.

To make division always possible, *fractional numbers* are created. $\frac{1}{2}$; $\frac{3}{5}$, $-\frac{2}{7}$ etc. are fractional numbers or *fractions*. Fractions are generally expressed in the form $\frac{p}{q}$, where p, q are integers and $q \neq 0$. The integers and the fractions together constitute what are known as *rational numbers*. Any integer can be expressed in the form of a fraction as $\frac{p}{q}$. For example, $6=\frac{6}{1}$ $(p=6, q=1)$, $-7=\frac{-7}{1}$ $(p=-7, q=1)$, etc.

Hence the rational numbers can be defined in the following way.

Def. A quantity which can be expressed in the form $\frac{p}{q}$, where p and q are integers and $q \neq 0$ is called a rational number.

Addition, subtraction, multiplication and division of rational numbers are defined as follows :

If $\frac{p}{q}$ $(q \neq 0)$ and $\frac{r}{s}$ $(s \neq 0)$ be two rational numbers, then

$$\frac{p}{q}+\frac{r}{s}=\frac{ps+qr}{qs}; \quad \frac{p}{q}-\frac{r}{s}=\frac{ps-qr}{qs}.$$

$$\frac{p}{q}\cdot\frac{r}{s}=\frac{pr}{qs}; \quad \frac{p}{q}\div\frac{r}{s}=\frac{ps}{qr} \text{ (if } r \neq 0).$$

Examples :

$$\frac{2}{3}+\frac{3}{7}=\frac{2.7+3.3}{3.7}=\frac{14+9}{21}=\frac{23}{21}$$

$$\left(-\frac{2}{3}\right)\left(-\frac{4}{7}\right)=\frac{-2}{3}\cdot\frac{-4}{7}=\frac{(-2).(-4)}{3.7}=\frac{8}{21}.$$

$$\frac{2}{3}-5=\frac{2}{3}-\frac{5}{1}=\frac{2-15}{3.1}=-\frac{13}{3}.$$

$$\left(\frac{5}{6}\right)\div\left(\frac{3}{4}\right)=\frac{5.4}{6.3}=\frac{20}{18}=\frac{10}{9}.$$

Note : (1) Division by zero is meaningless. For, Let $15\div 0=a$. $\therefore$ According to the definition of division, $15=a\times 0$. But we know that $a\times 0=0$, $\therefore$ $15=0$, which is absurd. $\therefore$ $15\div 0$ is meaningless.

Hence no number can be divided by 0. That deviation from this rule leads to absurd results is illustrated in the following example.

Let, $a=5$, $\therefore$ $a^2=25$ or, $a^2-5^2=0$.

Again, $a-5=0$, $\therefore$ $a^2-25=a-5$,

or, $(a-5)(a+5)=a-5$. Dividing both sides by $a-5$, we obtain $a+5=1$.

Putting the value of a, we get $10=1$, which is absurd. The absurdity occurs due to division of both sides by $a-5$ which is equal to 0.

See that $\dfrac{a^2-5^2}{a-5}=\dfrac{(a+5)(a-5)}{a-5}=a+5$, when $a\neq 5$.

For, $a=5$, $\dfrac{a^2-5^2}{a-5}$ is undefined.

Note : (2) Historically, negative numbers and fractional numbers originated in the efforts of measuring opposite quantities and part of an object respectively.

§ 1·5. Some properties of rational numbers :

(i) The sum, difference, product and quotient (when the divisor is not zero) of two rational numbers is a rational number.

(ii) If x, y, z be three rational numbers, then

(a) $x+y=y+x$, $xy=yx$

[Commutative laws of addition and multiplication.]

(b) $(x+y)+z=x+(y+z)$, $x(yz)=(xy)z$

[associative laws of addition and multiplication.]

(c) $(x+y).\ z=x.\ z+y.\ z$; $x(y+z)=x.y+x.z$

[distributive law.]

(iii) If x and y be two rational numbers then one and only one of $x>y$, $x=y$ and $x<y$ will be true. If z be any rational number such that $x<z$ and $z<y$, then $x<y$. [Law of order]

(iv) There are infinite number of rational numbers between any two rational numbers.

§ 1·6. Decimal expression of a rational number :

Let $\frac{p}{q}$ $(q\neq 0)$ be a rational number. In order to express this number in decimal we first divide p by q and then multiplying the remainder by 10 we again divide by q. The remainder of this division is multiplied by 10 and we again divide the product by q. In this way as we divide by q the product obtained by multiplying the successive remainders by 10, the quotients will be the successive digits after the decimal point.

If after a few steps the remainder be zero, then the decimal expression will be finite. If the remainder be never zero, then at each step, the remainders being less than q will be one of $1, 2, 3 \cdots q-1$. Since the numbers $1, 2, 3 \dots q-1$ are finite in number and the remainder is never 0, after a few steps the remainder will be equal to one of the preceding remainders. From this stage the division-process will be the same as before and the successive quotients will be equal to the preceding quotients, till the quotient at the beginning of the stage is repeated. In this way the quotients will be repeated and the decimal expression of the number will be a recurring decimal.

Hence we find that the decimal-expression of a rational number is either finite or infinite recurring decimal.

The above reasoning is explained in the following example.

Example. Find decimal expressions of the numbers $\frac{3}{8}$ and $\frac{10}{7}$.

```
3
8) 30 (3
   24
8) 60 (7
   56
8) 40 (5
   40
    ×
```

Here the remainder becomes 0 after three steps. Hence the decimal expression is finite and $\frac{3}{8} = \cdot 375$.

```
10
7) 10 (1
    7
7) 30 (4
   28
7) 20 (2
   14
7) 60 (8
   56
7) 40 (5
   30
7) 50 (7
   49
7) 10 (1
    7
    3
```

Dividing 10 by 7 and also the product of the successive remainders and 10 by 7 in succession, the remainders are respectively 3, 2, 6, 4, 5, 1, 3, etc. Notice that the remainders are less than 7 and the first remainder is repeated after $7-1$ divisions.

Now, the divisions will be as before i.e, the remainders and the quotients will be successively as before and after 6 more divisions 3 will be the remainder. In this way the process will be repeated.

$$\therefore \quad \frac{10}{7} = 1\cdot\dot{4}2857\dot{1}.$$

§ 1·7. Geometrical representation of rational numbers :

The rational numbers can be represented as points of a straight line in the following way.

Let O be any point on a straight line. The point O divides the straight line into two parts, we may take one part as the positive side and the other negative side. Generally the right side is taken to be positive. On the Positive side (i.e., on the right of O), we take another point I. The numbers 0 (zero) and 1 are indicated by the points O and I respectively. The point O is called the origin

and the length OI is called the unit length. Now any rational number can be represented by a point on the straight line $\overrightarrow{OI}$.

If $+a$ be any positive number, then the number will be represented by a point P on OI on the right of O such that $OP=a.OI$. As for example, in the figure the number 2 has been represented

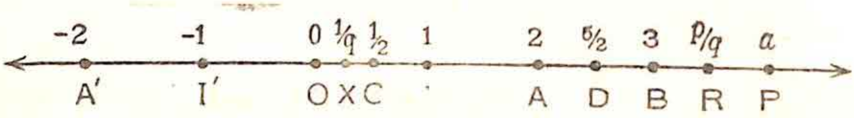


Fig. 1

by the point A; where $OA=2.OI$. The point B on the right of O, where $OB=3.OI$ represents the number 3.

A negative number $-b$ will be represented by a point on the left of O at a distance $b.OI$ from O. The point -1 is represented by the point I on the left of O such that $OI'=OI$. The number -2 is represented by the point A' on the left of O such that $OA'=2.OI$.

A fraction $\frac{p}{q}$ will be represented in the following manner. Let $q>O$ (this is always possible, for if the fraction be negative, then we shall take p as negative ; as for example, $-\frac{3}{4}=\frac{-3}{4}$). Divide the line segment $\overline{OI}$ in q equal parts ; let $\overline{OX}$ be one such part. Now, if $\frac{p}{q}$ be positive, then the number $\frac{p}{q}$ will be represented by a point R on the right of O such that $OR=p.OX$. Similarly, if $p<O$, then the number $\frac{p}{q}$ will be represented by a point R' on the left of O such that $OR'=OR$. To express the number $\frac{5}{2}$, first divide $\overline{OI}$ into two equal parts $\overline{OC}$ and $\overline{CI}$. Now take a point D on the right of O such that $OD=5.OC$. The point D will represent the number $\frac{5}{2}$.

In this way all the rational numbers can be represented by points on the straight line $\overleftrightarrow{OI}$. The straight line $\overleftrightarrow{OI}$ is called the *number line*.

§ 1·8. Irrational Numbers.

In the last article we have shown that corresponding to every rational number there is a point on the number line. Now, it will be shown that the converse is not true ; i.e., there are points on the number line which do not represent any rational number.

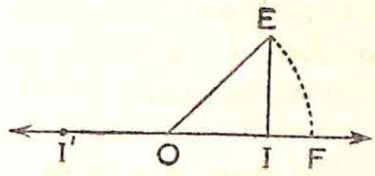


Fig. 2

At the point I on the number line, draw a perpendicular on the number line and on this perpendicular take a point E such that $OI=IE$. Join OE and take a point F on OI on the right of O such the $OF=OE$.

Now, $OF=OE=\sqrt{OI^2+IE^2}$

$$[\because \triangle OIE \text{ is right angled }]$$

$$=\sqrt{OI^2+OI^2} \text{ [as by construction } OI=IE \text{]}$$

$$=\sqrt{2.OI^2}=\sqrt{2}.OI.$$

$\therefore$ The distance of the point F, on the number line, from O is $\sqrt{2}$ times the length of the line segment $\overline{OI}$. So, the point F will represent the number $\sqrt{2}$. But $\sqrt{2}$ is not a rationnal number. This can be proved as follows.

If possible let $\sqrt{2}$ be a positive rational number and let $\sqrt{2}=\frac{m}{n}$, where m and n are two positive integers. Now, we may assume that m and n have no factor in common. For, if they have any factor in common, we can cancel that at the very beginning.

$\therefore \quad \sqrt{2}=\frac{m}{n}$; $\quad \therefore \quad n.\sqrt{2}=m$. Squaring both sides we get $2n^2=m^2$. $\therefore$ 2 is a factor of m^2 and so m^2 is an even positive integer. So m is also an even positive integer [for, if m be odd, then m can be taken in the form $m=2p+1$; and so $m^2=(2p+1)^2=4p^2+4p+1=2(2p^2+2p)+1$ which is an odd number]

$\therefore$ Let $m=2k$, where k is a positive integer.

$$\therefore \quad 2n^2=m^2=(2k)^2 \text{ or, } n^2=2k^2.$$

Hence as before, n^2 and so n is an even integer.

Hence 2 is a common factor of m and n.

But we assumed at the very beginning that m and n have no common factor. Hence we get an absurd result. The absurdity is due to the hypothesis that $\sqrt{2}$ is a rational number. Hence $\sqrt{2}$ cannot be a rational number.

So, on the number line the point F does not represent a rational number. In this way one can show that there are infinite number of points on the number line which do not represent rational numbers. With each of these points which cannot represent a rational number, is associated a new kind of number, called an *irrational number*.

Hence, with the introduction of these irrational numbers, we can say that every point on the number line represents a number either rational or irrational. The totality of the rational and irrational numbers is called the system of *real numbers*.

Since the irrational numbers cannot be represented in the form of a fraction i.e., in the form $\frac{p}{q}$ ($q \neq 0$ and p, q integers), they are said to be incommensurable. But every irrational number represents a definite magnitude and this magnitude can be represented by rational numbers to any desired degree of approximation. This is explained in the following example.

We have seen that $\sqrt{2}$ is not a fraction. But we can determine the approximate value of $\sqrt{2}$ to any desired degree.

$$1^2 < 2 < 2^2, \qquad \therefore \quad 1 < \sqrt{2} < 2$$

$\therefore$ the value of $\sqrt{2}$ lies between 1 and 2.

Now squaring $1{\cdot}1, 1{\cdot}2, 1{\cdot}3, 1{\cdot}4, \ldots\ldots, 1{\cdot}9$.

We see $(1{\cdot}4)^2 = 1{\cdot}96 < 2$ and $(1{\cdot}5)^2 = 2{\cdot}25 > 2$

$$\therefore \quad 1{\cdot}4 < \sqrt{2} < 1{\cdot}5, \text{ so } \sqrt{2} = 1{\cdot}4\ldots\ldots$$

Again, squaring $1{\cdot}41, 1{\cdot}42, 1{\cdot}43, \ldots\ldots, 1{\cdot}49$

We see, $(1{\cdot}41)^2 = 1{\cdot}9881 < 2$ and $(1{\cdot}42)^2 = 2{\cdot}0164 > 2$

$$\therefore \quad 1{\cdot}41 < \sqrt{2} < 1{\cdot}42, \text{ so } \sqrt{2} = 1{\cdot}41\ldots\ldots$$

Similarly, as $(1{\cdot}414)^2 = 1{\cdot}999396 < 2$ and

$$(1{\cdot}415)^2 = 2{\cdot}002225 > 2, \text{ so}$$

$$1{\cdot}414 < \sqrt{2} < 1{\cdot}415. \qquad \therefore \quad \sqrt{2} = 1{\cdot}414\ldots\ldots$$

Continuing this process one can find the value of $\sqrt{2}$ correct to any place of decimal. The decimal expression of $\sqrt{2}$ will

not be terminating or non-terminating recurring. For, if it were terminating or non-terminating recurring, then $\sqrt{2}$ would have been a rational number. But $\sqrt{2}$ is not a rational number.

Similarly, one can determine the approximate value of any irrational number correct to any place of decimal and that decimal expression of the number will be *non-terminating* and *non-recurring*. Conversely, it can be proved that any non-terminating and non-recurring decimal expression represents an irrational number.

From the preceding discussion about rational and irrational numbers we can say that if the decimal expression of a number be terminating or non-terminating and recurring, then the number is a rational number and if the decimal expression is non-terminating and non-recurring, then the number is an irrational number.

The sum and product of two real numbers are also real numbers and the real numbers satisfy for addition and multiplication the properties of rational numbers stated in § 1·5.

The totality of all real numbers between any two real numbers a and b $(a<b)$ is called an *interval.*

If the extreme points a and b be included in the interval, then the interval is called a *closed interval* and is denoted by $[a, b]$.

If the extreme points a and b are not included in the interval, then the interval is an *open interval* and it is denoted by (a, b). Hence if x be a number belonging to the closed interval, then $a \leqslant x \leqslant b$. If x be a number belonging to the open interval, then $a<x<b$. The totality of real numbers is frequently denoted as $-\infty<x<\infty$.

§ 1·9. Absolute value :

The absolute value of any number a is denoted by the symbol $|a|$ and is defined as follows :

$$\begin{aligned} |a| &= a, \text{ if } a \text{ be positive, i.e, } a>0 \\ &= 0, \text{ if } a=0 \\ &= -a, \text{ if } a \text{ be negative, i.e., } a<0 \end{aligned}$$

For example,

$$|3|=3\ (\because\ 3>0),\ |0|=0,$$
$$|5|=-(-5)=5\ (\because\ -5<0),\ |\sqrt{2}|=\sqrt{2}$$
$$|2-3|=|-1|=1 \text{ etc.}$$

$\therefore$ The absolute value of a number a is its numerical value. If $|x|=a$, then $x=+a$, or $-a$.

Example 1

Ex. 1. For which value of x, the following are undefined ?

(i) $\dfrac{x^3-8}{x-2}$ (ii) $\dfrac{\sin x}{x}$ (iii) $\dfrac{x^2+4x+1}{x^2-2x+1}$.

(i) when $x=2$, $x-2=0$. Since the denominator cannot be zero, so the quantity is undefined when $x=2$.

(ii) When $x=0$, the value of the quantity is of the form $\frac{0}{0}$ which is undefined. H ence $\dfrac{\sin x}{x}$ is undefined when $x=0$.

(iii) When $x=1$, the denominator becomes zero. Hence the quantity becomes undefined when $x=1$.

Ex. 2. Show that $\sqrt{3}$ is an irrational quantity.

If possible let $\sqrt{3}=\frac{p}{q}$, where p and q are two mutually prime positive integers (i.e., p and q have no factor in common) and $q\neq0$.

$$\because\quad \sqrt{3}=\frac{p}{q},\qquad \therefore\quad \sqrt{3}.q=p,$$

squaring we get $3\ q^2=p^2$.

$\therefore$ 3 is a factor of p^2. So, 3 is also a factor of p.

[For, if $p=3k+1$ or, $3k+2$, then p^2 will be$=3(3k^2+2k)+1$, or, $3(3k^2+4k+1)+1$].

$\therefore$ p can be written as $p=3k$, where k is a positive integer.

$$\therefore\quad p^2=9k^2,\qquad \text{or, } 3q^2=p^2=9k^2$$

or, $q^2=3k^2$. So 3 is a factor of q^2 and hence of q (as before). Hence 3 is a common factor of p and q. But this is

contradictory to our assumption p and q have no common factor. The contradiction is due to our assumption $\sqrt{3}=\frac{p}{q}$.

Hence $\sqrt{3}$ cannot be expressed in the form $\frac{p}{q}$ and $\sqrt{3}$ is irrational.

Ex. 3. Find the value of $3^{\frac{1}{3}}$ to the second place of decimal.

$$1^3=1<3 \text{ and } 2^3=8>3,$$

$$\therefore \; 1^3<3<2^3, \quad \text{or,} \quad 1<3^{\frac{1}{3}}<2$$

Now cubing $1{\cdot}1, 1{\cdot}2, \dots 1{\cdot}9$ we find,

$$(1{\cdot}4)^3=2{\cdot}147<3 \text{ and } (1{\cdot}5)^3=3{\cdot}375>3$$

$$\therefore \; 1{\cdot}4<3^{\frac{1}{3}}<1{\cdot}5.$$

cubing $1{\cdot}41, 1{\cdot}42, \dots 1{\cdot}49$ we find,

$$(1{\cdot}44)^3=2{\cdot}985984<3 \text{ and } (1{\cdot}45)^3=3{\cdot}193625>3$$

$$\therefore \; 1{\cdot}44<3^{\frac{1}{3}}<1{\cdot}45.$$

$$\therefore \; 3^{\frac{1}{3}}=1{\cdot}44\dots$$

Ex. 4. Show that $|ab|=|a|.|b|$

If $a\geqslant 0$, $b\geqslant 0$, then $ab\geqslant 0$.

$$\therefore \; |a|=a, \; |b|=b \text{ and } |ab|=ab=|a| \; |b|$$

If $a\geqslant 0$, $b<0$, then $ab\leqslant 0$ and $|a|=a$,

$$|b|=-b \quad \therefore \; |ab|=-ab=a(-b)=|a| \; |b|$$

Similarly, if $a<0$, $b\geqslant 0$, $|ab|=|a| \; |b|$

If $a<0$, $b<0$, then $ab>0$ and $|a|=-a$,

$$|b|=-b, \; |ab|=ab=(-a)(-b)=|a|, |b|.$$

Ex. 5. Prove that $|a+b|\leqslant|a|+|b|$

$$\because \; |a|=+a \text{ when } a>0$$

$$=-a \text{ when } a<0,$$

$$\text{so, } \; a\leqslant|a|, -a\leqslant|a|$$

Now if $a+b\geqslant 0$. then $|a+b|=a+b\leqslant|a|+|b|$

and if $a+b<0$, $|a+b|=-(a+b)=-a-b$

$$\leqslant|a|+|b|$$

So, for all values of a and b, $|a+b|\leqslant|a|+|b|$.

Ex. 6. Show that between two rational numbers there exist an infinite number of rational numbers.

Let a and b $(b>a)$ be two rational numbers and also let n be a positive integer. Now $\frac{b-a}{n}$ is a rational number. Since the rational numbers are closed with respect to addition, so $a+\frac{b-a}{n}, a+2\frac{b-a}{n}, a+3\frac{b-a}{n}, \cdots, a+(n-1)\frac{b-a}{n}$ are all rational numbers clearly each of these numbers lies between a and b. Hence between a and b, we obtain n number of rational numbers. Now n can be made large at pleasure. Hence between a and b there exist an infinite number of irrational numbers.

[Note : Between any two real numbers there exist an infinite number of real numbers.]

Ex. 7. Show that if $\frac{p}{q}$ be a rational number, then $\frac{p}{q}+\sqrt{2}$ and $\frac{p}{q}\sqrt{2}$ are irrational numbers.

If possible let $\frac{p}{q}+\sqrt{2}$ is rational and $\frac{p}{q}+\sqrt{2}=\frac{r}{s}$ [p, q, r, s are integers and $q\neq 0, s\neq 0$].

$$\therefore \quad \sqrt{2}=\frac{r}{s}-\frac{p}{q}=a \text{ rational number.}$$

But $\sqrt{2}$ is not a rational number. So, there must be something wrong. The only possible reason of the mistake is our assumption $\frac{p}{q}+\sqrt{2}$ is rational. So $\frac{p}{q}+\sqrt{2}$ is irrational.

Similarly, if possible let $\frac{p}{q}\sqrt{2}$ is irrational.

and $\frac{p}{q}\sqrt{2}=\frac{r}{s}$ $\quad \therefore \quad \sqrt{2}=\frac{r}{s}/\frac{p}{q}$ (assuming $p\neq 0$)

$=\frac{qr}{ps}$ which is rational. But $\sqrt{2}$ is irrational.

$$\therefore \quad \frac{p}{q}\sqrt{2} \text{ is irrational.}$$

Ex. 8. Find two irrational numbers whose (i) sum and (ii) product are rational numbers.

(i) $2+\sqrt{5}$ are $3-\sqrt{5}$ are two irrational numbers, but their sum $(2+\sqrt{5})+(3-\sqrt{5})=5$ is rational.

(ii) $3+\sqrt{5}$, $3-\sqrt{5}$ are two irrational numbers but their product $(3+\sqrt{5})(3-\sqrt{5})=9-5=4$.

Ex. 9. Prove that, if $|x-a|<b$, then $a-b<x<a+b$.

If $x-a>0$, then $x-a=|x-a|<b$

$\therefore\ x<a+b$.

Again, $\because\ x-a>0$. $\therefore\ x>a$ $\therefore\ a<x$.

$\therefore\ a-b<x$ [$\because$ b is positive]

$\therefore\ a-b<x<a+b$.

If $x-a<0$, then $|x-a|=a<x$.

So $a-x<b$ $\therefore\ a-b<x$.

Again, as $x-a<0$, $\therefore\ x<a$, $\therefore\ x<a+b$

[$\because$ b is positive]

Hence in this case also $a-b<x<a+b$.

Exercise 1

1. For which values of x the following are undefined ?

(i) $\dfrac{x^2}{x}$ (ii) $\dfrac{\tan x}{x}$ (iii) $\dfrac{1-\cos 2x}{\sin 2x}$ (iv) $\dfrac{x^5-32}{x-2}$

2. For which values of x, the following equations are not true ?

(i) $\dfrac{x}{x}=1$ (ii) $\dfrac{x^3+a^3}{x^2-a^2}=\dfrac{x^2-xa+a^2}{x+a}$

(iii) $\dfrac{1-x}{1+\sqrt{x}}=1-\sqrt{x}$ (iv) $\dfrac{1-\cos 2x}{\sin 2x}=\tan x$.

3. Prove that $\sqrt{5}$ is an irrational number.

4. Find the value of $\sqrt{7}$, correct to four places of decimals.

5. Find the value of $\sqrt[3]{2}$, correct to three places of decimals.

6. If r be a positive integer and not a perfect square, then show that $\frac{p}{q}\sqrt{r}$ is an irrational number. (p, q integers and $q \neq 0$).

7. Find eight rational numbers between $\frac{4}{5}$ and $\frac{10}{11}$.

8. Show that $\sqrt{3} > \sqrt{2}$. Find four irrational numbers between $\sqrt{2}$ and $\sqrt{3}$.

9. Show that corresponding to every irrational number a, one can always find an integer n and an irrational number b, for which $a = n + b$ and $0 < b < 1$.

10. Show that between any two irrational numbers there exist an infinite number of irrational numbers.

11. Between $\frac{1}{2}$ and 1

(i) how many real numbers are there ?

(ii) is there any rational number which is not an integer.

(iii) is there any rational number which is not a real number ? [A. I. H. S. '70]

12. Select suitable words from the brackets for filling up the blanks :

(i) The sum of two……is always a……
[rational numbers, irrational numbers].

(ii) The decimal expression of an irrational number is a……decimal. [terminating, non-terminating recurring, non-terminating non-recurring].

(iii) If the decimal expression of a number is terminating, then the number is……[rational, irrational].

13. Show that if the square of an odd integer is divided by 8, the remainder is always 1. [A. I. H. S. '72]

14. Prove that the product of three consecutive natural numbers is always divisible by 6.

15. Arrange the following numbers in order of magnitude.

(i) $\sqrt{3}, \sqrt{8}, \sqrt[3]{10}$, (ii) $\sqrt[3]{2}, \sqrt{3}, \sqrt[4]{7}$,

(iii) $\sqrt[4]{9}, \sqrt[6]{25}, \sqrt[3]{8}$.

CHAPTER TWO

VARIABLE AND FUNCTION

§ 2·1. Variable and Constant.

While discussing Mathematics, different Natural science such as Physics, Chemistry or different social sciences such as Economics we have to deal with different quantities, like mass, time, weight, temperature, price of commodity etc. The magnitudes of these quantities are generally expressed by real numbers. We generally get concerned with their numerical values without taking their units into consideration. So the quantities are taken as pure real numbers.

The quantities are denoted by the letters $a, b, c, x, y, z \cdots$ of the English alphabet.

In different mathematical discussions, we have, in most cases, to use more than one quantity. Of these quantities, some changes in value and others do not. Quantities whose values can undergo changes are called *variable quantities* or *variables* and which cannot undergo change are called *constants.*

For example, let a particle starting from a point O moves along a straight line with uniform velocity. Now if we denote time by t and the distance of the particle from O by s and also the velocity by a, then we find that the values of t and s change, but being uniform velocity, the value of a cannot undergo any change. So here t and s are variable and a is a constant.

From the above discussion we can define variables and constants as follows.

Def. If in a mathematical discussion, a quantity can assume more than one value, then the quantity is called a *variable quantity* or a *variable.*

If in a mathematical discussion a quantity cannot assume more than one value, then it is called a *constant.* Following are some examples of variables and constants which you should note carefully.

Ex. 1. If the area and radius of circle be denoted by A and r respectively, then $A=\pi r^2$. Here for different values of r, the values of A also become different, but π remains constant. Hence when r is variable, then A will be a variable quantity but π is always a constant.

Ex. 2. Let us consider the case when water is heated in a pot. Now if the temperature of water be denoted by t, then t changes with the gain in heat, so that t is a variable.

Ex. 3. Let a particle is falling due to gravity. If the distance of the particle from the earth's surface be x and its velocity be v, then both x and v are variables. But if the acceleration due to gravity be g, then g is constant.

Ex. 4. The equation of a parabola is $y^2=4ax$. Here x and y are variables but a is constant.

Note: (1) If a variable can assume only real values, then the variable is said to be a real variable. In this book we shall discuss only real variables. So, by a variable quantity or variable we shall mean a real variable.

(2) Conventionally; constants are denoted by the letters a, b, c, d, etc. of the upper half of the English alphabet and the variables are denoted by the letters x; y, z, u, v, w,$\cdots$of the lower half. Remember that this is just a convention and not a rule.

§ 2·2. Range of a variable.

A variable may not be capable of assuming all real values. The totality of the real values which a variable can assume is called the *range* of the variable. Note carefully the following examples

(1) Let t be the temperature of water contained in a vessel. Now the value of t cannot be more than 100°C ; for, if the temperature be more than 100°C, then the water becomes vapour. Also the temperature of water cannot be less than 0°C ; for in that case the water becomes ice. So, in this case we can say that the value of t can be any real number between 0 and 100. This is expressed as the closed interval $0 \leqslant t \leqslant 100$. Hence the range of t is the closed interval $0 \leqslant t \leqslant 100$.

(2) Let x be the number obtained by a student in Mathematics in the secondary examination. Now the student may obtain respectively 0 and 100 in the minimum or maximum. Again, as in the secondary examination marks are published in whole numbers, so the possible values of x in this case are the whole numbers between 0 and 100.

Hence the range of x is $\{0, 1, 2, 3, \cdots, 98, 99, 100\}$

(3) Let $y=\sqrt{(x-1)(2-x)}$ where y is real. Now y to be real, the value of $(x-1)(2-x)$ must be positive which is possible if and only if the value of x is greater than or equal to 1 but less than or equal to 2. Hence the range of x in this case is the closed interval $1 \leqslant x \leqslant 2$.

In example (1), the value of t may be any real number between 0 and 100. But in example (2), the value of x cannot be all the real numbers between 0 and 100. In example (1), the variable t is a *continuous variable* and in example (2), the variable x is a discontinuous or discrete variable. Hence continuous variable may be defined as follows.

Def. If a variable is capable of assuming all values within an interval, then the variable is said to be continuous in that interval.

Otherwise, the variable is said to be a discontinuous or a discrete variable.

[Note. In this book we shall generally be concerned with continuous variables and by a variable mean a continuous variable, except otherwise stated.]

Sometimes by imposing conditions we take a part of the natural range of a variable as its range. For example, if the equation of a Circle be $x^2+y^2=1$, then $y=\pm\sqrt{1-x^2}$. As y is real, so $1-x^2$ must be positive. Hence the value of x must lie between -1 and $+1$. Hence the natural range of the variable is $-1 \leqslant x \leqslant 1$.

Now, if we take into consideration only the portion of the Circle in the first quadrant, then x cannot be negative and then the range of x will be $0 \leqslant x \leqslant 1$.

Ex. If $y=\sin x$, then x may assume any real value, but the value of y will lie between -1 and $+1$. Hence the range of x is $-\infty<x<\infty$ and the range of y is $-1\leqslant y<1$.

If now the conditiou $0\leqslant x<\frac{\pi}{2}$ be imposed on x, than the range of y will be $0\leqslant x<1$.

§ 2·3. **Function.** In different mathematical discussions it is frequently found that two variables are connected in such a way that one is dependent on the other. For example, the area of a circle is dependent on the length of the radius. For every value of the radius, we get a definite value of the area, so the area of a circle is said to be a *function* of the radius of the circle. Hence functions can be defined as follows.

Def. If in a mathematical discussion two variables x and y be connected in such a way that for all possible values of x, there exists a definite value of y, then y is said to be a *function* of x.

Since y is dependent on x, so x is called the independent variable and y is called the dependent variable.

For example, let x and y be respectively the length of a side and the area of square. Then $y=x^2$.

Now, if $x=1$. then $y=1^2=1$

$x=1{\cdot}1$, then $y=1{\cdot}1^2=1{\cdot}21$

$x=2$, then $y=2^2=4$

$x=\sqrt{2}$, then $y=(\sqrt{2})^2=2$, etc.

Hence we find that for every value of x, we can get a definite value of y. Hence y is a function of x.

Let us take another example. Let a particle is moving in a straight line starting from the point O. Let at time t after start, the distance of the particle from O be s. Now at every instance, the particle will be at a definite distance from the point O. So for every possible values of t (here the range of t is 0 and all the positive real numbers i.e., $0\leqslant t\leqslant\infty$), s has a definite value. Hence s is a function of t. This is generally expressed as $s=f(t)$.

Note. If y be a function of x, then for all values of x in its range y will have definite values. Now, it may so happen that for different values of x, the value of y remains the same. In that case y is a *constant function* of x.

§ 2·4. Notation of function.

If y, be a function of x, then we write $y=f(x)$. (This is read as y is equal to f, x).

Hence y and $f(x)$ denote the same quantity. f is the first letter of the word function. $y=f(x)$ means y is the dependent variable and x is the independent variable and y is a function of x.

F(x). $\phi(x)$, $g(x)$ are also used to denote functions of x

The totality of the values of x for which values of $y=f(x)$ can be obtained. i.e , $y=f(x)$ is defined, is called the *domain of definition* of the function.

If $y=\sqrt{(x-1)(2-x)}$, i.e , $y=f(x)=\sqrt{(x-1)(2-x)}$, thsn the domain of definition of $f(x)$ is the closed interval $1\leqslant x\leqslant 2$; values of x outside this closed intervals make y imaginary.

A mathematical equation connecting x and y generally defines a function.

For example, from the equation $y=\sin x$, for every value of x we get a definite value of y. (If $x=0$, then $y=0$, if $x=\frac{\pi}{2}$, then $y=1$, etc.). Hence y is a function of x. Hence we shall write $f(x)=\sin x$.

If a be any real number, within the domain of definition of $y=f(x)$, then we denote the value of y when $x=a$ by $f(a)$.

If $y=f(x)=x^2-x+1$, then

$f(0)$ is the value of y at $x=0$.

Now, when $x=0$, then $y=0^2-0+1=1$.

$\therefore$ $f(0)=1$. Similarly, $f(1)=1^2-1+1=1$.

$f(1\cdot1)=(1\cdot1)^2-1\cdot1+1=1\cdot11$, $f(a)=a^2-a+1$, etc.

If $f(x)=\sin x$, then $f(0)=\sin 0=0$.

$$f\left(\frac{\pi}{2}\right)=\sin\frac{\pi}{2}=1,\quad f\left(\frac{\pi}{4}\right)=\sin\left(\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}} \text{ etc.}$$

If $f(x)=\dfrac{x^2-4}{x-2}$ then $f(0)=\dfrac{-4}{-2}=2,$

$$f(1)=\frac{1^2-4}{1-2}=\frac{-3}{-1}=3,\quad f(2)=\frac{4-4}{2-2}=\frac{0}{0}$$

which is not defined.

Hence at the point $x=2$, i.e., when the value of x is 2, the function is undefined.

$$f(2+h)=\frac{(2+h)^2-4}{(2+h)-2}=\frac{h^2+4h}{h}=h+4.$$

If x and y are connected by an equation of the form $f(x, y)=0$, then y is said to be an *implicit* function of x.

[$f(x, y)$ is a function depending on the two variables x, y]

The equation $f(x, y)=0$ can in many cases be solved in the form $y=\phi(x)$. A function expressed in the form $y=f(x)$, is called an *explicit* function of x. In $f(x, y)=x^3-2xy+4y+1=0$, y is an implicit function of x. Solving the equation $f(x, y)=0$, we can write $y=\dfrac{x^3+1}{2(x-2)}$. So the equation $f(x, y)=0$ is expressed in the explicit form $y=\phi(x)=\dfrac{x^3+1}{2(x-2)}$. But an implicit function cannot always be expressed in the explicit form.

For example, the equations $x^2+xy+3y^3+4x^3-3=0$, $e^y+x^2+y^2=0$ cannot be expressed in the explicit form $y=f(x)$.

If a function $y=f(x)$ can be expressed in the form $x=\phi(y)$, then $\phi(y)$ is said to be the inverse function of the function $f(x)$.

For example, if $y=2x+1$, then $x=\dfrac{y-1}{2}$.

So $x=\dfrac{y-1}{2}$ is the inverse function of the function $y=2x+1$. Similarly, the inverse function of $y=a^x$ is $x=\log_a y$. The inverse function of $y=\sin x$ is $x=\sin^{-1}y$. It may not be possible to determine the inverse of every function.

For example, $y=x^3+x^2+1$ cannot be expressed in the form $x=\phi(y)$. The inverse of the function $y=f(x)$ is frequently written as $x=f^{-1}(y)$.

§ 2·5. Graph of a Function :

Let $y=f(x)$ be a function of x. Two mutually orthogonal straight lines $\overleftrightarrow{XX'}$, $\overleftrightarrow{YY'}$ intersect at O. The unit distance on $\overleftrightarrow{XX'}$ is fixed by taking a point I on OX as in chapter one. So any real number, can now be represented by a point on $\overleftrightarrow{XX'}$. Similarly points on $\overleftrightarrow{YY'}$ can also represent the real numbers.

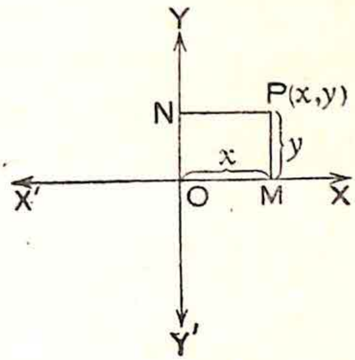


Fig. 3

Let the points M on $\overleftrightarrow{XX'}$ and N on $\overleftrightarrow{YY'}$ represent respectively a particular value of x and the corresponding value of the variable y. Now through M and N draw parallels to $\overrightarrow{OY}$ and $\overrightarrow{OX}$ respectively to intersect at the point P. Then the co-ordinates of P are (x, y). Now, as x varies in its range, then P will also move in the xy-plane and describe a curve in the plane. This curve is called the graph of the function $y=f(x)$. Hence as x varies in its range, the set of points on the xy-plane with abscissa x and ordinate y constitute the graph of the function $y=f(x)$.

Note (1) The *general rule* of drawing graph of a function is to determine values of y corresponding to a few values of x and plot points on the xy-plane, the points with the values of x as abscissa and the corresponding values of y as their ordinates. Now these points are joined by a free-hand line to determine the graph.

(2) Graphs are geometrical representations of functions. This geometrical representation makes it more convenient to discuss the properties of a function.

(3) A function may not have a mathematical form. But its graph may be drawn and with the help of this graph we may understand the nature of the function. Let at time t, the temperature of a patient be θ°C. Now at each instant the

patient has a temperature. Hence θ will have a value for every possible value of t. Hence θ is a function of t. It is difficult to connect θ and t by any mathematical relation, but the graph can be drawn.

Ex. 1. Draw the graph of the function $f(x)=2x+1$.

Let $y=f(x)$ and determine a few points on the graph from the following table.

x	0	1	$-\frac{1}{2}$
y	1	3	0

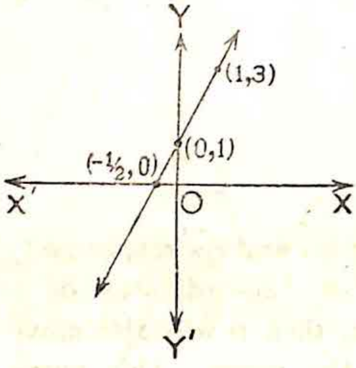


Fig. 4

Now, plot the points $(0, 1)$, $(1, 3)$, $(-\frac{1}{2}, 0)$ and draw the curve, (in this case a straight line) joining these points. The straight line is in this case the required graph.

From the graph note that,

(i) The function is not disconnected anywhere i.e., there is no gap in the graph.

(ii) The value of y increases as the value of x increases (such a function is called an increasing function).

Ex. 2. Draw the graph of the curve $f(x)=x^2$.

Let $y=x^2$ and determine a few points on the graph.

x	0	1	2	-1	-2	± 3
y	0	1	4	1	4	9

Now, plot the points $(0, 0)$, $(1, 1)$, $(2, 4)$, $(-1, 1)$, $(-2, 4)$, $(3, 9)$, $(-3, 9)$ and join them by a free hand line. The curve obtained is the required graph.

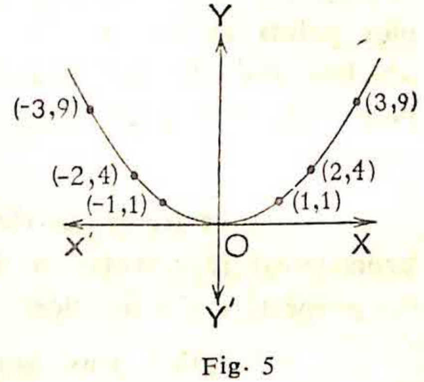


Fig. 5

Note that (i) as the graph entirely lies above the x-axis, the value of $f(x)$ cannot be negative, (ii) The minimum value of $f(x)$ is 0, (iii) $f(x)$ has no finite maximum value. (iv) There is no gap in the graph.

Ex. 3. Draw the graph of the function $f(x) = |x|$.

Let $y = |x|$

x	0	1	2	-1	-2
y	0	1	2	1	2

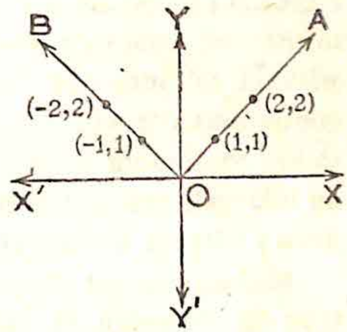


Fig. 6

The points (0, 0), (1, 1), (2, 2) etc. are points on the line $\overrightarrow{OA}$ and the points (0, 0), (−1, 1), (−2, 2) are points on the line $\overrightarrow{OB}$. Hence $\overrightarrow{OA}$ and $\overrightarrow{OB}$ are the graphs of the given function. Here note that (i) $f(x)$ is not negative and its minimum value is 0, (ii) the graph has two branches which are not disconnected and (iii) the graph is symmetrical about the y-axis.

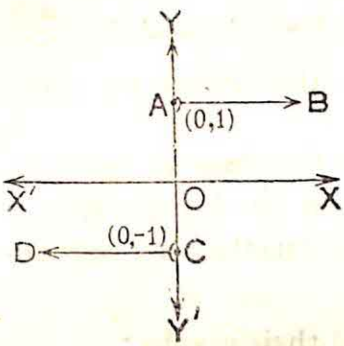


Fig. 7

Ex. 4. Draw the graph of $\frac{|x|}{x}$.

Here when $x > 0$, $y = f(x) = \frac{x}{x} = 1$

When $x < 0$, $y = f(x) = \frac{-x}{x} = -1$

and when $x = 0$, $y = f(x) = \frac{|0|}{0} = \frac{0}{0}$ which is indeterminate.

Let A and C be the points (0, 1) and (0, −1). $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are straight lines parallel to the x-axis. Now, $\overleftrightarrow{AB}$ is the graph of $y = 1$ and $\overleftrightarrow{CD}$ is the graph of $y = -1$.

Hence the rays $\overrightarrow{AB}$ and $\overrightarrow{CD}$ except the points A and C are the graphs of the function. [That the points A and C do not belong to the graph have been shown in the graph by) and (signs] Here note that (i) The function is not defined when $x = 0$, (ii) The function has two branches and (iii) The graph is disconnected at the point $x = 0$ and the graph cannot be drawn continuously without withdrawing the pen from the paper.

§ 2·6. Continuity of a function :

Let the domain of definition of the function $y = f(x)$ be an interval of x.

Now the function $f(x)$ is said to be continuous at a point $x=a$ of the interval, if on both sides of the point in the neighbouring region the graph of the function be continuous i.e., if on both sides in the neighbouring region of the point, the graph can be drawn without withdrawing the pen from the paper. If a function be continuous at every point of an interval, then the function is said to be continuous in that interval. Hence if $f(x)$ be continuous in an interval, the graph of the function within that interval can be drawn without withdrawing the pen from the paper.

If at any point there be any break in the graph of a function, then the function is said to be discontinuous at that point. In the neighbouring region of a point where a function is not continuous, the graph of the function cannot be drawn without withdrawing the pen from the paper.

In the last article (§ 2·5) the function of example 4, $\left(f(x)=\frac{|x|}{x}\right)$ is discontinuous at the point $x=0$. At all other points the graph is continuous.

There is no break in the graphs of the functions in examples 1, 2, 3. The graphs of these functions can be drawn without lifting the pen from the paper. Hence these functions are continuous everywhere.

§ 2·7. Basic elementary functions and their graphs :

The basic elementary functions are as follows :

I. Power function : Its form is $f(x)=x^n$, where n is a constant.

II. Exponential function : $f(x)=a^x$ where a is a positive number whose value is other than 1.

III. Logarithmic function :

$f(x)=\log_a x$, where $a>1$ but $a\neq 1$; logarithmic functions are inverses of exponential functions.

IV. Trigonometric functions :

$\sin x$, $\cos x$, $\tan x$, $\cot x$, $\sec x$ and $\operatorname{cosec} x$ are trigonometric functions of x.

V. Inverse trigonometric (or circular) functions :

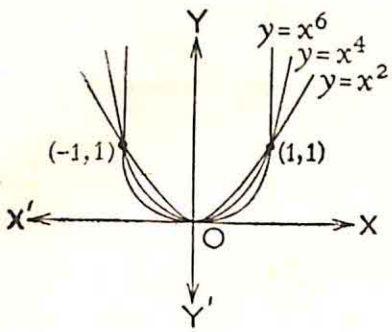
$\sin^{-1}x$, $\cos^{-1}x$, $\tan^{-1}x$, $\cot^{-1}x$, $\operatorname{cosec}^{-1}x$ and $\sec^{-1}x$ are inverse trigonometric functions.

We shall now discuss about the range, domain of definition and graph of these elementary functions.

I. Power function : $y=f(x)=x^n$.

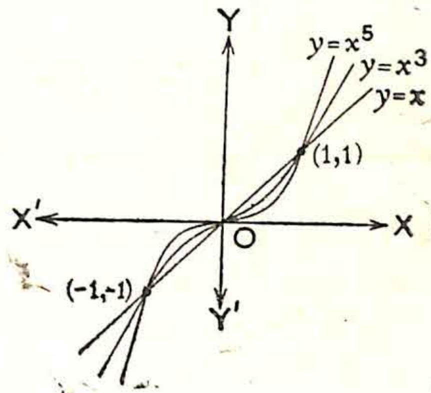
(*i*) If n be a positive integer, then the range of x is $-\infty<x<\infty$ and that of y is (*i*) $0\leqslant y\leqslant\infty$ when n is even and (*iii*) $-\infty<y<\infty$, when n is odd.

In figures 8 and 9, the graphs of x^n for different positive integral values of n are shown.



$n=\text{even}>O$

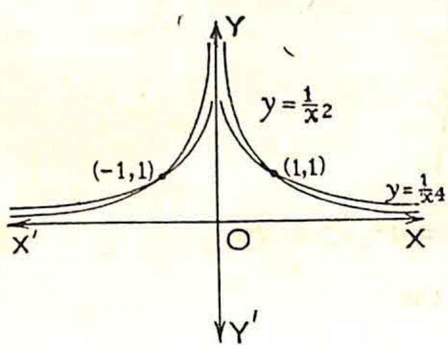
Fig. 8



$n=\text{odd}>O$

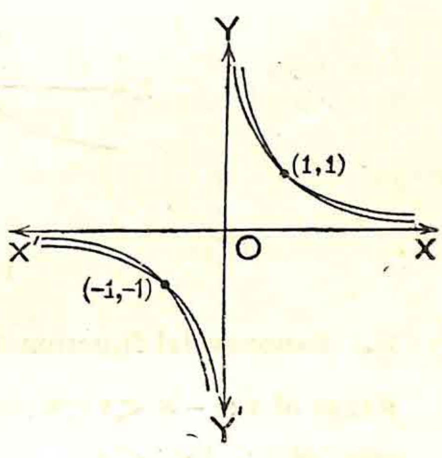
Fig. 9

(*ii*) If n be a negative integer, then the range of x and y will be the same as before, only $x\neq 0$.



$n=\text{even}<O$

Fig. 10



$n=\text{odd}<O$

Fig. 11

In figures 10 and 11, the graphs of x^n for different negative integral values of n are shown.

Figures 12, 13 and 14 are the graphs of x^n for different fractional values of n.

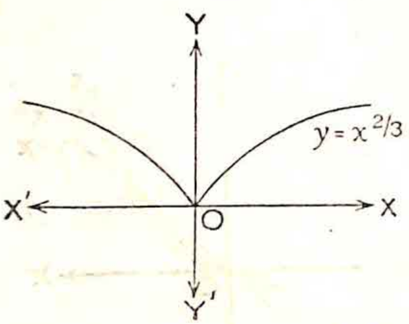


Fig. 12

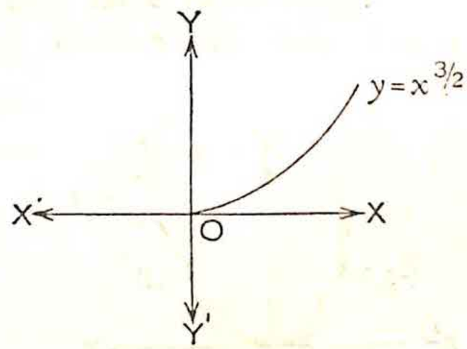


Fig. 13

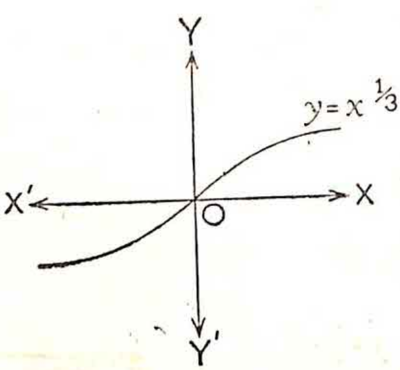


Fig. 14

II. Exponential function : $y=a^n$, $a>0$, $a\neq 1$.

Range of x : $-\infty<x<\infty$, and the

range of y : $0<y<\infty$

In Fig. 15. graphs of the function for different values of a are shown.

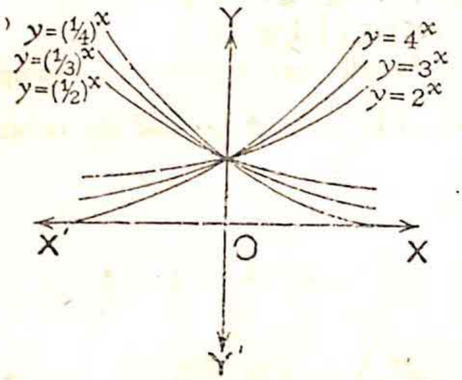


Fig. 15

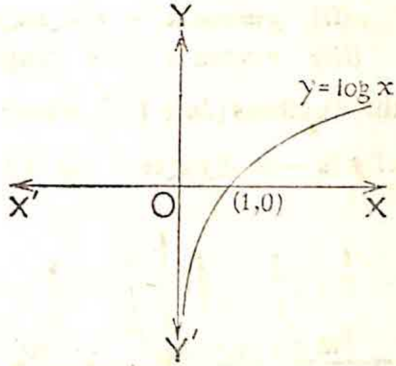


Fig. 16

III. Logarithmic functions : $y=f(x)=\log_a x$, $a\neq 1$, $a>0$.

The range of x is $0<x<\infty$ and that of y is $-\infty<y<\infty$.

The graph of the function is shown in fig. 16.

IV. Trigonometric functions : The trigonometric functions are periodic functions. A function is said to be a periodic function if there exists a constant a, such that $f(x+a)=f(x)$, i.e., the values of $f(x)$ at the points x and $x+a$ be the same. a is said to be the *period* of the function. You have learnt in trigonometry that,

$\sin(x+2\pi)=\sin x$, $\cos(x+2\pi)=\cos x$,
$\tan(x+2\pi)=\tan x$, $\cot(x+2\pi)=\cot x$,
$\sec(x+2\pi)=\sec x$, $\operatorname{cosec}(x+2\pi)=\operatorname{cosec} x$.

Hence the trigonometric functions are periodic, each with period 2π. The range and graphs of the trigonometric functions are given below,

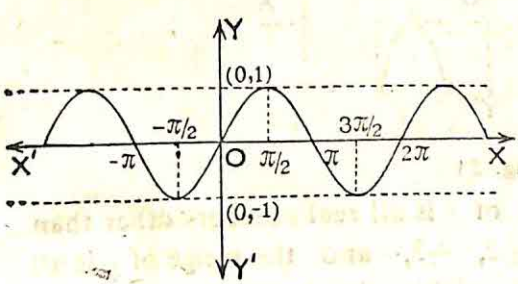


Fig. 17

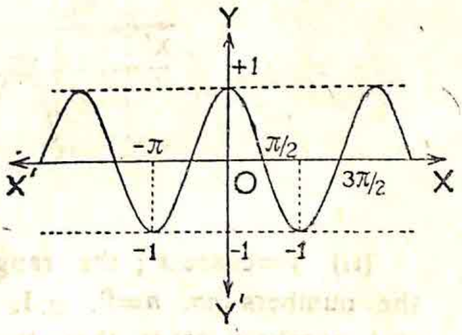


Fig. 18

(*i*) $y=f(x)=\sin x, -\infty<x<\infty, -1\leqslant y\leqslant 1$ [Fig. 17]

(*ii*) $y=\cos x, -\infty<x<\infty, -1\leqslant y\leqslant 1$ [Fig. 18]

(*iii*) $y=\tan x$; the range of x is all real numbers excepting the numbers $(2n+1)\frac{\pi}{2}$, where $n=0, \pm1, \pm2, \pm3,\cdots$and the range of y is $-\infty<y<\infty$ [Fig. 19].

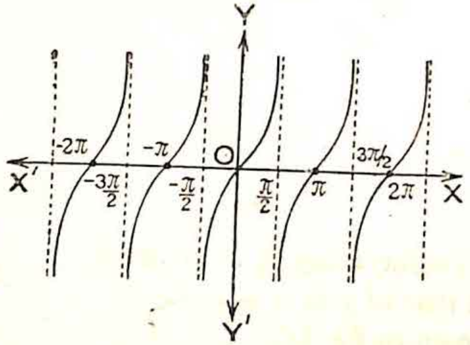


Fig. 19

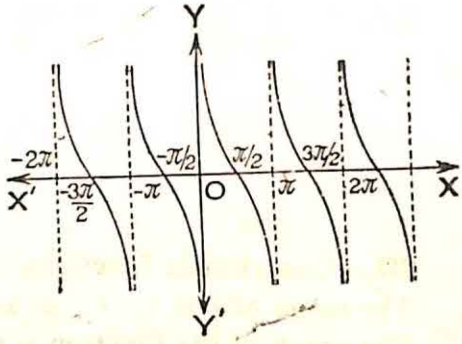


Fig. 20

(*iv*) $y=\cot x$; the range of x is all real numbers excepting the numbers $n\pi$, when $n=0, \pm1, \pm2, \pm3$, and the range of y is $-\infty<y<\infty$.

(*v*) $y=\sec x$; the range of x is all real numbers other than the numbers $(2n+1)\frac{\pi}{2}$, $n=0, \pm1, \pm2, \pm3$.........and the range of y is $-\infty<y\leqslant-1$ and $1\leqslant y<\infty$ i.e., all real numbers other than the numbers within the interval $-1<y<1$. [Fig. 21]

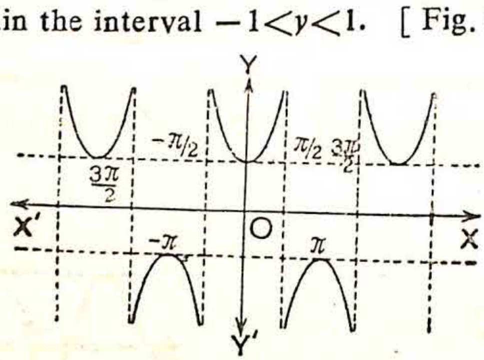


Fig. 21

(*vi*) $y=\operatorname{cosec} x$; the range of x is all real numbers other than the numbers $n\pi$, $n=0, \pm1, \pm2, \pm3,\cdots$and the range of y is all real numbers other than those within the interval $-1<y<1$. [Fig. 22]

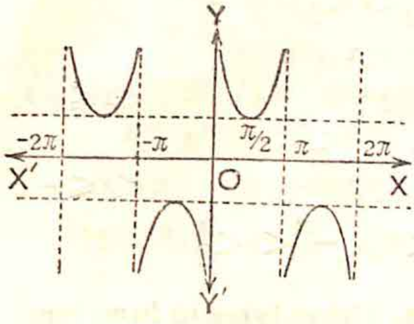


Fig. 22

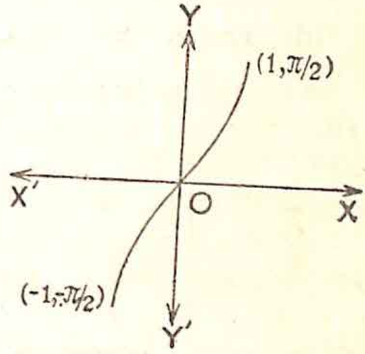


Fig. 23

V. **Inverse Trigonometric functions :**

(i) $y=\sin^{-1}x$; $-1\leqslant x<1$; $-\frac{\pi}{2}\leqslant y<\frac{\pi}{2}$.

Actually for every value of x within $-1\leqslant x\leqslant 1$, y will have an infinite number of values. But we shall consider only those values within the interval $-\frac{\pi}{2}\leqslant x\leqslant\frac{\pi}{2}$ [Fig. 23]

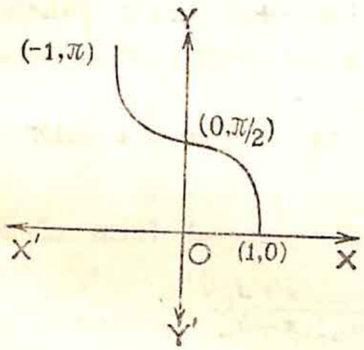


Fig. 24

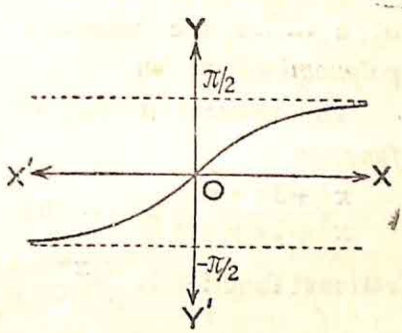


Fig. 25

(ii) $y=\cos^{-1}x$; $-1\leqslant x\leqslant 1$, $0\leqslant y\leqslant\pi$ (Fig. 24)

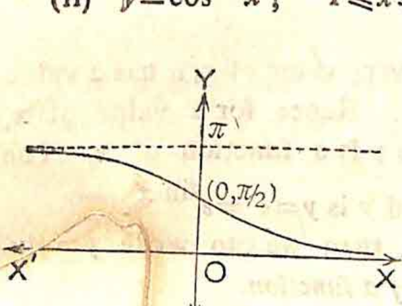


Fig. 26

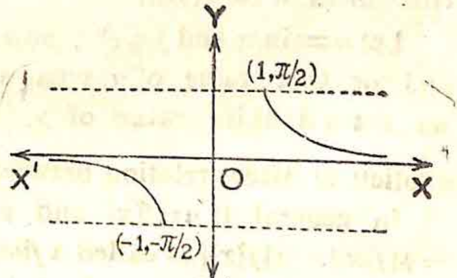


Fig. 27

(iii) $y=\tan^{-1}x$; $-\infty<x<\infty$, $-\frac{\pi}{2}<y<\frac{\pi}{2}$ (Fig. 25)

(iv) $y=\cot^{-1}x$; $-\infty<x<\infty$, $0<y<\pi$ (Fig. 26)

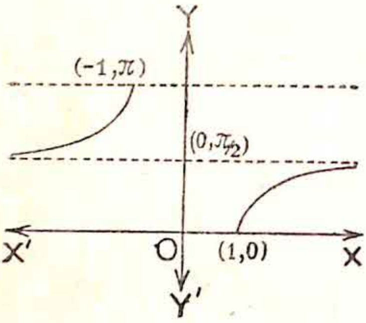


Fig. 28

(v) $y=\sec^{-1}x$; $-\infty<x\leqslant-1$, $1\leqslant x<\infty$, $0<y<\pi$ (Fig. 28)

(vi) $y=\operatorname{cosec}^{-1}x$; $-\infty<x\leqslant-1$, $1\leqslant x<\infty$, $-\frac{\pi}{2}<y<\frac{\pi}{2}$ (Fig. 27)

§ 2·8. Other types of functions ; function of a function.

By addition, subtraction, multiplication and division of the elementary functions one can obtain other types of functions. For example, from the elementary functions x^3, $\sin x$, e^x, etc. we obtain the functions $x^3-3\sin x$, $xe^x+\sin x$, $\frac{e^x}{x-\sin x}$, etc.

A function of the form $a_0x^n+a_1x^{n-1}+\cdots+a_{-1}x+a$ (where $a_0, a_1 \ldots\ldots a_n$ are constants and n is a positive integer) is called a *polynomial function.*

The quotient of two polynomial functions is said to be a *rational function.*

$\frac{x^2+3x+4}{x^2+2x+1}$ is a rational function. The general form of a rational function is $\frac{a_0x^n+a_1x^{n-1}+\cdots+a_{n-1}x+a_n}{b_0x^m+b_1x^{m-1}+\cdots+b_{m-1}x+b_m}$.

Polynomial functions and rational functions are called *algebraic functions.* Function other than algebraic functions are called *transcendental* function.

Let $u=\sin x$ and $y=e^u$; now, for every value of x, u has a value and for that value of u, y has a value. Hence for a value of x, we get a definite value of y. Hence y is a function of x. The explicit or direct relation between x and y is $y=e=e^{\sin x}$.

In general, if $u=f(x)$ and $y=\phi(u)$, then we can write $y=\phi(u)$ $=\phi\{f(x)\}$. $\phi\{f(x)\}$ is called a ***function of a function.***

A function is frequently defined by more than one mathematical relations.

For example, $f(x)=x$ when $x \geqslant 0$
$= -x$ when $x<0$

If $f(-x)=f(x)$, then $f(x)$ is called an *even function* and if $f(-x)=-f(x)$, then $f(x)$ is said to be an *odd function.* x^2, x^4, $\cos x$ etc. are examples of even functions. For, $(-x)^2=x^2$, $\cos(-x)=\cos x$. x, x^3, $\sin x$ etc. are odd functions, for $(-x) = -x$, $(-x)^2=-x^3$, $\sin(-x)=-\sin x$. The function x^2+x is neither odd nor even.

If in an interval, as x increases, the value of $f(x)$ also increases then $f(x)$ is said to be an increasing function in that interval. If as value of x increases, $f(x)$ decreases, then it is said to be a decreasing function. For example, in the interval $0 \leqslant x \leqslant \frac{\pi}{2}$, $f(x)=\sin x$ is increasing, but $f(x)=\cos x$ in decreasing in this interval (see the graphs of the two functions.)

Examples 2

Ex. 1. Draw the graph of $y=[x]$, where $[x]$ is the integral part of x.

$[1{\cdot}5]=1$, $[1{\cdot}6]=1$, $[{\cdot}6]=0$, as the integral part of $\cdot 6$ i.e., $0{\cdot}6$ is 0.

$\therefore$ When x lies in $1 \leqslant x<2$, then $y=[x]=1$,

When x lies in $2 \leqslant x<3$, then $y=[x]=2$. etc.

Fig. 29 is the graph of $y=[x]$.

The function is called a step function. The function is discontinuous at the points 0, 1, 2, 3, -1, -2, -3, etc.

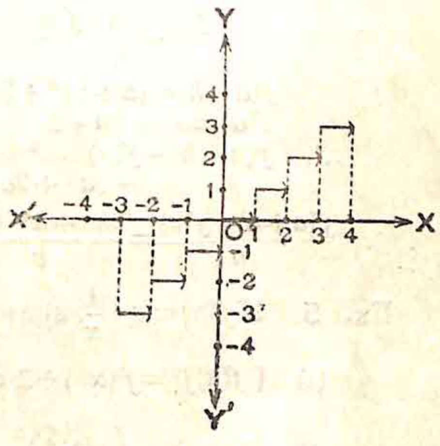


Fig. 29

Ex. 2. Determine the range of the function $y=\lfloor\underline{x}$. $\lfloor\underline{x}$ is defined for positive integral values of x. $\therefore$ The range of x is all the positive integers, i.e., $\{0, 1, 2, 3, \ldots\}$. The graph of $\lfloor\underline{x}$ is the totally of the disjoint points (0, 0), (1, 1), (2, 2), (3, 6), (4, 24), (5, 120)$\cdots\cdots$

Ex. 3. If $f(x)=\frac{1}{x^2}$ find the value of $f(x)-f(x+1)$ and hence find the sum of $\frac{3}{1^2.2^2}+\frac{5}{2^2.3^2}+\frac{7}{3^2.4^2}+\frac{9}{4^2.5^2}+\cdots$to n terms.

$$f(x)=\frac{1}{x^2}; \quad \therefore\ f(x+1)=\frac{1}{(x+1)^2}$$

So, $f(x)-f(x+1)=\frac{1}{x^2}-\frac{1}{(x+1)^2}=\frac{(x+1)^2-x^2}{x^2(x+1)^2}=\frac{2x+1}{x^2(x+1)^2}$

Now, putting $x=1$, we get $\frac{1}{1^2}-\frac{2}{2^2}=\frac{3}{1^2.2^2}$

putting $x=2$, ,, $\frac{1}{2^2}-\frac{1}{3^2}=\frac{5}{2^2.3^2}$

putting $x=3$, ,, $\frac{1}{3^2}-\frac{1}{4^2}=\frac{7}{3^2.4^2}$

... ...

putting $x=n$. ,, $\frac{1}{n^2}-\frac{1}{(n+1)^2}=\frac{2n+1}{n^2(n+1)^2}$

Adding, we get, $\frac{1}{1^2}-\frac{1}{(n+1)^2}=\frac{3}{1^2.2^2}+\frac{5}{2^2.3^2}+\cdots+\frac{2n+1}{n^2(n+1)^2}$

Hence the required sum is $1-\frac{1}{(n+1)^2}=\frac{2n+n^2}{(n+1)^2}$

Ex. 4. If $f(x)=x^2+2x+2$, find the value of

$$\frac{f(a+h)-f(a)}{h}$$

$$f(a+h)=(a+h)^2+2(a+h)+2$$
$$f(a)=a^2+2a+2$$
$$\therefore\ f(a+h)-f(a)=a^2+2ha+h^2+2a+2h+2 \\ -(a^2+2a+2)=2ha+h^2+2h$$

$$\therefore\ \frac{f(a+h)-f(a)}{h}=\frac{2ha+h^2+2h}{h}=2a+2+h=2(a+1)+h$$

Ex. 5. If $f(x)=x+\frac{1}{x}$, show that

(i) $[f(x)]^2=f(x^2)+2$ and (ii) $f\left(\frac{1}{x}\right)=f(x)$

(i) $[f(x)]^2=\left(x+\frac{1}{x}\right)^2=x^2+2.x.\frac{1}{x}+\frac{1}{x^2}=x^2+\frac{1}{x^2}+2$

Again, $f(x^2)=x^2+\frac{1}{x^2}$, $\therefore\ [f(x)]^2=f(x^2)+2.$

(ii) $f\left(\frac{1}{x}\right)=\frac{1}{x}+\frac{1}{\frac{1}{x}}=\frac{1}{x}+x=f(x)$

Ex. 6. If $y=f(x)=\frac{2x-3}{x-2}$, show that $x=f(y)$

$y=\frac{2x-3}{x-2}$, or, $y(x-2)=2x-3$,

or, $(y-2)x=2y-3$, or, $x=\frac{2y-3}{y-2}=f(y)$.

Ex. 7. If y be the length of a chord of a circle of radius a, at a distance x from the centre of the circle, express y as a function of x. What are the ranges of x and y?

From the figure, it is clear that

$a^2=x^2+\frac{y^2}{4}$,

or, $y=2\sqrt{a^2-x^2}$

The range of x is $0\leqslant x<a$ and that of y is $0<y\leqslant 2a$.

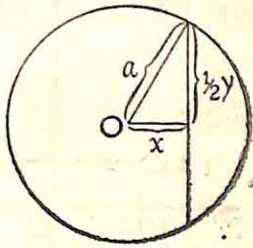


Fig. 30

Ex. 8. If $f(x)=\log_e x$ and $g(x)=e^x$, show that

$$f\{g(x)\}=g\{f\{x)\}$$

$f\{g(x)\}=f(u)$, [where $u=g(x)=e^x$]

$=\log_e u=\log_e e^x=x\log_e e=x.1=x$.

Again, $g\{f(x)\}=g\{\log_e x\}=e^{\log_e x}=x$.

$\therefore\ f\{g(x)\}=g\{f(x)\}$.

Ex. 9. If $f(x)=ax^2+bx+c$ and

$f(0)=1, f(1)=2$ and $f(-1)=0$, find the value of $f(2)$.

$f(x)=ax^2+bx+c$,

$\therefore\ 1=f(o)=a.o^2+b.o+c=c$, $\therefore\ c=1$.

$2=f(1)=a.1^2+b.1+c=a+b+c$

$\therefore\ a+b=2-c=2-1=1$.

$0=f(-1)=a.(-1)^2+b(-1)+c=a-b+c$

$\therefore\ a-b=-c=-1$.

Adding we get $2a=0$; $\therefore\ a=0$ and so $b=1$.

$\therefore\ f(x)=0.x^2+1.x+1+1=x+1$

$\therefore\ f(2)=2+1=3$.

Ex. 10. If $f(x)=2^x$, show that

$$f(x+1)-f(x-1)=\tfrac{3}{2}f(x) \text{ and } \frac{f(x+1)}{f(x-1)}=4.$$

$$f(x+1)=2^{x+1}=2.2^x\ ;\ f(x-1)=2^{x-1}=2^x.2^{-1}=\tfrac{1}{2}.2^x.$$

$$\therefore\ f(x+1)-f(x-1)=2.2^x-\tfrac{1}{2}.2^x=(2-\tfrac{1}{2}).2^x=\tfrac{3}{2}.2^x=\tfrac{3}{2}f(x)$$

$$\frac{f(x+1)}{f(x-1)}=\frac{2.2^x}{\frac{1}{2}.2^x}=4.$$

Ex. 11. Draw the graph of $f(x)=|x-1|+|x+1|$.

If $x<-1$, $x-1$ and $x+1$ are both negative.

$\therefore\ |x-1|+|x+1|=-(x-1)-(x+1)=-2x$.

If $-1\leqslant x\leqslant 1$, $x-1$ is negative or zero and $x+1$ is positive.

$\therefore\ |x-1|+|x+1|=-(x-1)+x+1=2.$

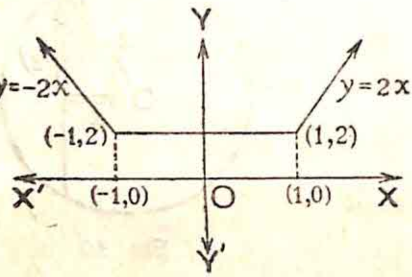


Fig. 31

If $x>1$, then both $x-1$ and $x+1$ are positive.

$\therefore\ |x-1|+|x+1|$
$=x-1+x+1=2x.$

Hence $f(x)=-2x$, when $x<-1$.
$=2$ when $-1\leqslant x\leqslant 1$
$=2x$ when $x>1$.

$\therefore$ The graph of $f(x)$ consists of three line segments. From the graph it is evident that the graph is continuous everywhere.

Ex. 12. Water can be brought into a cistern by two taps. Opening the first tap, level of water in the cistern can be increased by 1 cm/min. and opening the second tap, level can be increased by 2 cms/min. When the level of water is 10 cms., the first tap is opened. After 15 minutes of this, the second tap is opened and both the tap remain open for next 20 minutes. Then the first tap is closed and the second tap remains open for further 15 minutes. If y cms. be the level of water, x minutes after opening the first tap, express y as a function of x.

During the first 15 minutes, the level of water increases by 1 cm per minute from 10 cms. If $x<15$, the height of the water level is $10+x$. After 15 minutes, both the taps being open, the level of water increase by $2+1=3$ cms per minute, hence the level of water is increased by $3(x-15)$ cms in $(x-15)$ minutes. As both the taps remain open from 15th minute to $15+20=35$th minute, when $15\leqslant x\leqslant 35$, the height of water level is $25+3\,(x-15)$ cms. After 35 minutes, first tap being closed, the water level during next x minutes

time increases from 85 cms by $2(x-35)$ cms. $\therefore$ When $35 \leqslant x \leqslant 50$, the height of water level is $85+2(x-35)$ cms.

After 50 minutes, both the taps being closed the water level will not further increase.

$$\begin{aligned} \therefore \quad y &= 10+x, \text{ when } 0 \leqslant x \leqslant 15 \\ &= 25+3(x-15), \text{ when } 15 \leqslant x \leqslant 35 \\ &= 85+2(x-35), \text{ when } 35 \leqslant x \leqslant 50 \\ &= 115 \qquad \text{when } x \leqslant 50 \end{aligned}$$

Exercise 2

1. If $f(x)=x^3-2x^2+4x+1$, then find the values of $f(0)$, $f(1), f(2{\cdot}1), f(-2), f(a), f(-x), f(a+h)$.

2. If $f(\theta)=2\cos\theta+1$, then evaluate $f(0), f\left(\frac{\pi}{2}\right), f\left(\frac{\pi}{3}\right), f\left(\frac{\pi}{4}\right)$, $f\left(\frac{\pi}{2}+h\right), f(x)$.

3. If $f(x)=x^2$ find the values of $f(2)$, $f(2{\cdot}1)$, $f(2{\cdot}01)$, $f(2{\cdot}001)$, $\dfrac{f(2{\cdot}0001)-f(2)}{{\cdot}0001}$.

4. If $f(x)=\dfrac{(x-3)(x-4)(x+1)}{(2x+3)(x-1)}$, find

   $f(3), f(0), f(-1), f(-x), f\left(\frac{1}{x}\right)$ and $f(1)$

5. If $f(x)=a^x$, find the values of $f(0), f(\log_a x)$,

   $\dfrac{f(x+h)-f(x)}{h}$.

6. If $f(x)=\dfrac{x-1}{x^2+2}$, find $f(0), f(-1), f(2x), f\left(\frac{1}{x}\right), f(x+h)$.

7. If $f(x)=\dfrac{1-x^2}{1+x^2}$, show that $f(\tan x)=\cos 2x$,

   $f(\sqrt{\cos x})=\tan^2\dfrac{x}{2}$.

8. If $f(\theta)=\sin\theta$ and $g(\theta)=\cos\theta$, then show that

   (i) $f(\theta+\phi)=f(\theta)g(\phi)+f(\phi).g(\theta)$.

   (ii) $\{f(\theta)\}^2+\{g(\theta)\}^2=1$.

   (iii) $\{g(\theta)\}^2-\{f(\theta)\}^2=g(2\theta)$.

   (iv) $\dfrac{1-g(\theta)}{1+g(\theta)}=\left\{\dfrac{f\left(\frac{\theta}{2}\right)}{g\left(\frac{\theta}{2}\right)}\right\}^2$

9. If $f(x)=3^x$, show that $f(x+1)=3f(x)$,
$f(x+2)-f(x-2)=\frac{80}{9}f(x)$, $\frac{f(x+2)+f(x-1)}{f(x+1)}=\frac{28}{9}$.
$f(a.fb)=f(a+b)$.

10. If $f(x)=\log x$, then show that
$f(1)=0$, $f(e)=1$, $f(xy)=f(x)+f(y)$
$f(x^m)=mf(x)$, $f\left(\frac{x}{y}\right)=f(x)-f(y)$

11. If $y=f(x)=\frac{3x-7}{7x-3}$, show that

(i) $x=f(y)$, (ii) $f(x).f\left(\frac{1}{x}\right)=1$.

12. $f(x)=\frac{1-x}{1+x}$, show that $f^2(x)=x$, where $f^2(x)=f\{f(x)\}$

13. Show that if $f(\theta)=\sin\theta$ then $f(-\theta)=-f(\theta)$
and if $f(\theta)=\frac{\sin\theta}{\theta}$, then $f(-\theta)=f(\theta)$

14. Show that $f(x)=x^4+\sin^2 x+2$ is an even function and $g(x)=x^7-\sin^3 x$ is an odd function.

15. (i) If $f(x)$ and $g(x)$ be respectively an even and an odd function, can you say anything about $f(x)\ g(x)$ and $\frac{f(x)}{g(x)}$?

(ii) Show that for any function $f(x)$, (*a*) $f(x)+f(-x)$ is an even function, (*b*) $f(x)-f(-x)$ is an odd function.

16. Show that the sum and product of two even functions are even functions and the sum and product of two odd functions are respectively odd and even functions.

17. Determine the domain of definition of the following functions:

(i) $f(x)=\sqrt{1-x^2}$ (ii) $f(x)=\sqrt{4+x}+(5-x)^{\frac{1}{4}}$

(iii) $y=(x+a)^{\frac{1}{3}}-(x-b)^{\frac{1}{3}}$ (iv) $f(x)=\frac{a+x}{a-x}$.

(v) $y=x^{-\frac{1}{2}}$ (vi) $y=\log(2x+1)$.

18. Draw the graphs of the following functions.
(i) $y=-3x+1$, (ii) $y=\cos 2x$, (iii) $y=x^2-2x+2$,

(iv) $y=\frac{1}{x^2}$, (v) $y=\cos\left(x-\frac{\pi}{4}\right)$, (vi) $y=\cos x+2\sin x$,

(vii) $y=6$, (viii) $y=\frac{x^2-1}{x-1}$.

19. In the closed interval $[-1, 1]$, the function $f(x)$ is defined as follows, $f(x)=x^2$; $0\leqslant x\leqslant 1$

$$=2x, \ -1\leqslant x\leqslant 0.$$

Draw the graph of the function.

20. In the closed interval $[-2, +2]$, the function $f(x)$ is defined as follows :

$$\begin{aligned} f(x)&=-2x, \ -2\leqslant x<0 \\ &=2x+1, \ 0\leqslant x<1 \\ &=4x-1, \ 1\leqslant x\leqslant 2 \end{aligned}$$

Find the points of discontinuity, if any, of the function.

21. Show that $\frac{f(x+h)-f(x)}{h}$ is equal to

(i) $\cos\left(x+\frac{h}{2}\right)\frac{\sin\frac{h}{2}}{\frac{h}{2}}$, when $f(x)=\sin x$

(ii) $\frac{1}{\cos(x+h)\cos x}\cdot\frac{\sin h}{h}$, when $f(x)=\tan x$.

(iii) $-\frac{\cos\left(x+\frac{h}{2}\right)}{\sin(x+h).\sin x}\cdot\frac{\sin\frac{h}{2}}{\frac{h}{2}}$, when $f(x)=\operatorname{cosec} x$.

(iv) $e^x\,\frac{e^h-1}{h}$, when $f(x)=e^x$.

22. From the four corners of a square plate of side 16 cms. made of tin, four squares each of side x cm. in length are cut off and then the plate is folded to form a box open at the top. If v be the volume of the box, express v as a function of x. What is the range of x ? When $x=2$, then what is the value of v ?

23. If the area and length of a side of a square be A and x respectively, express A as a function of x.

24. If $f(\theta)=a+b\cos\theta$ and the maximum and minimum value of $f(\theta)$ be 3 and 1 respectively. Find the values of a and b.

25. If $f(x)=ax+\frac{b}{x}+c$, and $f(-2)=-\frac{1}{2}$, $f(-1)=2$, $f(1)=4$, find the value of $f(2)$

26. Find the inverse function of the following functions :

(i) $y=\sqrt{1+x^2}$, $x>0$ (ii) $y=\sqrt{1+x^2}+x$,

(iii) $y=x^2+\frac{1}{x^2}$, (iv) $y=\frac{1+x^2+x^4}{x^2}$.

27. Express the following equations in the form $x=\phi(y)$.

(i) $a+bx+cy+dxy=0$ (ii) $y=\frac{1-\cos x}{1+\cos x}$.

28. The fare of a taxi for a distance of less than or equal to one kilometre from start is 1 rupee 80 paise. For each of the subsequent distant of 100 metres or less than it, the fare is 18 paise. If the fare for a distance of x kilometres is y rupees, express y as a function of x.

29. A cistern has two pipes. By the first pipe water can be entered into the cistern to increase the level of water in the cistern by 2 cms. per minute. By the other pipe water can be taken out of the cistern to decrease the level by 1 cm. per minute. When the level of water is 50 cms., water is taken out of the cistern by opening the second pipe for half an hour, and then the first pipe is opened. Both the pipe remain open for the next half an hour and then the 2nd pipe is closed. One hour hence first pipe is also closed. If y cms. be the level of water x minutes after the opening of the second pipe, find the relation between x and y.

CHAPTER THREE
LIMIT

§ 3·1. The concept of Limit is most important in calculus. In chapter one, we have already said, that both the branches of calculus viz, Differential Calculus and Integral Calculus are based on this very important concept and it is this concept which has separated Calculus from Algebra. In this Chapter, we shall explain the concept of limit by intuitive and graphical methods. Formal mathematical definition of limit is outside the scope of this introductory discourse.

§ 3·2. **Meaning of 'variable x approaching a constant a'.**

Let x be a variable and the constant a is a point in its range.

Since x is a variable, so its value can be changed. Now the value of x may be so changed that the value may be nearer to a at pleasure. In such cases we say that x approaches a. To be more clear, x approaches a means the distance $|x-a|$ between x and a may be made smaller than any positive number however small. If x approaches a in such a way, we express it by the notation $x \to a$.

Hence the meaning of $x \to a$ is that x changes its value in such a way that atlast the distance $|x-a|$ between x and a will be smaller than any positive number, however small.

Now, let us explain the meaning of $x \to a$ with the help of the number line. Let the point A on the number line represent the number a and the point P on the number line represent any position of the variable x.

Now the meaning of x approaches a is that the point P gradually approaches the point A along the number line in such a way that if we take a point A′ on the line very near to A, then P will finally cross the point A′. If we take a point A″, more near to A, then the point P will also cross the point A″, and come very near to A. In the figure, the points A′, A″ have been taken on the

Fig. 32

right of A. Similarly if we take points B$'$, B$''$ etc. on the left of A, then P will also cross those points from the left and come nearer to A.

Example. The meaning of $x \to 2$ is that x changes its value in such a manner that the distance $|x-2|$ between x and 2 can be made smaller than any positive number, however small.

Let ·1 be a positive number.

The value of x will be finally such that $|x-2|$ will be less than ·1. i.e., $|x-2| < \cdot 1$ which means $2 - \cdot 1 < x < 2 + \cdot 1$ i.e., the value of x will lie within the interval $1 \cdot 9 < x < 2 \cdot 1$.

Similarly if we choose the positive number ·00005, then finally the value of x will be such that $|x-2| < \cdot 00005$,

i.e., the value of x will lie within the interval $1 \cdot 99995 < x < 2 \cdot 00005$.

Note : (i) By $x \to a$, it is meant that x assumes values very near to a. It is immaterial whether x assumes the value a or not. But generally, we presume that x does not take the value a.

(ii) If x approaches the value a from values greater than a, i.e., on the number line x approaches A from the right side of A, then we write $x \to a+$ and say that x approaches a from the right.

Similarly, if x approaches the value a from values less than a i.e., on the number line x approaches A from the left side of A, then we write $x \to a-$ and say that x approaches a from the left.

$x \to a$ means both $x \to a+$ and $x \to a-$. So x approaches or tends to a means x approaches the value a from both right and left.

(iii) $x \to a$ refers to a particular mode of change of values of the variable x. If when x change its value in this particular way, how a function of x will change its value, is the main point of discussion of the present chapter. In the next article we shall discuss about it.

(iv) To determine the mode of change of the value of a function $f(x)$, we generally take numbers close to the point a

and determine the mode of change of the value of the function. These values are taken according to convenience taking care that the values of x approach a, both from the right and the left.

For example, if we are to determine the mode of variation in the value of a function as $x \to 2$, we shall have to determine the values of the function at points very close to the point 2.

The points may be taken as the points 1·9, 1·99, 1·999, etc. and 2·1, 2·01, 2·001 etc. x approaches 2 from the left through the points 1·9, 1·99, 1·999 etc. and from the right through the points 2·1, 2·01, 2·001 etc. Notice that the points have been so chosen that the distance $|x-2|$ between x and 2 becomes smaller and smaller and ultimately can be made smaller than any positive number small at pleasure.

Instead of taking the points 1·9, 1·99, 1·999 etc. and 2·1, 2·01, 2·001 etc., we may chose any other set of points such as 1·95, 1·995, 1·9995···and 2·5, 2·05, 2·005, 2·0005,···. For these values also, x gradually approaches 2 both from the left and the right. Hence we can choose the points in any manner. Only taking care that the points lie on both sides of the point $x=2$ and that the distance between x and 2 gradually becomes smaller and smaller.

Ex. To study the nature of variation in value of a function as $x \to 0$ we may take the following values of the variable.

$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}$ etc. and $-1, -\frac{1}{2}, -\frac{1}{3}, -\frac{1}{4}, -\frac{1}{5}$ etc. As x takes the values $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}$ etc. we find $x \to 0+$ and as x takes the values $-1, -\frac{1}{2}, -\frac{1}{3}, -\frac{1}{4}$, etc. we find $x \to 0-$. Note that if x assumes the above values the distance $|x-0| = |x|$, of x from the point $x=0$, i.e., the origin, can be gradually made smaller than any positive number however small.

If ·00005 be any preassigned positive number, then if we take $x = \pm\frac{1}{200001}$, $\pm\frac{1}{200002}$ etc., then $|x| < \frac{1}{20000} = \cdot 00005$.

In this way corresponding to any preassigned positive number, we can find a value of x such that $|x|$ is less than that number.

§ 3·3. Discussion about limit in the intuitive method.

Let y be a variable depending on x. So, y is a function of x.

Now if x approaches a given number, let us discuss how the value of y changes.

First let us take a few examples.

Ex. 1. Let $y=2x+1$ and $x\to 1$, i.e., the value of x approaches 1.

The table below shows the values of $y=2x+1$ for values of x close to 1.

First let $x\to 1+$

x	1·1	1·01	1·001	1·0001
$y=2x+1$	3·2	3·02	3·002	3·0002

, etc.

From the table it is evident that as x approaches 1 from the right, then y also gradually assumes values close to 3. Actually by making x sufficiently close to 1, the distance between y and 3 i.e., $|y-3|$ can be made small at pleasure. So, as $x\to 1+$, then $y\to 3$, i.e., as x approaches 1 from the right and side, then y approaches the value 3. In the language of Calculus, one can say that as x approaches 1 from the right the limiting value of y is 3. 3 is said, in this case, the right hand limit of $y=2x+1$ and in symbols this is expressed as

$$\lim_{x\to 1+} y=3 \text{ or } \operatorname*{Lt}_{x\to 1+} 2x+1=3$$

Similarly, let us form a table showing the corresponding values of x and y as x approaches the value 1 from the left, i.e., as x assumes values very close to 1 but less than 1.

x	·9	·99	·999	·9999
y	2·8	2·98	2·998	2·9998

, etc.

Evidently as x approaches 1 from the left, then the value of y approaches 3. Actually making x sufficiently close to 1 we can make the distance $|y-3|$ small at pleasure. In such a situation we say that as x approaches 1 from the left hand side, the limit of y is 3. 3 is called the left hand limit of y and in symbols this is expressed as $\operatorname*{Lim}_{x\to 1-} y=3$ or $\operatorname*{Lt}_{x\to 1-} 2x+1=3.$

Here the left hand and right hand limits are equal. This common limit is said to be the limit of $y=2x+1$ as x approaches 1 and is denoted as

$$\lim_{x\to1} y=3 \text{ or, } \lim_{x\to1} (2x+1)=3.$$

Hence $\lim_{x\to1} y=3$ means that by taking x sufficiently close to 1 the distance $|y-3|$ between y and 3 can be made small at pleasure.

In general, if $y=f(x)$ be any function of x, then $\lim_{x\to a} f(x)=l$ means that by taking x sufficiently close to a one can make the distance $|f(x)-l|$, small at pleasure.

From the above discussion we can define limit as follows.

Def. When a variable x approaches a constant a, belonging to its range, then the constant 'l' is said to be the limit of $f(x)$ if by making the value of x sufficiently close to a the distance $|f(x)-l|$ between $f(x)$ and l can be made smaller than any preassigned positive number, small at pleasure and this is denoted by the symbol

$$\lim_{x\to a} f(x)=l \quad \text{or} \quad \operatorname*{Lt}_{x\to a} f(x)=l.$$

The definition is explained with a few more examples.

Ex. Show that $\lim_{x\to2} 5x=10$.

Here $f(x)=5x$ and x approaches 2.

Hence we shall have to discuss the values of $5x$ for values of x close to 2. In the following tables values of $5x$ for values of x close to 2 are tabulated.

x	1·9	1·99	1·999	1·9999	
$f(x)=5x$	9·5	9·95	9·995	9·9995,	etc.

x	2·1	2·01	2·001	2·0001	
$f(x)=5x$	10·5	10·05	10·005	10·0005,	etc.

From the above table it is evident that as x approaches the value 2, whether from the left hand side or the right hand side, the value of y approaches 10. Actually by making x sufficiently

close to 2 we can make the distance $|5x-10|$ of $5x$ and 10 smaller than any pre-assigned positive number however small. For example, let $\cdot 1$ be a preassigned positive number. Now there are values of x close to 2 such that $|5x-10|<\cdot 1$. Actually if $|5x-10|<\cdot 1$, then $|x-2|<\frac{\cdot 1}{5}=\cdot 02$. So, when $|x-2|<\cdot 02$ i.e., for all values of x within the interval $1\cdot 98<x<2\cdot 02$, $|5x-10|$ will be less than $\cdot 1$.

Again, if the preassigned positive number be $\cdot 005$, then for all values of x within the interval $1\cdot 999<x<2\cdot 001$; $|5x-10|<\cdot 005$. Similarly it can be shown that corresponding to any preassigned positive number, small at pleasure, we can find values of x close to 2 such that $|5x-10|$ will be less than that preassigned positive number. Hence we can say that as x tends (approaches) to 2 i.e. $x\to 2$, then the limiting value of $f(x)=5x$ is 10.

In symbols this is denoted as $\lim\limits_{x\to 2} 5x=10$.

Let us now make the above discussion graphically.

In the graph of $y=5x$, the point A on the x-axis is the point $x=2$. $A_1, A_2, \cdots, A_n$ etc are points on the x-axis situated on the right of A. The points $P_1, P_2, \cdots, P_n$ etc. are on the graph of $y=5x$ whose x co-ordinates are respectively $OA_1, OA_2, \cdots OA_n$ etc.

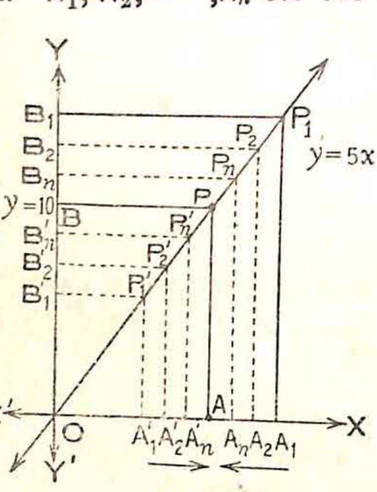


Fig. 33

$B_1, B_2, \cdots B_n$ are points on the y-axis with ordinates same as those of $P_1, P_2, \cdots P_n$. The point B has ordinate 10. Now as x approaches the point A i.e., the point 2 through the points $A_1, A_2, \cdots, A_n$ then y approaches the point B i.e. the point $y=10$ through the points $B_1, B_2, \cdots B_n$. Hence as x approaches the point A from the right, then y approaches the point B. Similarly as x approaches the point A from the left then y approaches the point B $(y=10)$. This has also been shown in the figure by taking the points $A_1', A_2', \cdots, A'$ on the

x-axis and corresponding points B_1', B_2',..., B_n' on the y-axis. Hence as x approaches the point $x=2$ whether from the right or left then y approaches, in both cases, the point B i.e., the point $y=10$.

In such cases we say, $\lim\limits_{x\to 2} y=10$, or, $\lim\limits_{x\to 2} f(x)=10$.

Let us now consider a function whose limit at a particular point cannot be determined.

Ex. Let $f(x)=\dfrac{|x|}{x}$ when $x\neq 0$ and $x\to 0$.

Here $f(x)$ is undefined at the point $x=0$, but its value can be determined at any point as close to the point 0 as we like.

First, let $x\to 0+$. Values of $f(x)$ on the right of the point $x=0$ are given in the table below.

x	$\cdot 1$	$\cdot 01$	$\cdot 001$	$\cdot 0001$
$f(x)=\dfrac{\lvert x\rvert}{x}$	1	1	1	1

For, $f(\cdot 1)=\dfrac{|\cdot 1|}{\cdot 1}=\dfrac{\cdot 1}{\cdot 1}=1$, etc.

Evidently, at all points on the right of the point $x=0$, the value of $|f(x)-1|$ is 0.

Hence one can say, that the value of $|f(x)-1|$ can be made smaller than any positive number, however small.

$$\therefore \lim_{x\to 0+}\frac{|x|}{x}=1.$$

Let now $x\to 0-$. The following table gives the values of $f(x)$ at points on the left of the point $x=0$.

x	$-\cdot 1$	$-\cdot 01$	$-\cdot 001$	$-\cdot 0001$
$f(x)=\dfrac{\lvert x\rvert}{x}$	-1	-1	-1	-1

For, $f(-\cdot 1)=\dfrac{|-\cdot 1|}{-\cdot 1}=\dfrac{\cdot 1}{-\cdot 1}=-1$, etc.

Hence the value of $|f(x)-(-1)|$ on the left of the point $x=0$ is always 0.

$\therefore$ One can say that the value of $|f(x)-(-1)|$ can be made smaller than any positive number however small.

$$\therefore \lim_{x\to 0-}\frac{|x|}{x}=-1.$$

So, we find that in this case the left hand and right hand limits are not equal. Hence $\lim\limits_{x \to 0} \frac{|x|}{x}$ does not exist.

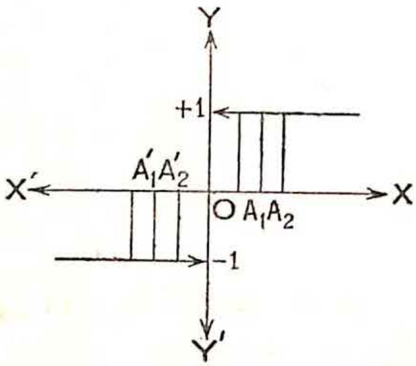


Fig. 34

The graph of the function $\frac{|x|}{x}$ has two branches. One branch is in the first quadrant and the other in the third quadrant. Both the branches are parallel to the x-axis and are at a distance of 1 unit from the x-axis. In the first branch notice that as x approaches the point o i.e., $x=0$ through the points A_2 A_1,$\cdots$the value of y always remains 1 and hence it can be said that y approaches 1. Similarly as x approaches 0 from the left through the points A_1', A_2',...the value of y remains -1. Hence it can be said that y approaches -1.

Notice that the graph breaks at the point $x=0$ and the function has no limiting value at that point. At all other points the graph is continuous (i.e., without any break) and the limit of the function at those points can be determined.

Ex. $f(x)=[x]$ and $x \to 2$, where $[x]$ is the integral part of x.

First let $x \to 2+$.

The values of $[x]$ for values of x greater than 2 and close to 2 are tabulated below.

x	2·1	2·01	2·001	
$f(x)=[x]$	2	2	2	etc.

$\therefore$ Evidently $\lim\limits_{x \to 2+} [x]=2$.

Now, let $x \to 2-$. Values of $f(x)$ at points less than 2 and close to 2 are tabulated below.

x	1·9	1·99	1 999	
$f(x)=[x]$	1	1	1	etc.

$\therefore$ Evidently $\lim\limits_{x \to 2-} [x]=1$.

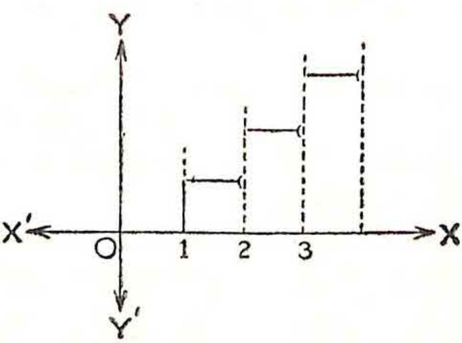


Fig. 35

Here the left hand limit and right hand limit are not equal. Hence as x approaches 2, then $[x]$ has no limiting value.

The graph of the function $[x]$ is broken at the points $x=1, 2, 3,$ etc. and the limit of the function at these points does not exist. The graph is continuous at all other points and the limit of the function at these points exists.

Note : (i) $\underset{x \to a}{\text{Lt}} f(x)=l$ or, $\underset{x \to a}{\text{L}} f(x)=l$ are alternative notations for $\lim_{x \to a} f(x)=l$.

$\lim_{x \to a} f(x)=l$ is read as limit x tends to a, $f(x)$ is equal to l.

This means as x tends to a, the limiting value of $f(x)$ is l.

(ii) To determine $\lim_{x \to a} f(x)$, values of $f(x)$ on both sides of the point a are to be determined. The value at the point a is not required. Actually $\lim_{x \to a} f(x)$ and $f(a)$ are different though in some cases they may be equal.

The value of $\lim_{x \to a} f(x)$ is dependent on the values of $f(x)$ very close to a but not at a, whereas $f(a)$ depends only on the value a of x. In the next article we shall discuss the relation between $\lim_{x \to a} f(x)$ and $f(a)$.

(iii) If $\lim_{x \to a} f(x)=l$, then $\lim_{x \to a+} f(x)=\lim_{x \to a-} f(x)=l$.

$\therefore$ If $\lim_{x \to a+} f(x) \neq \lim_{x \to a-} f(x)$, then the limit of $f(x)$ at the point a does not exist.

Hence to find the limit of a function at a point, one should first determine the right hand and left hand limits and examine whether they are equal or not.

Ex. 1. Show that $\lim_{x \to a} x=a$.

Here $f(x)=x$ and we are to show that $l=a$.

Now, $|f(x)-l| = |x-a|$. Now it is self-evident that $|f(x)-l| = |x-a|$ can be made smaller than any positive

number, small at pleasure as $x \to a$. Actually, $f(x)$ will be so close to l as x is close to a; for, $|f(x)-l| = |x-a|$.

$\therefore \lim_{x \to a} f(x) = l$, i.e., putting the value $\lim_{x \to a} x = a$.

Ex. 2. Show that $\lim_{x \to a} c = c$.

Here $f(x)=c$ and $l=c$. $\therefore |f(x)-l| = |c-c| = 0$.

Now 0 being less than any positive number, $|f(x)-l|$ can be made smaller than any positive number however small, whatever x may be.

$\therefore \lim_{x \to a} f(x) = l$ or, $\lim_{x \to a} c = c$.

Ex. 3. Show that $\lim_{x \to 2} x^2 = 4$. Find the interval, for values of x within which $|x^2-4|$ will be less than $\cdot 01$.

Here $f(x)=x^2$ and $l=4$.

Now, we tabulate below the values of $f(x)$ for values of x close to 2.

x	$2 \cdot 1$	$2 \cdot 01$	$2 \cdot 001$	$2 \cdot 0001$
$f(x)=x^2$	$4 \cdot 41$	$4 \cdot 0401$	$4 \cdot 004001$	$4 \cdot 00040001$, etc.

Evidently as $x \to 2+, f(x) \to 4$.

x	$1 \cdot 9$	$1 \cdot 99$	$1 \cdot 999$	$1 \cdot 9999$
$f(x)=x^2$	$3 \cdot 61$	$3 \cdot 9601$	$3 \cdot 996001$	$3 \cdot 99960001$, etc.

$\therefore$ As x approaches 2 from the left or from the right $f(x)$ tends to 4 and the value of $|f(x)-4|$ can be made smaller than any positive number however small by making x sufficiently close to 2.

$\therefore \lim_{x \to 2} f(x) = 4$ i.e., $\lim_{x \to 2} x^2 = 4$.

Now, if $|x^2-4| < \cdot 01$, then $|x+2|\ |x-2| < \cdot 01$.

Again, as x is close to 2, we can say $2 < |x+2|$

$\therefore 2|x-2| < |x+2|\ |x-2| < \cdot 01$

or, $|x-2| < \frac{\cdot 01}{2}$, or, $|x-2| < \cdot 005$

i.e., $1 \cdot 995 < x < 2 \cdot 005$

Hence $|x^2-4|$ will be less than $\cdot 01$ if x lies in the interval $1 \cdot 995 < x < 2 \cdot 005$.

Ex. 4. $f(x)=\frac{x^2}{x}$. Find the limiting value of $f(x)$ at the point $x=0$.

Here the function is undefined at the point $x=0$.

$$\left(\text{For, } f(0)=\frac{0}{0}\right).$$

$\therefore$ $f(0)$ has got no value. But $f(x)$ has values at points close to 0. We tabulate below the values of the function at points close to 0.

$x=$	1	·01	·001	etc.
$\frac{x^2}{x}=$	1	·01	·001	

$x=$	-1	$-·1$	$-·01$	$-·001$	etc.
$\frac{x^2}{x}=$	-1	$-·1$	$-·01$	$-·001$	

From the above tables it is evident that as x approaches 0 whether from the left or from the right the values of $\left|\frac{x^2}{x}-0\right|$ become smaller and smaller.

$$\therefore \quad \lim_{x\to 0}\frac{x^2}{x}=0.$$

The graph of the function $y=\frac{x^2}{x}$ will be the same as the graph of $y=x$, the only difference between them is that the point (0,0) i.e., the origin is not a point of the graph of $y=\frac{x^2}{x}$.

Ex. 5. Show that $\lim_{x\to 2}(2x+4)=8$. In which intervals should x lie in order that the value of $|(2x+4)-8|$ will be less than (i) ·1 and (ii) ·004 ?

Here $f(x)=2x+4$, $l=8$ and $x\to 2$. Values of $f(x)$ at points close to 2 are tabulated below.

x	2·1	2·01	2·001	2·0001	etc.
$f(x)$	8·2	8·02	8·002	8·0002	

and

x	1·9	1·99	1·999	etc.
$f(x)$	7·8	7·98	7·998	

Evidently, as x approaches 2 whether from the left or from the right, $f(x)$ approaches 8 and taking the value of x sufficiently close

to 2, the value of $|f(x)-8|$ can be made smaller than any pre-assigned positive number however small.

$\therefore \lim_{x\to 2} f(x)=8$ or, $\lim_{x\to 2}(2x+4)=8$.

Now, if $|(2x+4)-8|<\cdot 1$, then $|2(x-2)|<\cdot 1$,
or, $|x-2|<\frac{\cdot 1}{2}$, or, $|x-2|<\cdot 05$, or, $1\cdot 95<x<2\cdot 05$.

$\therefore$ For values of x within the interval $1\cdot 95<x<2\cdot 05$, the value of $|(2x+4)-8|$ will be less than $\cdot 1$.

Again, if $|(2x+4)-8|<\cdot 0004$, then $|x-2|<\frac{\cdot 0004}{2}$,

or, $|x-2|<\cdot 0002$.

$\therefore$ For values of x within the interval $1\cdot 9998<x<2\cdot 0002$, the value of $|(2x+4)-8|$ will be less than $\cdot 0004$.

Ex. 6. Show that $\lim_{x\to\sqrt[3]{2}} x^3=2$.

Here $a=\sqrt[3]{2}=1\cdot 259\ldots$ and $f(x)=x^3$, $l=2$.

Now,

x	1	1·2	1·25	1·259
x^3	1	1·728	1·953125	1 995616979

etc.

Evidently, as $x\to\sqrt[3]{2}-$, then $x^3\to 2$.

Again

x	1·3	1·26
x^3	2·197	2·00376

etc.

Evidently as $x\to\sqrt[3]{2}+$, then $x^3\to 2$

$$\therefore \lim_{x\to\sqrt[3]{2}} x^3=2.$$

Ex. 7. Show that $\lim_{x\to 5}(x-5)=0$.

Here $f(x)=x-5$ and $x\to 5$. Hence we are to find the values of $f(x)$ for values of x close to 5.

x	4·9	4·99	4·999	4·9999
$f(x)$	−·1	−·01	−·001	−·0001

etc.

x	5·1	5·01	5·001
$f(x)$	·1	·01	·001

etc.

Hence it is evident from the above tables that as x approaches 5 whether from the left or from the right, $f(x)$ tends to zero.

$\therefore \lim_{x\to 5} f(x)=0$, i.e., $\lim_{x\to 5}(x-5)=0$.

Ex. 8. Show that $\lim\limits_{x\to 0} \sin x=0.$

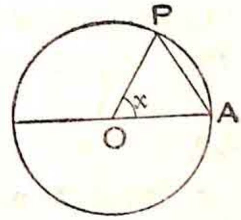


Fig. 36

Let OA be a radius of a circle with centre O, and let P be a point such that $m\angle AOP=x$ (radian). Now, area of $\triangle OAP<$area of the sector OAP. or, $\frac{1}{2}OA\ OP.\sin x<\frac{1}{2}OA^2x$,

or, $\sin x<x$

clearly when $x<0$, $|\sin x|<|x|$.

For any value of x,

$|\sin x-0|=|\sin x|<|x|$

Now, taking x very near to 0, $|\sin x-0|$ can be made small at pleasure.

$$\therefore \lim_{x\to 0} \sin x=0.$$

Ex. 9. Show that $\lim\limits_{x\to a} \sin x=\sin a$

$\sin x-\sin a=2\sin\frac{1}{2}(x-a)\cos\frac{1}{2}(x+a)$

$\therefore\ |\sin x-\sin a|=2\,|\sin\frac{1}{2}(x-a)|\ |\cos\frac{1}{2}(x+a)|$

$\leqslant 2.\frac{1}{2}\,|x-a|\cdot 1$ [$\because \cos\frac{1}{2}(x-a)\leqslant 1$ and $|\sin x|\leqslant|x|$]

i.e., $\leqslant |x-a|$

$\therefore$ as x is nearer to a, $\sin x$ is more near to $\sin a$ and $|\sin x-\sin a|$ can be made small at pleasure.

$$\therefore \lim_{x\to a} \sin x=\sin a$$

Ex. 10. Show that, $\lim\limits_{x\to 0}\left(x\sin\frac{1}{x}\right)=0.$

$\left|x\sin\frac{1}{x}-0\right|=|x|\left|\sin\frac{1}{x}\right|<|x|\ \left(\because\ \left|\sin\frac{1}{x}\right|\leqslant 1\right)$

Taking x sufficiently near to 0, $\left|x\sin\frac{1}{x}-0\right|$ can be made small at pleasure.

$$\therefore \lim_{x\to 0} x\sin\frac{1}{x}=0.$$

Exercise 3(A)

1. Show that

(i) The values $5, 4, \frac{11}{3}, \frac{7}{2}, \frac{17}{5},\cdots$ make $x\to 3+$

(ii) The values $2, \frac{5}{2}, \frac{8}{3}, \frac{11}{4}, \frac{14}{5},\cdots$ make $x\to 3-$

2. Tabulate the values of $f(x)=3x+4$ at points close to the point $x=3$ and show that $\lim\limits_{x\to 3}(3x+4)=13.$

In which interval must the values of x lie in order to make $|f(x)-13|<{\cdot}0005$?

3. Find the nature of the changes in values of $f(x)=|x|$ as $x\to 0$.

4. Find the nature of the changes in values of

$$f(x)=x,\ x>0$$
$$=x+1,\ x\leqslant 0,\ \text{as } x\to 0$$

Does $\lim\limits_{x\to 0} f(x)$ exist ?

Draw the graph of $y=f(x)$ and show that the graph is broken at the point $x=0$.

5. Show that $\lim\limits_{x\to 5} 3=3.\ \lim\limits_{x\to a} 4=4,\ \lim\limits_{x\to a} c=c.$

6. Show that $\lim\limits_{x\to 4} x=4,\ \lim\limits_{x\to 2\cdot 1} x=2{\cdot}1.$

7. Show that $\lim\limits_{x\to 2\cdot 4} x^2=5{\cdot}76.$

8. Show that $\lim\limits_{x\to 0} \dfrac{x^3}{x}=0.$

Draw the graph of $y=\dfrac{x^3}{x}$ and show that though the graph is broken at the origin, the distance between the two branches is zero.

9. Show that $\lim\limits_{x\to 1} \dfrac{x^2-1}{x-1}=2.$ Draw the graph of the function.

10. Show that for the function

$$f(x)=3x \text{ when } x\geqslant 1$$
$$=x \text{ when } x<1,$$

$\lim\limits_{x\to 1} f(x)$ does not exist.

Draw the graph of the function $y=f(x)$ to show that the graph is broken at the point $x=1$. Show also that the distance between the disjoint branches is 2.

11. Show that (i) $\lim\limits_{x\to 3}(x-3)=0$ (ii) $\lim\limits_{x\to 0}(x-a)=0.$

12. Show that, (i) $\lim_{x \to a} \cos x = \cos a$,

(ii) $\lim_{x \to \pi/2} \sin x = 1$, (iii) $\lim_{x \to 0} \sin \frac{x}{2} = 0$.

13. Prove that, (i) $\lim_{x \to 0} x \cos \frac{1}{x} = 0$ (ii) $\lim_{x \to 0} x^2 \sin \frac{1}{x} = 0$.

§ 3·4. **Continuity :** In the previous chapter it has been said that a function is said to be continuous at a point if the graph of the function be continuous i.e., unbroken on both sides of the point and closed to it including the point itself.

Let the function $f(x)$ be continuous at a point $x=a$. Let P be the point on the graph of the function corresponding to $x=a$. As the function is constinuous at $x=a$, so the graph of the function from one side of the point to the other side is continuous. Let $A_1, A_2, A_3 \cdots$etc. be points on the x-axis on the right of the point A $(x=a)$. Let the perpendiculars on the x-axis at these points intersect the graph of the function at the points $P_1, P_2, P_3, \cdots$etc.

Let also the straight lines parallel to the x-axis through the points $P_1, P_2, P_3, \cdots$, etc. intersect the y-axis at the points $B_1, B_2, B_3 \cdots$etc.

Draw a perpendicular on the x-axis at the point $A(x=a)$. It will cut the graph at the point P and the straight line parallel to the x-axis through P cut the y-axis at the point B. Then B is the point $y=f(a)$. From the graph it is evident that as the variable x crossing the

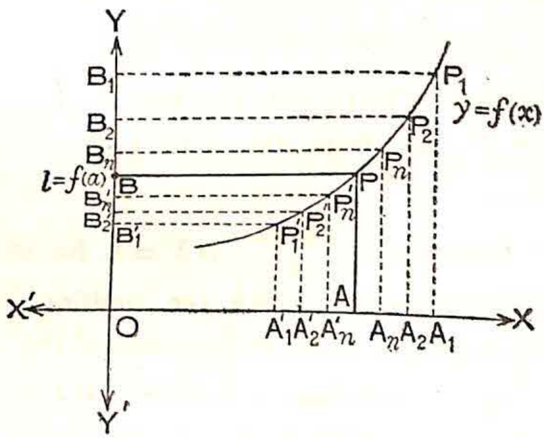


Fig. 37

points $A_1, A_2, \cdots, A_n$, etc. approaches the point $A(x=a)$, then the variable y crossing the points $B_1, B_2 \cdots, B_n$, etc. approaches the point B or $y=f(a)$. By taking x sufficiently close to a from the right, y can be made as close to $f(a)$ as we like. Hence we can say that $\lim_{x \to a+} f(x) = f(a)$.

Similarly by taking points $A_1', A_2', \cdots, A_n'$ on the x-axis to the left of the point A it can be shown that as x approaches the point A from the left through the points $A_1', A_2', \cdots, A_n'$, then y crossing the corresponding points $B_1', B_2', \cdots, B_n'$ (see figure 37) on the y-axis approaches the point B (i.e., $y=f(a)$). Hence we can say $\lim_{x\to a-} f(x)=f(a)$.

So, we find that if the function $f(x)$ be continuous at the point $x=a$, then $\lim_{x\to a+} f(x)=\lim_{x\to a-} f(x)=f(a)$.

Here, since the left hand and right hand limits are equal, so $\lim_{x\to a} f(x)$ exists and in this case $\lim_{x\to a} f(x)=f(a)$.

Thus at a point of continuity, the limiting value of $f(x)$ is the value of $f(x)$ at that point. Hence if a function be continuous at a point, then the limiting value of the function at that point can be obtained by substituting for x the value of that point in the function. For example the graph of the function $f(x)=2x+1$ is not broken anywhere. $\therefore$ $f(x)$ is continuous at the point $x=1$. So, the limiting value of $f(x)$ at $x=1$ i.e., $\lim_{x\to 1} f(x)$ is the value of the function at that point, i.e., $f(1)$.

$$\therefore \lim_{x\to 1} f(x)=f(1)=2\cdot 1+1=3.$$

Similarly, $\lim_{x\to 2} x^2 =2^2=4$. for the function is continuous at the point $x=2$. Hence the limiting value of the function at the point $x=2$, is equal to the value of the function at the point.

Note. We have so far discussed about continuity of a function graphically and seen that at the points of continuity, the limiting values of $f(x)$ are respectively equal to the values of the functions at these points. This is intuitive definition of continuity. We give below mathematical definition of continuity.

Def. If $\lim_{x\to a} f(x)=f(a)$, then the function $f(x)$ is said to be continuous at the point $x=a$. The point $x=a$ is called a point of continuity of the function.

If a function be defined in the closed interval [a, b], then the function is said to be continuous at the end points a and b if $\lim_{x \to a+} f(x)=f(a)$ and $\lim_{x \to b-} f(x)=f(b)$ respectively.

[Note that since $f(x)$ is defined in the closed interval [a, b], we are not concerned with points on the left of a or on the right of a].

If a function be continuous at all points of a closed interval [a, b], then the function is said to be continuous in that interval.

Ex. Evaluate : $\lim_{x \to \frac{\pi}{2}} \sin x$

The graph of the function $\sin x$ is continuous at the point $x=\frac{\pi}{2}$ and so the function is continuous at this point.

So, $\lim_{x \to \frac{\pi}{2}} \sin x=\sin\left(\frac{\pi}{2}\right)=1.$

Actually the graph of the function $\sin x$ is continuous everywhere and so at any point $x=a$, $\lim_{x \to a} \sin x=\sin a$, where a is any real number.

In particular,

$\lim_{x \to 0} \sin x=\sin 0=0.$ $\lim_{x \to \frac{\pi}{3}} \sin x=\sin\frac{\pi}{3}=\frac{\sqrt{3}}{2}$

$\lim_{x \to -\frac{\pi}{6}} \sin x=\sin\left(-\frac{\pi}{6}\right)=-\frac{1}{2}.$

$\lim_{x \to \frac{3\pi}{2}} \sin x=\sin\frac{3\pi}{2}=-1$ etc.

Graphs of most of the elementary functions, discussed in chapter II, are continuous, we shall now discuss their limiting values.

The graphs of the functions e^x, $\sin x$, $\cos x$ are continuous everywhere and hence these functions are continuous for all values

of x. Hence, at any point $x=a$, the limiting values of the functions are equal to the values of the functions at the point.

$$\therefore \lim_{x\to a} e^x=e^a, \lim_{x\to a} \sin x=\sin a, \lim_{x\to a} \cos x=\cos a$$

for all real numbers a.

The graph of the function $\tan x$ is broken at the points $x=\pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}$, etc., but is continuous at all other points.

$\therefore \lim_{x\to a} \tan x=\tan a$, where a is a real number other than

$$(2n+1)\frac{\pi}{2}\ [n=0, \pm 1, \pm 2,\cdots]$$

Similarly, $\lim_{x\to a} \cot x=\cot a$, if $a\neq n\pi [n=0, \pm 1, \pm 2,]$

$\lim_{x\to a} \sec x=\sec a$, if $a\neq(2n+1)\frac{\pi}{2}\ [n=0, \pm 1, \pm 2,\cdots]$

and $\lim_{x\to a} \operatorname{cosec} x=\operatorname{cosec} a$, if $a\neq n\pi\ [n=0, \pm 1, \pm 2,\cdots]$

$$\lim_{x\to a} \log x=\log a, \text{ if } a>0.$$

$\lim_{x\to a} x^n=a^n$, for all real values of a, if $n\geqslant 0$ and for all real values of a other than 0, if $n<0$.

§ 3·5. Discontinuity.

If a function $f(x)$ be not continuous at the point $x=a$, then the function is said to be discontinuous at the point. The graph of the function is broken at the point.

If a function $f(x)$ is continuous at a point $x=a$, then

$$\lim_{x\to a+} f(x)=\lim_{x\to a-} f(x)=f(a).$$

Hence if $f(x)$ is discontinuous at $x=a$,

then either (i) $\lim_{x\to a+} f(x)\neq \lim_{x\to a-} f(x)$

i.e., $\lim_{x\to a} f(x)$ does not exist.

or, (ii) $\lim_{x\to a+} f(x)=\lim_{x\to a-} f(x)\neq f(a)$

i.e., $\lim_{x\to a} f(x)$ exists but the limiting value is not equal to $f(a)$.

In the first kind of discontinuity the graph of the function will be broken at the point of discontinuity and the two branches will have some definite distance.

In the second kind of discontinuity also the graph will be broken at the point of discontinuity, but the distance between the two branches will be less than any positive number however small, i.e., the distance will be zero. In this kind of discontinuity, by properly choosing the value of $f(x)$ at the point of discontinuity $x=a$, the discontinuity can be reduced to a continuity. So this kind of discontinuity is called removable discontinuity.

The above discussion is explained in the following examples :

Examples 3B

Ex. 1. A function $f(x)$ is defined as follows

$$
\begin{aligned}
f(x) &= -x \text{ when } x \leqslant 0 \\
&= x \text{ when } 0 < x < 1 \\
&= 2-x \text{ when } x \geqslant 1
\end{aligned}
$$

Show that the function is continuous at the points $x=0$ and $x=1$. [C. U. 1942]

$$\operatorname*{Lim}_{x \to 0-} f(x) = \lim_{x \to 0-} (-x) = 0$$

$$\lim_{x \to 0+} f(x) = \lim_{x \to 0+} (x) = 0$$

$$\therefore \lim_{x \to 0-} f(x) = \lim_{x \to 0+} f(x) \quad \therefore \lim_{x \to 0} f(x) \text{ exists}$$

and $\lim\limits_{x \to 0} f(x) = 0$.

Also $f(0)=0 \quad \therefore \lim\limits_{x \to 0} f(x) = f(0)$

$\therefore$ The function is continuous at $x=0$

Next $\lim\limits_{x \to 1-} f(x) = \lim\limits_{x \to 1-} x = 1$

$$\lim_{x \to 1+} f(x) = \lim_{x \to 1+} (2-x) = 1$$

$$\therefore \lim_{x \to 1-} f(x) = \lim_{x \to 1-} f(x) = 1$$

$\therefore \lim\limits_{x \to 1} f(x)$ exists and equals 1.

Also $f(1)=2-1=1$.

$$\therefore \lim_{x \to 1} f(x) = f(1)$$

Hence $f(x)$ is continuous at $x=1$

Ex. 2. A function $f(x)$ is defined as follows

$$\begin{aligned} f(x) &= x^2 \text{ when } 0 < x < 1 \\ &= x \text{ when } 1 \leqslant x < 2 \\ &= \tfrac{1}{4}\, x^3 \text{ when } 2 \leqslant x < 3. \end{aligned}$$

Show that the function is continuous at $x=1$ and $x=2$.

[C. U. 1941]

$$\lim_{x\to 1-} f(x) = \lim_{x\to 1-} x^2 = 1$$

$$\lim_{x\to 1+} f(x) = \lim_{x\to 1+} x = 1$$

$$\therefore \quad \lim_{x\to 1-} f(x) = \lim_{x\to 1+} f(x) = 1$$

$$\therefore \quad \lim_{x\to 1} f(x) \text{ exists and equals } 1$$

Also $f(1)=1$ $\quad \therefore \quad \lim\limits_{x\to 1} f(x) = f(1)$

Hence $f(x)$ is continuous at $x=1$

Again, $\lim\limits_{x\to 2-} f(x) = \lim\limits_{x\to 2-} (x) = 2$

$$\lim_{x\to 2+} f(x) = \lim_{x\to 2+} (\tfrac{1}{4}\, x^3) = 2.$$

$$\therefore \quad \lim_{x\to 2-} f(x) = \lim_{x\to 2+} f(x) = 2$$

$$\therefore \quad \lim_{x\to 2} f(x) \text{ exists and equals } 2.$$

Also $f(2) = \frac{1}{4}.(2)^3 = 2.$

$$\therefore \quad \lim_{x\to 2} f(x) = f(2)$$

Hence the function is continuous at $x=2$.

Ex. 3. A function $f(x)$ is defined as follows

$$\begin{aligned} f(x) &= -1 \text{ when } x < 0 \\ &= \ \ 0 \text{ when } x = 0 \\ &= \ \ 1 \text{ when } x > 0. \end{aligned}$$

Test the continuity of the function at $x=0$.

$$\lim_{x\to 0-} f(x) = \lim_{x\to 0-} -1 = -1$$

$$\lim_{x\to 0+} f(x) = \lim_{x\to 0+} 1 = 1$$

$\therefore \lim\limits_{x \to 0-} f(x) \neq \lim\limits_{x \to 0+} f(x)$

Hence $\lim\limits_{x \to 0} f(x)$ does not exist.

So the function is discontinuous at $x=0$.

Ex. 4. A function $f(x)$ is defined as follows :

$$f(x)=(2x+1), \text{ when } x \geqslant 1$$
$$=2x-1, \text{ when } x<1.$$

Here $\lim\limits_{x \to 1+} f(x) = \lim\limits_{x \to 1+} (2x+1) = 3.$

[$\because$ $x \to 1+$, so $x>1$ $\quad\therefore$ by definition $f(x)=2x+1$]

$$\lim_{x \to 1-} f(x) = \lim_{x \to 1-} (2x-1) = 1.$$

[Here, $\because$ $x \to 1-$, so $x<1$ and hence by definition $f(x)=2x-1$].

$\therefore \lim\limits_{x \to 1+} f(x) \neq \lim\limits_{x \to 1-} f(x)$.

Hence the function is discontinuous at the point $x=1$ and this discontinuity is not removable.

In the graph, notice that the graph is broken at the point $x=1$ and the distance between the disjoint branches is $AB=3-1=2$ which is a finite number.

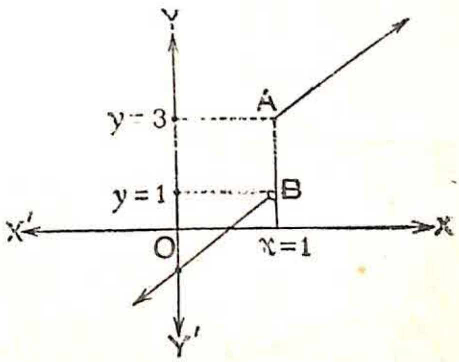


Fig. 38

Ex. 5. Let us consider the function

$$f(x)=\frac{x^2-1}{x-1}.$$

The function is undefined at the point $x=1$ and so is discontinuous at the point.

Now, $\lim\limits_{x \to 1} \dfrac{x^2-1}{x-1} = \lim\limits_{x \to 1} (x+1) = 2$ [as $x \to 1$, we can take $x \neq 1$]

Hence $\lim\limits_{x \to 1} f(x)$ is not equal to $f(1)$ which is undefined.

Let us now redefine the function as follows :

$$f(x)=\frac{x^2-1}{x-1} \text{ when } x\neq 1 \text{ and}=2, \text{ when } x=1.$$

Now according to this new definition of the function

$$\lim_{x\to 1} f(x)=2=f(1).$$

Hence the function is now continuous at the point $x=1$. Thus we see that in this case by redefining the function properly at the point in which it is not defined, the discontinuity of the function at the point can be removed.

Hence the discontinuity of the original function at the point $x=1$ is a removable discontinuity.

The graph of the function $f(x)=\frac{x^2-1}{x-1}$ has two branches. The branches are disjoint at the point $x=1$: but the distance between the branches is less than any positive number, however small, i.e., 0. In reality, as the function is undefined at the point $x=1$, its graph has a break only at the point. Now if we take $f(x)=2$ at the point, then there will be no break in the graph and it will be continuous at the point $x=1$.

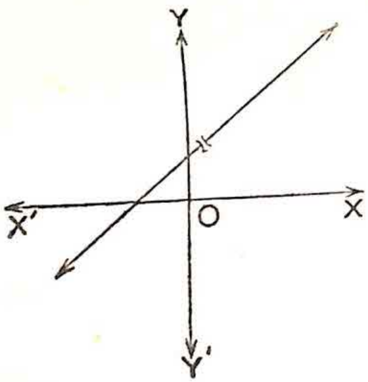


Fig. 39

Ex. 6. $f(x)=\frac{\sin x}{x}$ when $x\neq 0$.

$=2$ when $x=0$.

[Here x is in radians]

Here $f(0)=2$ and values of $f(x)$ at points close to 0 are tabulated below.

$x=$	1	·1	01	·001	
$\sin x=$	·84147	·0998334	·00999984	·00099999983	etc.
$\frac{\sin x}{x}=$	·84147	·998334	·999984	·99999983	

$x=$	1	$-\cdot 1$	$-\cdot 01$	$-\cdot 001$
$\sin x=$	$-\cdot 84144$	$-\cdot 0998334$	$-\cdot 00999984$	$\cdot 00099999983$ etc.
$\dfrac{\sin x}{x}=$	$\cdot 84147$	$\cdot 998334$	$\cdot 999984$	$\cdot 99999983$

Hence it is evident that as x approaches the point $x=0$, whether from the right or from the left, $\dfrac{\sin x}{x}$ approaches the value 1.

$$\therefore \lim_{x \to 0} \frac{\sin x}{x}=1.$$

Now, $f(0)=2$

$$\therefore \lim_{x \to 0} f(x) \neq f(0).$$

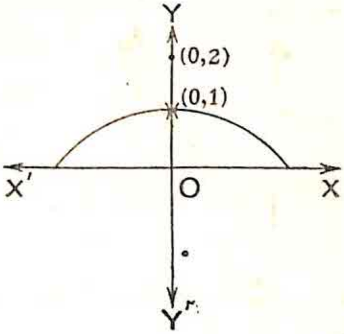


Fig. 40

Hence the function $f(x)$ is discontinuous at the point $x=0$. Now, if instead of 2 the value of the function at $x=0$ be taken as 1, then we shall have

$$\lim_{x \to 0} f(x)=f(0)$$

and the function will be continuous at the origin. So in this case by properly redefining the function at the origin the discontinuity can be removed.

The graph of the function has three branches, one on each side of the y-axis and the third being the point $(0, 2)$. The main two branches are disconnected at the point $(0, 1)$ and distance between these two branches is less than any positive number, however small, i.e., the distance is zero. Now if the value of the function at the point $x=0$, be taken as 1 instead of 2, then the point $(0, 1)$ will be a point on the graph of the function in place of the point $(0, 2)$ and the function will be continuous at the point $x=0$.

The limit $\displaystyle\lim_{x \to 0} \frac{\sin x}{x}=1$ is very useful. We shall discuss this limit in § 3·7.]

Ex. 6. A function $f(x)$ is defined as follows :

$$f(x)=2x,\ x\leqslant\tfrac{1}{2}$$
$$=2x+1,\ \tfrac{1}{2}<x<1$$
$$=3x,\ x>1.$$

Find the points of discontinuity of the function $f(x)$.

Let $y=f(x)$. Now the graph of the function $y=f(x)$ has three branches one for $x\leqslant\frac{1}{2}$, one branch for $\frac{1}{2}<x<1$ and the third for $x\geqslant 1$. The graph of the function is shown below.

From the graph notice that the graph is broken at the points $x=\frac{1}{2}$ and $x=1$. The branches disjoint at $x=\frac{1}{2}$ are at a distance 1 unit from each other. Hence the discontinuity of the function at the point $x=\frac{1}{2}$ is not removable. The distance between the branches disjoint at the point $x=1$ is less than any positive number, however small, i.e., the distance is zero. So, the discontinuity of the function at the point $x=1$ is removable. In all other points the graph is continuous and hence $x=\frac{1}{2}$ and $x=1$ are the only two points of discontinuity of the function.

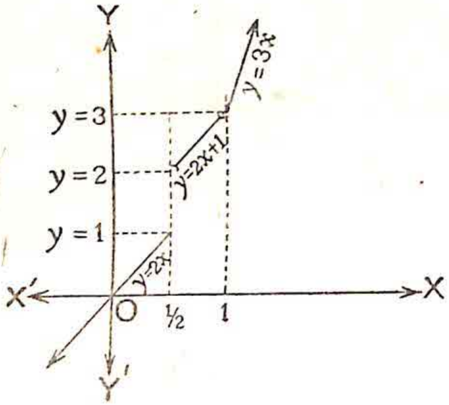


Fig. 41

Ex. 7. A function $f(x)$ is defined as follows :

$$f(x)=-1,\quad x<0$$
$$=x,\quad 0\leqslant x<1$$
$$=2-x,\quad 1\leqslant x<2$$
$$=2x,\quad x\geqslant 2.$$

Find the points of discontinuity of the function.

The graph of the function has four branches. The graph is drawn here. It is evident from the graph, that it is broken at the points $x=0$ and $x=2$. The disjoint branches at these points are respectively at distances 1 and 4 from each other. So, the

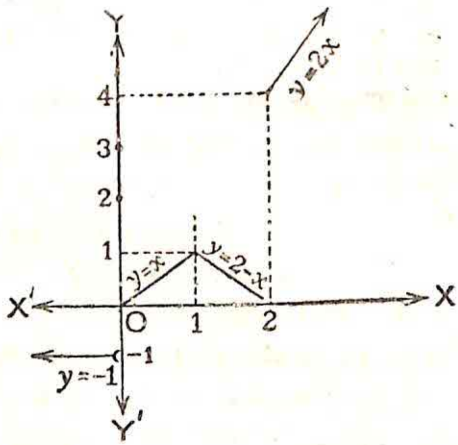


Fig. 42

discontinuities of the function at these points are not removable. The function is continuous at all other points.

Ex. 8. Show that, $f(x)=x \sin \frac{1}{x}$ is discontinuous at $x=0$. Find the nature of the discontinuity.

$\because \quad \left| x \sin \frac{1}{x} \right| = | x | \left| \sin \frac{1}{x} \right| \leqslant | x | \quad \to 0 \quad$ as $x \to 0$,

$\lim\limits_{x \to 0} x \sin \frac{1}{x} = 0.$ [see Ex. 10, § 3·3]

But, at $x=0$, $\sin \frac{1}{x}$ is indeterminate, hence at $x=0$, $x \sin \frac{1}{x}$ i.e., $f(0)$ is undefined. Therefore the function is discontinuous at $x=0$. If now the function is defined as

$$f(x)=x \sin \frac{1}{x}, \text{ when } x \neq 0$$
$$=0 \quad , \text{ when } x=0,$$

then the discontinuity at $x=0$ is removed, for according to the new definition, $\lim\limits_{x \to 0} f(x) = \lim\limits_{x \to 0} x \sin \frac{1}{x} = 0 = f(0)$.

$\therefore$ Discontinuity at $x=0$ is removable.

Exercise 3C

1. (*a*) Draw the graph of the following functions to evaluate the following limits :

(i) $\lim\limits_{x \to \sqrt{2}} x^2$ (ii) $\lim\limits_{x \to 4} (3x+4)$

(iii) $\lim\limits_{x \to \frac{\pi}{2}} \operatorname{cosec} x$ (iv) $\lim\limits_{x \to 4} e^x$.

(*b*) Determine the points of discontinuity of the functions :

(i) $\dfrac{1}{x-2}$ (ii) $\dfrac{x^2-5x+6}{x^2-3x+2}$ (iii) $\dfrac{\sin x}{\cos x}$.

2. Draw the graph of the function

$| x-1 |$ and hence find $\lim\limits_{x \to 1} f(x)$.

3. (i) $f(x)=x$ when $x<0$

$=2x+1$ when $0 \leqslant x < 1$

$=3x$ when $x \geqslant 1$

Find the points of discontinuity of the function if any.

(ii) $f(x)=\dfrac{x^2-25}{x-5}$ when $x \neq 5$.

Test the function for continuity at $x=5$

(iii) A function $\phi(x)$ is defined in the interval $-1 \leqslant x \leqslant 2$ as follows

$$\begin{aligned}\phi(x)&=3+2x \quad \text{when } -1 \leqslant x<0\\ &=3-2x \quad \text{when } 0 \leqslant x \leqslant \tfrac{3}{2}\\ &=-3+2x \text{ when } \tfrac{3}{2} \leqslant x \leqslant 2.\end{aligned}$$

Test the continuity of the function at $x=0$ and $x=\frac{3}{2}$.

4. Draw the graph of the function 2^x and determine the points of discontinuity of the function.

5. Find the limit $\lim\limits_{x \to 3} \dfrac{x^2-9}{x-3}$. What kind of discontinuity the function has at the point $x=3$.

6. A function $f(x)$ is defined as follows :

$$\begin{aligned}f(x)&=-x, \qquad x<0\\ &=x, \qquad 0 \leqslant x<1\\ &=2x, \qquad 1 \leqslant x<2\\ &=3x, \qquad x \geqslant 2.\end{aligned}$$

Deter mine the points of discontinuity of the function and also the types of discontinuity.

7. $f(x)=\dfrac{x^2-4}{x-2}$ is not defined at $x=2$. For what value of $f(2)$, the discontinuity at $x=2$ can be changed into a continuity ?

8. A function $\phi(x)$ is defined as follows :

$$\begin{aligned}\phi(x)&=\tfrac{1}{2}-x, \text{ when } x<\tfrac{1}{2}\\ &=\tfrac{1}{2}, \qquad \text{when } x=\tfrac{1}{2}\\ &=\tfrac{3}{2}-x, \text{ when } x>\tfrac{1}{2}\end{aligned}$$

Show that $\phi(x)$ is discontinuous at $x=\frac{1}{2}$.

§ 3·6. Limit Theorems.

The li mits of different functions can frequently be determined with the help of the following limit theorems. We state below the following limit theorems without proof.

If $f(x)$ and $g(x)$ be two functions of x and $\lim\limits_{x \to a} f(x)=l$, $\lim\limits_{x \to a} g(x)=m$, where l and m are two finite quantities, then,

(1) $\lim\limits_{x \to a} c.f(x) = cl = c. \lim\limits_{x \to a} f(x)$, where c is a constant.

(2) $\lim\limits_{x \to a} \{f(x) + g(x)\} = l + m = \lim\limits_{x \to a} f(x) + \lim\limits_{x \to a} g(x)$

(3) $\lim\limits_{x \to a} \{f(x) - g(x)\} = l - m = \lim\limits_{x \to a} f(x) - \lim\limits_{x \to a} g(x)$

(4) $\lim \{f(x)g(x)\} = \lim\limits_{x \to a} f(x). \lim\limits_{x \to a} g(x)$

(5) $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{l}{m} = \dfrac{\lim\limits_{x \to a} f(x)}{\lim\limits_{x \to a} g(x)}$, where $m \neq 0$.

(6) $\lim\limits_{x \to a} f\{g(x)\} = f(m) = f\{\lim\limits_{x \to a} g(x)\}$, if $f(x)$ be a continuous function.

Note 1. The theorems can also be stated as follows :

If the limits of two functions at a point exist, then the limits of the sum, difference, product and quotient of the functions are respectively equal to the sum, difference, product and quotient of the limits of the functions at the point. (In case of quotient, the limit in the denominator must not be zero.)

2. The theorems (2), (3) and (4) are true for more than two functions ; i.e., if the limits of functions $f_1(x), f_2(x), f_3(x), \cdots\cdots$ at the point a exist, then

$$\lim_{x \to a} \{f_1(x) \pm f_2(x) \pm f_3(x) \pm \cdots\cdots\}$$

$$= \lim_{x \to a} f_1(x) \pm \lim_{x \to a} f_2(x) \pm \lim_{x \to a} f_3(x) + \cdots\cdots$$

and $\lim\limits_{x \to a} \{f_1(x).f_2(x).f_3(x)\cdots\cdots\}$

$$= \lim_{x \to a} f_1(x). \lim_{x \to a} f_2(x). \lim_{x \to a} f_3(x)\cdots\cdots$$

3. If $\lim\limits_{x \to a} f(x)$ and $\lim\limits_{x \to a} g(x)$ do not exist, then the theorems will not be true.

For example, $\dfrac{x^4 - 16}{x - 4}$ can be written as $\dfrac{x^4}{x-4} - \dfrac{16}{x-4}$.

But $\lim\limits_{x \to 4} \dfrac{x^2 - 16}{x - 4} \neq \lim\limits_{x \to 4} \dfrac{x^2}{x-4} - \lim\limits_{x \to 4} \dfrac{16}{x-4}$

The limit on the left hand side is 8 and none of the two limits on the right exist.

Ex. 1. Show that, $\lim\limits_{x\to 2}(x^2+2x)=\lim\limits_{x\to 2} x^2+\lim\limits_{x\to 2} 2x.$

We tabulate below the values of x^2, $2x$ and x^2+2x at points close to $x=2$

$x=$	1·9	1·99	1·999	1·9999
$x^2=$	3·61	3·96	3·996	3·9996
$2x=$	3·80	3·98	3·998	3·9998
$x^2+2x=$	7·41	7·94	7·994	7·9994 etc.

$x=$	2·1	2·01	2·001	2·0001
$x^2=$	4·41	4·0401	4·004001	4·00040001
$2x=$	4·20	4·02	4·002	4·0002
$x^2+2x=$	8·61	8·0601	8·006001	8·00060001 etc.

From the above tables it is evident that as x approaches the point 2 whether from the left or from the right, each of x^2 and $2x$ tends to 4 and x^2+2x approach 8.

Hence $\lim\limits_{x\to 2} x^2=4$; $\lim\limits_{x\to 2} 2x=4$; $\lim\limits_{x\to 2}(x^2+2x)=8.$

$$\therefore \quad \lim (x^2+2x)=8=4+4.$$

$$=\lim_{x\to 2} x^2+\lim_{x\to 2} 2x$$

Ex. 2. Show that $\lim\limits_{x\to 1}(2x+1)^2=\left\{\lim\limits_{x\to 1}(2x+1)\right\}^2$

We tabulate below the values of $2x+1$ and $(2x+1)^2$ at points close to the point $x=1$.

x	$\cdot 9$	$\cdot 99$	$\cdot 999$	$\cdot 9999$
$2x+1$	$2\cdot 8$	$2\cdot 98$	$2\cdot 998$	$2\cdot 9998$
$(2x+1)^2$	$7\cdot 84$	$8\cdot 8804$	$8\cdot 98804$	$8\cdot 99880004$ etc.

x	$1\cdot 1$	$1\cdot 01$	$1\cdot 001$	$1\cdot 0001$
$2x+1$	$3\cdot 2$	$3\cdot 02$	$3\cdot 002$	$3\cdot 0002$
$(2x+1)^2$	$10\cdot 24$	$9\cdot 1204$	$9\cdot 012004$	$9\cdot 0012003$ etc.

It is evident from the tables that as x tends to 1 whether from the left or from the right the functions $(2x+1)$ and $(2x+1)^2$ respectively tend to the limits 3 and 9. Hence,

$$\lim_{x\to 1} (2x+1)^2 = 9 = 3^2 = \left\{\lim_{x\to 1} (2x+1)\right\}^2$$

From the above theorems we may say that if two functions be continuous at a point, then the sum, difference, product and quotient of the functions (the limit in the denominator in case of quotient will not be zero) will also be continuous at the point.

For, let $f(x)$ and $g(x)$ be two functions continuous at the points $x=a$

Then $\lim\limits_{x\to a} f(x)=f(a)$ and $\lim\limits_{x\to a} g(x)=g(a)$.

$$\therefore \lim_{x\to a}\left\{f(x)\pm g(x)\right\} = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$$

[As the two limits exist]

$$= f(a) \pm g(a).$$

$$\lim_{x\to a}\left\{f(x)g(x)\right\} = \lim_{x\to a} f(x). \lim_{x\to a} g(x) = f(a)g(a)$$

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{f(a)}{g(a)} \text{ if } g(a)\neq 0.$$

The following examples illustrate the use of the above limit theorems.

Ex. 3. $\lim\limits_{x\to 2\cdot 2} 5x = 5 \lim\limits_{x\to 2\cdot 2} x$ [Theorem (1)] $= 5\times 2\cdot 2 = 11$.

Ex. 4. $\lim\limits_{x\to 2} (x^2-4x-5) = \lim\limits_{x\to 2} x^2 - \lim\limits_{x\to 2} 4x - \lim\limits_{x\to 2} 5$

[Theorems 2 and 3]

$= \lim\limits_{x\to 2} x \lim\limits_{x\to 2} x - 4 \lim\limits_{x\to 2} x - \lim\limits_{x\to 2} 5$ [Theorems (4) and (1)]

$= 2.2 - 4.2 - 5 = 4 - 8 - 5 = -9$.

Ex. 5. Show that $\lim\limits_{x\to a} x^n = a^n$ [n is any positive integer]

$\lim\limits_{x\to a} x^n = \lim\limits_{x\to a} (x.x.x\cdots \text{to } n \text{ factors})$

$= \lim\limits_{x\to a} x \lim\limits_{x\to a} x \lim\limits_{x\to a} x \ldots\ldots \text{to } n$ factors [See note 2]

$= a.a.a\cdots$ to n factors $= a^n$

[The above limit is true for all values of n]

Ex. 6. If $f(x) = a_0x^n + a_1x^{n-1} + \ldots + a_{n-1}x + a_n$ be a polynomial, then show that $\lim\limits_{x\to a} f(x) = f(a)$

$\lim\limits_{x\to a} f(x) = \lim\limits_{x\to a} (a_0x^n + a_1x^{n-1} + \ldots + a_{n-1}x + a_n)$

$= \lim\limits_{x\to a} a_0x^n + \lim\limits_{x\to a} a_1x^{n-1} + \cdots + \lim\limits_{x\to a} a_{n-1}x + \lim\limits_{x\to a} a_n$

[By theorems (2) and Note (2)]

$= a_0 \lim\limits_{x\to a} x^n + a_1 \lim\limits_{x\to a} x^{n-1} + \cdots \ldots a_{n-1} \lim\limits_{x\to a} x + a_n,$

$= a_0\, a^n + a_1 a^{n-1} + \cdots + a_{n-1}\, a + a_n$

$= f(a)$.

[From the above example, it can be said that all polynomials are continuous everywhere and hence the limit of the function at any point is the value of the function at the point. For example, $\lim\limits_{x\to 3} (x^4 - 3x^2 + 4) = 3^4 - 3.3^2 + 4 = 81 - 27 + 4 = 58$.]

Ex. 7. $\lim\limits_{x\to 4} x^{-3} = \lim\limits_{x\to 4} \dfrac{1}{x^3} = \dfrac{\lim\limits_{x\to 4} 1}{\lim\limits_{x\to 4} x^3} = \dfrac{1}{4^3} = \dfrac{1}{64}$

Ex. 8. $\lim\limits_{x\to -1} \dfrac{x^2-3x+4}{x^2+x+1} = \dfrac{\lim\limits_{x\to -1} (x^2-3x+4)}{\lim\limits_{x\to -1} (x^2+x+1)}$

$= \dfrac{(-1)^2-3(-1)+4}{(-1)^2+(-1)+(1)} = \dfrac{1+3+4}{1-1+1} = 8$

Ex. 9. $\lim\limits_{x\to 1} \sin (x^2+2x-3)$

$= \sin \left\{ \lim\limits_{x\to 1} (x^2+2x-3) \right\}$ [By theorem 6 as $\sin x$ is a continuous function]

$= \sin (1^2+2.1-3) = \sin 0 = 0.$

Ex. 10. $\lim\limits_{x\to 1} 2^{x^2-4x+1} = 2^{\lim\limits_{x\to 1} x^2-4x+1}$

[As 2^x is a continuous function]

$= 2^{1^2-4.1+1} = 2^{-2} = \frac{1}{4}.$

Exercise 3(D)

1. Evaluate the following limits :

(i) $\lim\limits_{x\to -1} 3x$ (ii) $\lim\limits_{y\to 2} (2y+5)$

(iii) $\lim\limits_{y\to 1} (100y^2+10y+10)$ (iv) $\lim\limits_{x\to 4} x^3$

(v) $\lim\limits_{x\to -2} x^{-3}$ (vi) $\lim\limits_{x\to 1} \dfrac{1}{x^2}$

(vii) $\lim\limits_{x\to 0} \dfrac{x^2-4}{x-2}$ (viii) $\lim\limits_{x\to 1} (x^2+1)^2$

(ix) $\lim\limits_{x\to 0} \cos (x^2-x)$ (x) $\lim\limits_{x\to 0} e^{x^3+4x+4}$

(xi) $\lim\limits_{x\to 3} \dfrac{x^2-3x+4}{x^2+4x-8}$ (xii) $\lim\limits_{x\to 1} \dfrac{x^2+2x+1}{x^{-2}+1}.$

2. Prove that

(i) $\lim\limits_{x\to 1} \dfrac{x^6-1}{x-1} = 6$ (ii) $\lim\limits_{x\to 0} \dfrac{\sqrt{1+x}-1}{x} = \frac{1}{2}.$

(iii) $\lim\limits_{x\to 1} \dfrac{x^2-1}{\sqrt{3x+1}-\sqrt{5x-1}} = -4$

(iv) $\lim\limits_{x\to 0} e^{\frac{\sin x}{x}} = 1.$

§ 3·7. Some important limits.

We have earlier shown that if a function be continuous at a point, then the limit of the function at the point is equal to the value of the function at that point. Determination of limits of functions at points in which the functions are not defined (i.e., of the form $\frac{0}{0}$) is very important. We state below the limits of some functions at such points.

(i) $\lim\limits_{x\to a} \dfrac{x^n - a^n}{x-a} = na^{n-1}$, where n is any constant.

(ii) $\lim\limits_{x\to 0} \dfrac{\sin x}{x} = 1$, where x is given in radians.

(iii) $\lim\limits_{x\to 0} \dfrac{e^x - 1}{x} = 1$. (iv) $\lim\limits_{x\to 0} \dfrac{\log(1+x)}{x} = 1$.

[Note : (1) The above limits can be intuitively proved easily. In fact the intuitive proof of (ii) has been given in § 3·5.

(2) The above limits are true if instead of x any other variable is taken. For example,

$$\lim_{h\to 0}\frac{\sin h}{h}=1 \; ; \quad \lim_{k\to 0}\frac{e^k-1}{k}=1 \; ; \text{ or } \lim_{y\to a}\frac{y^n-a^n}{y-a}=na^{n-1} \text{ etc.}$$

(3) In the limit (i), putting $x-a=h$ we get

$$\lim_{x\to a}(x-a)=0. \qquad \text{or, } \lim_{x\to a} h = 0$$

i.e., when $x\to a$, then $h\to 0$.

(i) can be written as

$$\lim_{h\to 0}\frac{(a+h)^n - a^n}{h} = na^{n-1}.$$

In fact in any limit, putting $x-a=h$.

$\lim\limits_{x\to a} f(x)$ can be expressed in the form $\lim\limits_{h\to 0} f(a+h)$

$$\text{i.e., } \lim_{x\to a} f(x) = \lim_{h\to 0} f(a+h)$$

The following examples illustrate the application of the above limits.

Ex. 1. $\lim\limits_{x\to 2}\dfrac{x^8-2^8}{x-2} = 8.2^7 = 8\times 128 = 1024$

[Application of (i)]

Ex. 2. $\lim\limits_{x\to 3}\dfrac{x^3-27}{x^2-9} = \lim\limits_{x\to 3}\dfrac{(x^3-27)/(x-3)}{(x^2-9)/(x-3)}$

$[\because\ x\to 3, \ \therefore\ x-3\neq 0]$

$$=\frac{\lim\limits_{x\to 3}\dfrac{x^3-27}{x-3}}{\lim\limits_{x\to 3}\dfrac{x^2-9}{x-3}}=\frac{\lim\limits_{x\to 3}\dfrac{x^3-3^3}{x-3}}{\lim\limits_{x\to 3}\dfrac{x^2-3^2}{x-3}}=\frac{3.3^2}{2.3}=\frac{9}{2}$$

Ex. 3. $$\lim_{x\to a}\frac{x^{\frac{5}{2}}-a^{\frac{5}{2}}}{x^{\frac{3}{8}}-a^{\frac{3}{8}}}=\lim_{x\to a}\frac{\dfrac{x^{\frac{5}{2}}-a^{\frac{5}{2}}}{x-a}}{\dfrac{x^{\frac{3}{8}}-a^{\frac{3}{8}}}{x-a}}\quad [\because\ x-a\neq 0]$$

$$=\frac{\lim\limits_{x\to a}\dfrac{x^{\frac{5}{2}}-a^{\frac{5}{2}}}{x-a}}{\lim\limits_{x\to a}\dfrac{x^{\frac{3}{8}}-a^{\frac{3}{8}}}{x-a}}$$

[Since the limits of both numerator and denominator exist and the limit of the denominator is not zero.]

$$=\frac{\frac{5}{2}a^{\frac{5}{2}-1}}{\frac{3}{8}a^{\frac{3}{8}-1}}=\tfrac{5}{2}\times\tfrac{8}{3}a^{\frac{5}{2}-\frac{3}{8}}=\tfrac{20}{8}a^{\frac{17}{8}}$$

Ex. 4. $$\lim_{h\to 0}\frac{(1+h)^n-1}{h}=\lim_{x\to 1}\frac{x^n+1}{x-1}=n1^{n-1}=n$$

[Putting $1+h=x$. As $h\to 0$, $x\to 1$]

Ex. 5. $$\lim_{x\to 0}\frac{\tan x}{x}=\lim_{x\to 0}\left\{\frac{\sin x}{x}.\frac{1}{\cos x}\right\}$$

$$=\lim_{x\to 0}\frac{\sin x}{x}\lim_{x\to 0}\frac{1}{\cos x}=1.\frac{1}{\lim\limits_{x\to 0}\cos x}=1.\frac{1}{1}=1.$$

Ex. 6. $$\lim_{x\to 0}\frac{\sin 3x}{x}=\lim_{x\to 0}\frac{3\sin 3x}{3x}$$

$$=3\lim_{x\to 0}\frac{\sin 3x}{3x}=3\lim_{h\to 0}\frac{\sin h}{h}=3.1=3.$$

[Putting $3x=h$ and as $x\to 0$, then $h=3x\to 0$]

Ex. 7. $$\lim_{x\to 0}\frac{\sin\alpha x}{\sin\beta x}=\lim_{x\to 0}\left\{\frac{\dfrac{\sin\alpha x}{\alpha x}}{\dfrac{\sin\beta x}{\beta x}}.\frac{\alpha x}{\beta x}\right\}$$

$$=\lim_{x\to 0}\left\{\frac{\dfrac{\sin\alpha x}{\alpha x}}{\dfrac{\sin\beta x}{\beta x}}.\frac{\alpha}{\beta}\right\}=\frac{\alpha}{\beta}.\frac{\lim\limits_{x\to 0}\dfrac{\sin\alpha x}{\alpha x}}{\lim\limits_{x\to 0}\dfrac{\sin\beta x}{\beta x}}=\frac{\alpha}{\beta}.\frac{1}{1}=\frac{\alpha}{\beta}.$$

Ex. 8. $\lim_{x\to 0}\left(\frac{\sin 2x}{x}\right)^2=\lim_{h\to 0}\left(\frac{\sin h}{\frac{h}{2}}\right)^2$

[Putting $h=2x$ and as $x\to 0$ then $h\to 0$]

$=4\lim_{h\to 0}\left(\frac{\sin h}{h}\right)^2=4\left\{\lim_{h\to 0}\left(\frac{\sin h}{h}\right)\right\}^2=4.1^2=4.$

Ex. 9. $\lim_{h\to 0}\frac{1-\cos^2 h}{h^2}=\lim_{h\to 0}\frac{2\sin^2 h}{h^2}$

$=2\lim_{h\to 0}\left\{\left(\frac{\sin h}{h}\right).\left(\frac{\sin h}{h}\right)\right\}$

$=2\lim_{h\to 0}\frac{\sin h}{h}\cdot\lim_{h\to 0}\frac{\sin h}{h}=2.1.1=2.$

Ex. 10. $\lim_{h\to 0}\frac{e^{h^2}-1}{h}=\lim_{h\to 0}\left\{\frac{e^{h^2}-1}{h^2}.\frac{h^2}{h}\right\}$

$=\lim_{h\to 0}\left\{\frac{e^{h^2}-1}{h^2}.h\right\}=\lim_{h\to 0}\frac{e^{h^2}-1}{h^2}.\ \lim h=1.0=0.$

[$\because$ when $h\to 0$ then $h^2\to 0$]

Ex. 11. $\lim_{x\to 0}\frac{e^{\sin x}-1}{x}=\lim_{x\to 0}\left\{\frac{e^{\sin x}-1}{\sin x}.\frac{\sin x}{x}\right\}$

$=\lim_{h\to 0}\frac{e^{\sin x}-1}{\sin x}\cdot\lim_{x\to 0}\frac{\sin x}{x}$

$=\left\{\lim_{h\to 0}\frac{e^h-1}{h}\right\}.\ 1,$ [Putting $\sin x=h$. As $x\to 0$ then $h\to 0$.]

$=1.1=1.$

Ex. 12. $\lim_{x\to 0}\frac{x\sin x}{1-\cos 2x}=\lim_{x\to 0}\frac{x\sin x}{2\sin^2 x}=\lim_{x\to 0}\frac{x}{2\sin x}$

$=\frac{1}{2}.\frac{1}{\lim_{x\to 0}\frac{\sin x}{x}}=\frac{1}{2}.\frac{1}{1}=\frac{1}{2}$

Ex. 13. $\lim_{x\to 0}\frac{\log(1+3x)}{x}=\lim_{x\to 0}\frac{\log(1+3x)}{3x}.3$

$=3\lim_{y\to 0}\frac{\log(1+y)}{y}$, where $y=3x\to 0$ as $x\to 0$

$=3.1=3.$

Ex. 14. $\lim\limits_{x\to0}(1+x)^{\frac{1}{x}}=\lim\limits_{x\to0} e^{\frac{\log(1+x)}{x}}$

$=e^{\lim\limits_{x\to0}\frac{\log(1+x)}{x}}$ $\left[\begin{array}{l}\text{by § 3·6 as } e^x\\ \text{continuous.}\end{array}\right]$

$=e^1=e.$

Ex. 15. $\lim\limits_{x\to0}\frac{\log(1+ax)}{\sin bx}=\lim\limits_{x\to0}\left\{\frac{\log(1+ax)}{ax}\cdot\frac{\sin bx}{bx}\cdot\frac{a}{b}\right\}$

$=\frac{a}{b}\lim\limits_{x\to0}\frac{\log(1+ax)}{ax}\cdot\lim\limits_{x\to0}\frac{\sin bx}{bx}=\frac{a}{b}.1.1=\frac{a}{b}$

Ex. 16. $\lim\limits_{x\to0}\frac{e^{\alpha x}-e^{\beta x}}{x}=\lim\limits_{x\to0} e^{\beta x}.\frac{e^{(\alpha-\beta)x}-1}{(\alpha-\beta)x}.(\alpha-\beta)$

$=(\alpha-\beta)\lim\limits_{x\to0}e^{\beta x}.\lim\limits_{y\to0}\frac{e^y-1}{y}$, $[y=(\alpha-\beta)x\to0 \text{ as } x\to0]$

$=(\alpha-\beta).1=\alpha-\beta.$

Exercise 3 (E)

1. Prove that

(i) $\lim\limits_{x\to5}\frac{x^4-625}{x-5}=500$

(ii) $\lim\limits_{x\to4}\frac{x^3-64}{x^2-16}=6$

(iii) $\lim\limits_{x\to a}\frac{x^{\frac{3}{7}}-a^{\frac{3}{7}}}{x^{\frac{2}{5}}-a^{\frac{2}{5}}}=\frac{15}{14}a^{\frac{1}{35}}$

(iv) $\lim\limits_{x\to a}\frac{x^n-a^n}{x^m-a^m}=\frac{n}{m}a^{n-m}$

(v) $\lim\limits_{x\to a}\frac{x^{-4}-a^{-4}}{x^{-7}-a^{-7}}=\frac{4}{7}a^3$

(vi) $\lim\limits_{x\to\sqrt{2}}\frac{x^{\frac{5}{2}}-2^{\frac{5}{4}}}{\sqrt{x}-2^{\frac{1}{4}}}=10$

(vii) $\lim\limits_{x\to0}\frac{\sin\frac{1}{2}x}{x}=\frac{1}{2}$

(viii) $\lim\limits_{x\to\frac{\pi}{2}}\frac{\cos x}{\frac{\pi}{2}-x}$

(ix) $\lim\limits_{h\to0}(h.\operatorname{cosec} h)=1$

(x) $\lim\limits_{h\to0}\frac{\sin^2 h\cos h}{h^2}=1$

(xi) $\lim\limits_{x\to0}\frac{\tan 5x}{x}=5$

(xii) $\lim\limits_{h\to0}\frac{1-\cos h}{h^2}=\frac{1}{2}$

(xiii) $\lim\limits_{\theta\to0}\frac{\tan m\theta}{n\theta}=\frac{m}{n}$

(xiv) $\lim\limits_{h\to0}\frac{e^{h^3}-1}{h}=0$

(xv) $\lim\limits_{x\to0}\frac{e^{\tan x}-1}{x}=1$

(xvi) $\lim_{h\to 0} \frac{e^{2+h}-e^2}{h}=e^2$ (xvii) $\lim_{h\to 0} \frac{e^{x+h}-e^x}{h}=e^x$

(xviii) $\lim_{x\to 0} \frac{\log(1+ax)}{x}=a$ (xix) $\lim_{x\to 0} \frac{\log(1+2x)}{\tan 3x}=\frac{2}{3}$

(xx) $\lim_{x\to 0} (1+2x)^{\frac{1}{x}}=e^2$.

2. Evaluate :

(i) $\lim_{x\to 2} \frac{(2x)^4-256}{2(x-2)}$ (ii) $\lim_{x\to -2} \frac{x^3+8}{x+2}$

(iii) $\lim_{x\to 1} \frac{x^{\frac{7}{2}}-1}{x^{\frac{5}{2}}-1}$ (iv) $\lim_{h\to 0} \frac{(2+h)^4-16}{h}$

(v) $\lim_{x\to 0} (\tan 3x \operatorname{cosec} 3x)$ (vi) $\lim_{x\to 0} \frac{2x}{\sin 3x}$

(vii) $\lim_{x\to 0} \frac{\log(1+\sin x)}{x}$ (viii) $\lim_{x\to 0} (1+ax)^{\frac{1}{x}}$.

§ 3·8. Meaning of the symbol ∞.

Let $y=\frac{1}{x}$ and $x\to 0+$; Now let us tabulate the value of $\frac{1}{x}$ for values of x close to 0.

x	1	·1	·01	·00001	10^{-9}	etc.
y	1	10	100	100000	10^9	

Evidently one may notice that as the value of x approaches from the right the value 0, then the value of $\frac{1}{x}$ gradually increases. Actually taking the value of x sufficiently close to 0, one can make the value of $y=\frac{1}{x}$ greater than any positive number however large. In such cases we say that y is tending to infinity and it is denoted by the symbol $y\to\infty$.

Hence $x\to\infty$ means that the value of x so changes that it can be made greater than any preassigned positive number however large.

Now if $f(x)$ be a function of x, and if $f(x)\to l$, as $x\to\infty$, then we say that the limiting, value of $f(x)$ is l as x tends to infinity. In symbols it is written as $\lim_{x\to\infty} f(x)=l$.

Example. Show that $\lim\limits_{x \to \infty} \frac{1}{x} = 0$

For large values of x values of $\frac{1}{x}$ are shown in the table below :

x	10	100	500	10^5	10^9	
$\frac{1}{x}$	·1	·01	·002	·00001	10^{-9}	etc.

Evidently as the values of x increase, $\frac{1}{x}$ tends to 0.

Hence $\lim\limits_{x \to \infty} \frac{1}{x} = 0$. Here note that for any value of x, the value of $\frac{1}{x}$ will not be zero.

From the above example we find that as $x \to \infty$, then $y \to 0+$. Hence for any function $f(x)$,

$$\lim_{x \to \infty} f(x) = \lim_{y \to 0+} f\left(\frac{1}{y}\right) \quad \left[\text{Putting } x = \frac{1}{y}\right]$$

Now the right-hand limit can be determined by methods previously discussed.

Note : (1) ∞ is not a number ; it is only a symbol which indicates a particular mode of change in values of variables. (2) $x \to -\infty$ means that x assumes values which can be less than any negative number (whose absolute value may be however large) however small. It can be shown that

$$\lim_{x \to 0-} \frac{1}{x} = -\infty$$

(3) If $\lim\limits_{x \to a} f(x) = \infty$, we say that as $x \to a$, then the limiting value of $f(x)$ does not exist.

Ex. 1. Show that $\lim\limits_{x \to \infty} \frac{3}{x^2} = 0$.

Let $y = \frac{1}{x}$; as $x \to \infty$, then $y \to 0+$

$\therefore$ Given limit $= \lim\limits_{x \to \infty} \frac{3}{x^2} = \lim\limits_{y \to 0+} 3y^2 = 3.0^2 = 0$.

Ex. 2. $\lim\limits_{x\to\infty}\left(2+\dfrac{1}{x^3}\right)=2$

$$\lim_{x\to\infty}\left(2+\frac{1}{x^3}\right)=\lim_{y\to0+}(2+y^3)=\lim_{y\to0+}2+\lim_{y\to0+}y^3$$
$$=2+0=2.$$

Examples

Ex. 1. Evaluate the limit.

$$\lim_{h\to-2}\left\{3h^4-e^h+\sin(h+2)+2+\frac{4}{h^2}\right\}$$

$$\lim_{h\to-2}\left\{3h^4-e^h+\sin(h+2)+2+\frac{4}{h^2}\right\}$$
$$=\lim_{h\to-2}3h^4-\lim_{h\to-2}e^h+\lim_{h\to-2}\sin(h+2)+\lim_{h\to-2}2+\lim_{h\to-2}\frac{4}{h^2}$$
$$=3.(-2)^4-e^{-2}+\sin(-2+2)+2+\frac{4}{(-2)^2}$$
$$=3.16-\frac{1}{e^2}+\sin 0+2+\tfrac{4}{4}=51-\frac{1}{e^2}.$$

Ex. 2. Evaluate the limits.

(i) $\underset{x\to1}{\text{Lt}}\ \dfrac{x^2-3x+2}{x^2-4x+3}$. (ii) $\underset{x\to3}{\text{Lt}}\ \dfrac{x^2-7x+12}{x^2-8x+15}$

(i) $$\underset{x\to1}{\text{Lt}}\ \frac{x^2-3x+2}{x^2-4x+3}=\underset{x\to1}{\text{Lt}}\ \frac{(x-1)(x-2)}{(x-1)(x-3)}$$
$$=\underset{x\to1}{\text{Lt}}\ \frac{x-2}{x-3}=\frac{\underset{x\to1}{\text{Lt}}\ (x-2)}{\underset{x\to1}{\text{Lt}}\ (x-3)}=\frac{-1}{-2}=\frac{1}{2}$$

(ii) $$\underset{x\to3}{\text{Lt}}\ \frac{x^2-7x+12}{x^2-8x+15}=\underset{x\to3}{\text{Lt}}\ \frac{(x-3)(x-4)}{(x-3)(x-5)}$$
$$=\underset{x\to3}{\text{Lt}}\ \frac{x-4}{x-5}=\frac{\underset{x\to3}{\text{Lt}}\ (x-4)}{\underset{x\to3}{\text{Lt}}\ (x-5)}=\frac{-1}{-2}=\frac{1}{2}.$$

Ex. 3. Evaluate the limits.

$$\lim_{h\to1}\frac{h^4-1}{h-1}=\lim_{h\to1}\frac{(h^2+1)(h+1)(h-1)}{h-1}$$

$$= \lim_{h\to 1} (h^2+1)(h+1)$$

$$= \lim_{h\to 1} (h^2+1) \lim_{h\to 1} (h+1) = 2.2 = 4.$$

Ex. 4. $\lim\limits_{x\to 0} \dfrac{x}{\sqrt{1+x}-1}$

[Here theorem 4 cannot be used as the limiting value of the denominator is zero.]

$$= \lim_{x\to 0} \frac{x(\sqrt{1+x}+1)}{(\sqrt{1+x}-1)(\sqrt{1+x}+1)} = \lim_{x\to 0} \frac{x(\sqrt{1+x}+1)}{1+x-1}$$

$$= \lim_{x\to 0} (\sqrt{1+x}+1) = \lim_{x\to 0} (1+x)^{\frac{1}{2}} + \lim_{x\to 0} 1$$

$$= \sqrt{\lim_{x\to 0} (1+x)} + 1 = \sqrt{1+0}+1 = 1+1 = 2.$$

Ex. 5. Evaluate the following limits :

(i) $\mathop{\mathrm{Lt}}\limits_{x\to 0} \dfrac{\sqrt{(1+x)}-\sqrt{(1-x)}}{x}$

(ii) $\mathop{\mathrm{Lt}}\limits_{x\to 0} \dfrac{\sqrt{1+2x}-\sqrt{1-3x}}{x}$ (iii) $\mathop{\mathrm{Lt}}\limits_{x\to 0} \dfrac{a-\sqrt{a^2-x^2}}{x^2}$

(i) $\mathop{\mathrm{Lt}}\limits_{x\to 0} \dfrac{\sqrt{(1+x)}-\sqrt{(1-x)}}{x}$

$$= \mathop{\mathrm{Lt}}_{x\to 0} \frac{\{\sqrt{(1+x)}+\sqrt{(1-x)}\}\{\sqrt{(1+x)}-\sqrt{(1-x)}\}}{\{\sqrt{(1+x)}+\sqrt{(1-x)}\}x}$$

$$= \mathop{\mathrm{Lt}}_{x\to 0} \frac{(1+x)-(1-x)}{\{\sqrt{(1+x)}+\sqrt{(1-x)}\}.x}$$

$$= \mathop{\mathrm{Lt}}_{x\to 0} \frac{2x}{\{\sqrt{(1+x)}+\sqrt{(1-x)}\}x} = \mathop{\mathrm{Lt}}_{x\to 0} \frac{2}{\sqrt{(1+x)}+\sqrt{(1-x)}}$$

$$= \mathop{\mathrm{Lt}}_{x\to 0} \frac{2}{\{\sqrt{(1+x)}+\sqrt{(1-x)}\}} = \frac{2}{2} = 1.$$

(ii) $\mathop{\mathrm{Lt}}\limits_{x\to 0} \dfrac{\sqrt{1+2x}-\sqrt{1-3x}}{x}$

$$= \mathop{\mathrm{Lt}}_{x\to 0} \frac{\{\sqrt{1+2x}+\sqrt{1-3x}\}\{\sqrt{1+2x}-\sqrt{1-3x}\}}{\{\sqrt{1+2x}+\sqrt{1-3x}\}x}$$

$$= \mathop{\mathrm{Lt}}_{x\to 0} \frac{(1+2x)-(1-3x)}{\{\sqrt{1+2x}+\sqrt{1-3x}\}x} = \mathop{\mathrm{Lt}}_{x\to 0} \frac{5x}{\{\sqrt{1+2x}+\sqrt{1-3x}\}x}$$

$$=\underset{x\to 0}{\text{Lt}}\ \frac{5}{\sqrt{1+2x}+\sqrt{1-3x}}=\frac{5}{\underset{x\to 0}{\text{Lt}}\ \{\sqrt{1+2x}+\sqrt{1-3x}\}}=\frac{5}{2}.$$

(iii) $$\underset{x\to 0}{\text{Lt}}\ \frac{a-\sqrt{a^2-x^2}}{x^2}=\underset{x\to 0}{\text{Lt}}\ \frac{(a+\sqrt{a^2-x^2})(a-\sqrt{a^2-x^2})}{(a+\sqrt{a^2-x^2})\,x^2}$$

$$=\underset{x\to 0}{\text{Lt}}\ \frac{a^2-(a^2-x^2)}{(a+\sqrt{a^2-x^2})x^2}=\underset{x\to 0}{\text{Lt}}\ \frac{x^2}{(a+\sqrt{a^2-x^2})x^2}$$

$$=\underset{x\to 0}{\text{Lt}}\ \frac{1}{a+\sqrt{a^2-x^2}}=\frac{1}{\underset{x\to 0}{\text{Lt}}\ (a+\sqrt{a^2-x^2})}=\frac{1}{2a}.$$

Ex. 6. Evaluate the limits :

$$\lim_{x\to 0}\frac{\sqrt{1+x^3}-\sqrt{1+x}}{\sqrt{1+x^4}-\sqrt{1+x}}$$

Given limit $$=\lim_{x\to 0}\left\{\frac{(\sqrt{1+x^3}-\sqrt{1+x})(\sqrt{1+x^3}+\sqrt{1+x})}{\sqrt{1+x^3}+\sqrt{1+x}}\times\frac{\sqrt{1+x^4}+\sqrt{1+x}}{(\sqrt{1+x^4}-\sqrt{1+x})(\sqrt{1+x^4}+\sqrt{1+x})}\right\}$$

$$=\lim_{x\to 0}\left\{\frac{1+x^3-1-x}{\sqrt{1+x^3}+\sqrt{1+x}}\cdot\frac{\sqrt{1+x^4}+\sqrt{1+x}}{1+x^4-1-x}\right\}$$

$$=\lim_{x\to 0}\left\{\frac{x(x^2-1)}{\sqrt{1+x^3}+\sqrt{1+x}}\cdot\frac{\sqrt{1+x^4}+\sqrt{1+x}}{x(x^3-1)}\right\}$$

$$=\lim_{x\to 0}\frac{(x+1)(\sqrt{1+x^4}+\sqrt{+x})}{(\sqrt{1+x^3}+\sqrt{1+x})(x^2+x+1)}$$

$$=\frac{\lim\limits_{x\to 0}(x+1)(\sqrt{1+x^4}+\sqrt{1+x})}{\lim\limits_{x\to 0}(\sqrt{1+x^3}+\sqrt{1+x})(x^2+x+1)}$$

$$=\frac{(0+1)(\sqrt{1+0}+\sqrt{1+0})}{(\sqrt{1+0}+\sqrt{1+0})(0+0+1)}=\frac{1\cdot(\sqrt{1}+\sqrt{1})}{(\sqrt{1}+\sqrt{1}).1}=1$$

Ex. 7. Evaluate the limits :

(i) $\lim\limits_{x\to 2}\dfrac{x^8-2^8}{x-2}$ (ii) $\lim\limits_{x\to 3}\dfrac{x^3-27}{x^2-9}$ (iii) $\lim\limits_{x\to 0}\dfrac{x^{\frac{5}{2}}-a^{\frac{5}{2}}}{x^{\frac{3}{8}}-a^{\frac{3}{8}}}$

(i) $$\lim_{x\to 2}\frac{x^8-2^8}{x-2}=8.2^{8-1}=8.2^7=8.128=1024.$$

(ii) $$\lim_{x\to3}\frac{x^3-27}{x^2-9}=\lim_{x\to3}\frac{\dfrac{x^3-27}{x-3}}{\dfrac{x^2-9}{x-3}}=\frac{\lim\limits_{x\to3}\dfrac{x^3-3^3}{x-3}}{\lim\limits_{x\to3}\dfrac{x^2-3^2}{x-3}}$$

$$=\frac{3.3^{3-1}}{2.3^{2-1}}=\frac{3.3^2}{2.3}=\frac{27}{6}=\frac{9}{2}.$$

(iii) $$\lim_{x\to a}\frac{x^{\frac{5}{2}}-a^{\frac{5}{2}}}{x^{\frac{3}{8}}-a^{\frac{3}{8}}}=\lim_{x\to a}\frac{\dfrac{x^{\frac{5}{2}}-a^{\frac{5}{2}}}{x-a}}{\dfrac{x^{\frac{3}{8}}-a^{\frac{3}{8}}}{x-a}}\quad[\because\ x-a\neq0]$$

$$=\frac{\lim\limits_{x\to a}\dfrac{x^{\frac{5}{2}}-a^{\frac{5}{2}}}{x-a}}{\lim\limits_{x\to a}\dfrac{x^{\frac{3}{8}}-a^{\frac{3}{8}}}{x-a}}=\frac{\frac{5}{2}a^{\frac{5}{2}-1}}{\frac{3}{8}a^{\frac{3}{8}-1}}=\tfrac{5}{2}\times\tfrac{8}{3}a^{\frac{5}{2}-1-\frac{3}{8}+1}=\tfrac{20}{3}a^{\frac{17}{8}}.$$

Ex. 8. Evaluate the limit :

$$\lim_{x\to0}\frac{a_0x^n+a_1x^{n-1}+a_2x^{n-2}+\cdots\cdots a_{n-1}x+a_n}{b_0x^m+b_1x^{m-1}+b_2x^{m-2}+\cdots\cdots b_{m-1}x+b_m}\ [b_m\neq0]$$

Given limit $$=\frac{\lim\limits_{x\to0}(a_0x^n+a_1x^{n-1}+\cdots\cdots+a_{n-1}x+a_n)}{\lim\limits_{x\to0}(b_0x^m+b_1x^{m-1}+\cdots+(b_{m-1}x+b_m)}$$

$$=\frac{a_0.0+a_1.0+\cdots\cdots+a_{n-1}.0+a_n}{b_0.0+b_1.0+\cdots\cdots+b_{m-1}.0+b_m}=\frac{a_n}{b_m}.\quad[\because\ b_m\neq0\ ]$$

Note : If $b_n=0$, then the limit does not exist.

Ex. 9. Show that $\lim\limits_{h\to0}\dfrac{(1+h)^n-1}{h}=n.$

$\lim\limits_{h\to0}\dfrac{(1+h)^n-1}{h}=\lim\limits_{x\to1}\dfrac{x^n-1}{x-1}$ [Let $1+h=x$; when $h\to0$, then $x\to1$]

$=n.1^{n-1}=n.$

Ex. 10. Evaluate :

(i) $\lim\limits_{x\to0}\dfrac{\tan x}{x}$ (ii) $\lim\limits_{x\to0}\dfrac{\sin 3x}{x}$ (iii) $\lim\limits_{x\to0}\dfrac{3x}{\sin x}.$

(iv) $\lim\limits_{x\to0}\dfrac{\sin\alpha x}{\sin\beta x}$ (v) $\lim\limits_{x\to0}\left(\dfrac{\sin 2x}{x}\right)^2$

(vi) $\lim\limits_{h\to0}\dfrac{1-\cos 2h}{h^2}$ (vii) $\lim\limits_{x\to0}\dfrac{x\sin x}{1-\cos 2x}.$

(i) $\lim_{x\to 0} \dfrac{\tan x}{x} = \lim_{x\to 0} \left\{\dfrac{\sin x}{x} \cdot \dfrac{1}{\cos x}\right\} = \lim_{x\to 0} \dfrac{\sin x}{x} \cdot \lim_{x\to 0} \dfrac{1}{\cos x}$

$= 1 \cdot \dfrac{1}{\lim_{x\to 0} \cos x} = 1 \cdot \frac{1}{1} = 1.$

(ii) $\lim_{x\to 0} \dfrac{\sin 3x}{x} = \lim_{x\to 0} 3 \cdot \dfrac{\sin 3x}{3x} = 3 \cdot \lim_{x\to 0} \dfrac{\sin 3x}{3x} = 3 \cdot \lim_{h\to 0} \dfrac{\sin h}{h}$,

[$3x = h$; when $x\to 0$, then $h\to 0$]

$= 3.1 = 3.$

(iii) $\lim_{x\to 0} \dfrac{3x}{\sin x} = \lim_{x\to 0} \dfrac{3}{\dfrac{\sin x}{x}} = \dfrac{3}{\lim_{x\to 0} \dfrac{\sin x}{x}} = \dfrac{3}{1} = 3.$

(iv) $\lim_{x\to 0} \dfrac{\sin \alpha x}{\sin \beta x} = \lim_{x\to 0} \left\{\dfrac{\dfrac{\sin \alpha x}{\alpha x}}{\dfrac{\sin \beta x}{\beta x}} \cdot \dfrac{\alpha x}{\beta x}\right\}$

$= \lim_{x\to 0} \left\{\dfrac{\dfrac{\sin \alpha x}{\alpha x}}{\dfrac{\sin \beta x}{\beta x}} \cdot \dfrac{\alpha}{\beta}\right\} = \dfrac{\alpha}{\beta} \cdot \dfrac{\lim_{x\to 0} \dfrac{\sin \alpha x}{\alpha x}}{\lim_{x\to 0} \dfrac{\sin \beta x}{\beta x}} = \dfrac{\alpha}{\beta} \cdot \dfrac{1}{1} = \dfrac{\alpha}{\beta}.$

(v) $\lim_{x\to 0} \left(\dfrac{\sin 2x}{x}\right)^2 = \lim_{h\to 0} \left(\dfrac{\sin h}{\dfrac{h}{2}}\right)^2$ [$h = 2x$ (say) when $x\to 0$, then $h\to 0$]

$= 4 \lim_{h\to 0} \left(\dfrac{\sin h}{h}\right)^2 = 4\left\{\lim_{h\to 0} \left(\dfrac{\sin h}{h}\right)\right\}^2 = 4 \cdot 1^2 = 4.$

(vi) $\lim_{h\to 0} \dfrac{1 - \cos 2h}{h^2} = \lim_{h\to 0} \dfrac{2 \sin^2 h}{h^2}$

$= 2 \lim_{h\to 0} \left\{\left(\dfrac{\sin h}{h}\right) \cdot \left(\dfrac{\sin h}{h}\right)\right\}$

$= 2 \cdot \lim_{h\to 0} \dfrac{\sin h}{h} \cdot \lim_{h\to 0} \dfrac{\sin h}{h} = 2.1.1 = 2.$

(vii) $\lim_{x\to 0} \dfrac{x \sin x}{1 - \cos 2x} = \lim_{x\to 0} \dfrac{x \sin x}{2 \sin^2 x} = \lim_{x\to 0} \dfrac{x}{2 \sin x}$

$= \dfrac{1}{2} \cdot \dfrac{1}{\lim_{x\to 0} \dfrac{\sin x}{x}} = \dfrac{1}{2} \cdot \dfrac{1}{1} = \dfrac{1}{2}.$

Ex. 11. Show that,

(i) $\lim_{x\to 0} \dfrac{\sin x^0}{x} = \dfrac{\pi}{180}$ (ii) $\lim_{x\to 0} \dfrac{\tan ax}{\sin bx} = \dfrac{a}{b}$

(iii) $\lim\limits_{x\to a}\dfrac{\sin(x-a)}{x-a}=1$ (iv) $\lim\limits_{\theta\to 0}\tan 4\theta \operatorname{cosec} 2\theta=2.$

(v) $\lim\limits_{x\to\frac{\pi}{4}}\dfrac{\sin x-\cos x}{\frac{\pi}{4}-x}=-\sqrt{2}.$

(vi) $\lim\limits_{x\to 0}\dfrac{x(\cos 2x-\cos 3x)}{\sin^3 x}=\dfrac{5}{2}.$

(i) $\lim\limits_{x\to 0}\dfrac{\sin x^0}{x}=\lim\limits_{x\to 0}\dfrac{\sin\frac{\pi x}{180}}{x}$ $\left[\begin{array}{l}\because\ \pi^c=180^\circ,\\ \therefore\ x^0=\dfrac{\pi x}{180}\end{array}\right]$

$=\lim\limits_{x\to 0}\dfrac{\sin y}{\frac{180}{\pi}y}$ $\left[\dfrac{\pi x}{180}=y\ (\text{say})\right]$ when $x\to a$ then $y\to 0$

$=\dfrac{\pi}{180}\lim\limits_{x\to 0}\dfrac{\sin y}{y}=\dfrac{\pi}{180}.\ 1=\dfrac{\pi}{180}.$

(ii) $\lim\limits_{x\to 0}\dfrac{\tan ax}{\sin bx}=\lim\limits_{x\to 0}\left(\dfrac{\sin ax}{ax}\cdot\dfrac{bx}{\sin bx}\cdot\dfrac{1}{\cos ax}\cdot\dfrac{a}{b}\right)$

$=\dfrac{a}{b}\lim\limits_{x\to 0}\dfrac{\sin ax}{ax}\cdot\dfrac{1}{\lim\limits_{x\to 0}\dfrac{\sin bx}{bx}}\cdot\dfrac{1}{\lim\limits_{x\to 0}\cos ax}=\dfrac{a}{b}.1.\dfrac{1}{1}.\dfrac{1}{1}=\dfrac{a}{b}.$

(iii) Let $x-a=y$; $\therefore$ when $x\to a$, then $y\to 0$.

Now, $\lim\limits_{x\to 0}\dfrac{\sin(x-a)}{x-a}=\lim\limits_{y\to 0}\dfrac{\sin y}{y}=1.$

(iv) $\lim\limits_{\theta\to 0}\tan 4\theta \operatorname{cosec}\theta=\lim\limits_{\theta\to 0}\dfrac{\sin 4\theta}{\cos 4\theta\sin 2\theta}$

$=\lim\limits_{\theta\to 0}\dfrac{2\sin 2\theta\cos 2\theta}{\cos 4\theta\sin 2\theta}=2\lim\limits_{\theta\to 0}\dfrac{\cos 2\theta}{\cos 4\theta}$

$=2.\dfrac{\lim\limits_{\theta\to 0}\cos 2\theta}{\lim\limits_{\theta\to 0}\cos 4\theta}=2.\dfrac{1}{1}=2.$

(v) $\lim\limits_{x\to\frac{\pi}{4}}\dfrac{\sin x-\cos x}{\frac{\pi}{4}-x}=\lim\limits_{x\to\frac{\pi}{4}}\dfrac{\sqrt{2}\left(\frac{1}{\sqrt{2}}\sin x-\frac{1}{\sqrt{2}}\cos x\right)}{\frac{\pi}{4}-x}$

$=\lim\limits_{x\to\frac{\pi}{4}}\dfrac{\sqrt{2}\sin\left(x-\frac{\pi}{4}\right)}{\frac{\pi}{4}-x}$ $\left[\because\ \cos\dfrac{\pi}{4}=\sin\dfrac{\pi}{4}=\dfrac{1}{\sqrt{2}}\right]$

$$=-\sqrt{2}\lim_{y\to 0}\frac{\sin y}{y} \quad \left[\text{Let, } y=x-\frac{\pi}{4} \text{ when } x\to\frac{\pi}{4} \text{ then } y\to 0\right]$$

$$=-\sqrt{2}.\,1=-\sqrt{2}.$$

(vi) Given limit$=\lim_{x\to 0}\dfrac{x.2\sin\frac{5x}{2}.\sin\frac{x}{2}}{\sin^3 x}$

$$=\lim_{x\to 0}\left\{\frac{x}{\sin x}.\frac{\sin\frac{5x}{2}}{\frac{5x}{2}}.\frac{\frac{5x}{2}}{\sin x}.\frac{\sin\frac{x}{2}}{\frac{x}{2}}.\frac{\frac{x}{2}}{\sin x}.2\right\}$$

$$=\lim_{x\to 0}\frac{5}{2}\left(\frac{x}{\sin x}\right)^3.\lim_{x\to 0}\frac{\sin\frac{5x}{2}}{\frac{5x}{2}}.\lim_{x\to 0}\frac{\sin\frac{x}{2}}{\frac{x}{2}}$$

$$=\tfrac{5}{2}.1^3.1.1.=\tfrac{5}{2}.$$

Ex. 12. Evaluate : $\lim_{h\to 0}\dfrac{e^{h^2}-1}{h}$.

Given limit$=\lim_{h^2\to 0}\left[\dfrac{e^{h^2}-1}{h^2}.h\right]$

$$=\lim_{y\to 0}\frac{e^y}{y}\ \lim_{h\to 0} h\ [h^2=y \text{ (say)}]=1.0=0.$$

Ex. 13. Show that $\lim_{x\to 0}\dfrac{e^{\alpha x}-e^{\beta x}}{x}=\alpha-\beta.$

Given limit$=\lim_{x\to 0}\dfrac{\left(e^{\alpha x}-1\right)-\left(e^{\beta x}-1\right)}{x}$

$$=\lim_{x\to 0}\left\{\frac{e^{\alpha x}-1}{x}-\frac{e^{\beta x}-1}{x}\right\}=\lim_{x\to 0}\frac{e^{\alpha x}-1}{x}-\lim_{x\to 0}\frac{e^{\beta x}-1}{x}$$

$$=\lim_{x\to 0}\alpha.\frac{e^{\alpha x}-1}{\alpha x}-\lim_{x\to 0}\beta.\frac{e^{\beta x}-1}{\beta x}$$

$$=\alpha.\lim_{\alpha x\to 0}\frac{e^{\alpha x}-1}{\alpha x}-\beta.\lim_{\beta x\to 0}\frac{e^{\beta x}-1}{1}$$

$$=\alpha.1-\beta.\,1=\alpha-\beta$$

Ex. 14. Show that $\lim_{x\to 0}\dfrac{e^{\sin x}-1}{x}=1$

Given limit $=\lim_{x\to 0}\left\{\frac{e^{\sin x}-1}{\sin x}\cdot\frac{\sin x}{x}\right\}$

$=\lim_{x\to 0}\frac{e^{\sin x}-1}{\sin x}\cdot\lim_{x\to 0}\frac{\sin x}{x}$

$=\lim_{y\to 0}\frac{e^{y}-1}{y}\cdot 1$ [Let $y=\sin x$; $\therefore$ When $x\to 0$, then $y=\sin x\to 0$]

$=1.1=1.$

Ex. 15. Evaluate the limits

(i) $\lim_{x\to 0}\frac{\log(1+3x)}{x}$ (ii) $\lim_{x\to 0}\frac{\log(1+ax)}{\sin bx}$

(i) Given limit $=\lim_{3x\to 0}\left\{\frac{\log(1+3x)}{3x}.3\right\}$

$=3.\lim_{3x\to 0}\frac{\log(1+3x)}{3x}=3.1=3.$

(ii) Given limit $=\lim_{x\to 0}\left\{\frac{\log(1+ax)}{ax}\cdot\frac{bx}{\sin bx}\cdot\frac{a}{b}\right\}$

$=\frac{a}{b}.\lim_{ax\to 0}\frac{\log(1+ax)}{ax}\cdot\frac{1}{\lim_{bx\to 0}\frac{\sin x}{bx}}$

$=\frac{a}{b}.1.\frac{1}{1}=\frac{a}{b}.$

Ex. 16. Show that (i) $\lim_{x\to 0}\frac{a^{x}-1}{x}=\log a$

(ii) $\lim_{x\to 0}\frac{a^{x}-b^{x}}{x}=\log\frac{a}{b}.$

(i) $\lim_{x\to 0}\frac{a^{x}-1}{x}=\lim_{x\to 0}\left\{\frac{e^{x\log a}-1}{x\log a}.\log a\right\}$

$=\log a.\lim_{y\to 0}\frac{e-1}{y}$ [$x\log a=y$ (say) ; So, when $x\to 0$, then $y\to 0$]

$=\log a.1=\log a.$

(ii) $\lim_{x\to 0}\frac{a^{x}-b^{x}}{x}=\lim_{x\to 0}\frac{(a^{x}-1)-(b^{x}-1)}{x}$

$=\lim_{x\to 0}\frac{a^{x}-1}{x}-\lim_{x\to 0}\frac{b^{x}-1}{x}$

$=\log a-\log b$ [by (i) above]

$=\log\frac{a}{b}.$

Ex. 17. Show that $\lim_{x\to 0}(1+x)^{\frac{1}{x}}=e.$

$$\lim_{x\to 0}(1+x)^{\frac{1}{x}}=\lim_{x\to 0} e^{\log(1+x)^{\frac{1}{x}}}$$

$$=e^{\lim_{x\to 0}\frac{\log(1+x)}{x}}\quad [\text{By § 3·5 Formula } (b) \text{ as } e^x \text{ is continuous}]$$

$=e^1=e.$

Ex. 18. Find the value of $\lim_{x\to 0}\frac{\sin\log(1+x)}{\log(1+\sin x)}$

$$\text{Given limit}=\lim_{x\to 0}\left\{\frac{\sin\log(1+x)}{\log(1+x)}\cdot\frac{\log(1+x)}{x}\cdot\frac{x}{\sin x}\cdot\frac{\sin x}{\log(1+\sin x)}\right\}$$

$$=\lim_{x\to 0}\frac{\sin\log(1+x)}{\log(1+x)}\cdot\lim_{x\to 0}\frac{\log(1+x)}{x}\cdot\lim_{x\to 0}\frac{x}{\sin x}\cdot\lim_{x\to 0}\frac{\sin x}{\log(1+\sin x)}.$$

$$=\lim_{y\to 0}\frac{\sin y}{y}.1.\frac{1}{\lim_{x\to 0}\frac{\sin x}{x}}\cdot\frac{1}{\lim_{z\to 0}\frac{\log(1+z)}{z}}$$

[Let $y=\log(1+x)$; $z=\sin x$ then $x\to 0$, then $y\to\log 1=0$ and $z\to\sin 0=0$]

$$=1.1.\frac{1}{1}.\frac{1}{1}=1.$$

Ex. 19. Show that

(i) $\lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h}=\cos x$ [H. S. 1979]

(ii) $\lim_{h\to 0}\frac{f(1+h)-f(1)}{h}=-1$ when $f(x)=\frac{1}{x}.$

(iii) If $f(x)=x^2+\frac{1}{x-1}+3$ then, $\lim_{h\to 0}\frac{f(h)-f(0)}{h}=-1.$

(i) $$\lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h}=\lim_{h\to 0}\frac{2\cos\left(x+\frac{h}{2}\right)\sin\frac{h}{2}}{h}$$

$$=\lim_{h\to 0}\left\{\frac{\sin\frac{h}{2}}{h}\cos\left(x+\frac{h}{2}\right)\right\}=\lim_{h\to 0}\frac{\sin\frac{h}{2}}{\frac{h}{2}}\cdot\lim_{h\to 0}\cos\left(x+\frac{h}{2}\right)$$

$$=\lim_{k\to 0}\frac{\sin k}{k}.\cos x\quad\left[k=\frac{h}{2}\text{ (say) when } h\to 0,\text{ then } k=\frac{h}{2}\to 0\right]$$

$=1.\cos x=\cos x.$

(ii) $f(x)=\frac{1}{x}$; $\therefore$ $f(1+h)=\frac{1}{1+h}$ and $f(1)=\frac{1}{1}=1.$

$$\therefore \quad \lim_{h\to 0}\frac{f(1+h)-f(1)}{h}=\lim_{h\to 0}\frac{\frac{1}{1+h}-1}{h}=\lim_{h\to 0}\frac{-h}{h(1+h)}$$

$$=-\lim_{h\to 0}\frac{1}{1+h}=-\frac{1}{\lim\limits_{h\to 0}(1+h)}=-\frac{1}{1+0}=-1.$$

(iii) $f(x)=x^2+\frac{1}{x-1}+3.$

$$\therefore \quad f(h)=h^2+\frac{1}{h-1}+3 \text{ and } f(0)=\frac{1}{-1}+3=2.$$

$$\therefore \quad \text{Given limit}=\lim_{h\to 0}\frac{h^2+\frac{1}{h-1}+3-2}{h}=\lim_{h\to 0}\frac{h^2+\frac{1}{h-1}+1}{h}$$

$$=\lim_{h\to 0}\left\{h+\frac{1+(h-1)}{h(h-1)}\right\}=\lim_{h\to 0}h+\lim_{h\to 0}\frac{1}{h-1}$$

$$=0-1=-1.$$

Ex. 20. Show that $\underset{x\to 0}{\text{Lt}}\left(x\cos\frac{1}{x}\right)=0.$

$$\left|x\cos\frac{1}{x}\right|=|x|\ \left|\cos\frac{1}{x}\right|. \quad \text{Now, } \left|\cos\frac{1}{x}\right|\leqslant 1$$

$$\therefore \quad \left|x\cos\frac{1}{x}\right|\leqslant|x|$$

Now, when $x\to 0$, then taking the values of x sufficiently close to 0 we can make $|x|$ less than any positive number, however small. So $\left|x\cos\frac{1}{x}\right|$ can be made smaller than any positive number, however small.

$$\therefore \quad \underset{x\to 0}{\text{Lt}}\left(x\cos\frac{1}{x}\right)=0.$$

Ex. 21. Evaluate : $\lim\limits_{x\to 0}\dfrac{x^2\sin\left(\frac{1}{x}\right)}{\sin x}$

$$\text{Given limit}=\lim_{x\to 0}\frac{x\sin\left(\frac{1}{x}\right)}{\frac{\sin x}{x}}=\frac{\lim\limits_{x\to 0}x\sin\left(\frac{1}{x}\right)}{\lim\limits_{x\to 0}\frac{\sin x}{x}}=\frac{0}{1}=0.$$

Ex. 22. Evaluate : $\lim\limits_{x\to\infty} \dfrac{\sin x}{x}$.

Let $x=\dfrac{1}{y}$ $\therefore$ $y=\dfrac{1}{x}$ when $x\to\infty$, then $y\to 0+$

$$\therefore \quad \lim_{x\to\infty} \frac{\sin x}{x} = \lim_{y\to 0+} y\left(\sin \frac{1}{y}\right)=0.$$

$$\left[\because \quad \lim_{y\to 0} y \sin\left(\frac{1}{y}\right)=0, \quad \therefore \quad \lim_{y\to 0+} y \sin\left(\frac{1}{y}\right)=0\right]$$

Ex. 23. Evaluate

(i) $\lim\limits_{x\to\infty} \dfrac{3x^2-4x+5}{x^2-3x+1}$ (ii) $\lim\limits_{x\to\infty} e^{\frac{1}{x}}$

(iii) $\lim\limits_{x\to\infty} \{\sqrt{x^4-x+2}-x^2\}$

(i) $$\lim_{x\to\infty} \frac{3x^2-4x+5}{x^2-3x+1} = \lim_{y\to 0+} \frac{\dfrac{3}{y^2}-\dfrac{4}{y}+5}{\dfrac{1}{y^2}-\dfrac{1}{y}+1}$$

[Let $x=\dfrac{1}{y}$; when $x\to\infty$, then $y\to 0+$]

$$=\lim_{y\to 0+} \frac{3-4y+5y^2}{1-3y+y^2}=\frac{3-4.0+5.0^2}{1-0+0}=\frac{3}{1}=3.$$

(ii) $$\lim_{x\to 0} e^{\frac{1}{x}} = \lim_{y\to 0+} e^{y} = e^{0} = 1.$$

(iii) $\lim\limits_{x\to\infty} \{\sqrt{x^4-x^2+2}-x^2\}$

$$= \lim_{x\to\infty} \frac{\{\sqrt{x^4-x^2+2}-x^2\}\{\sqrt{x^4-x^2+2}+x^2\}}{\sqrt{(x^4-x^2+2}+x^2)}$$

$$=\lim_{x\to\infty} \frac{x^4-x^2+2-x^4}{x^2\left\{\sqrt{1-\dfrac{1}{x^2}+\dfrac{2}{x^4}}+1\right\}}$$

$$= \lim_{x\to\infty} \frac{-1+\dfrac{2}{x^2}}{\sqrt{1-\dfrac{1}{x^2}+\dfrac{2}{x^4}}+1}$$

$$= \lim_{y\to 0+} \frac{-1+2y^2}{\sqrt{1-y^2+2y^4}+1} \quad \left[y=\frac{1}{x} \text{ (say)}\right]$$

$$= \frac{-1}{\sqrt{1}+1} = -\frac{1}{2}.$$

Ex. 24. Show that

(i) $\lim\limits_{x\to\infty} \frac{1^2+2^2+3^2+\cdots+x^2}{x^3} = \frac{1}{3}$

(ii) $\lim\limits_{x\to\infty} \left[\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\cdots+\frac{1}{x(x+1)}\right] = 1.$

(i) Given limit $= \lim\limits_{x\to\infty} \frac{x(x+1)(2x+1)}{6x^3}$

[where x is a positive integer]

$$= \lim_{y\to 0+} \frac{\frac{1}{y}\left(\frac{1}{y}+1\right)\left(\frac{2}{y}+1\right)}{\frac{6}{y^3}} \quad \left[y=\frac{1}{x}\text{(say)}\right]$$

$$= \lim_{y\to 0+} \frac{(1+y)(2+y)}{6} = \frac{1\cdot 2}{6} = \frac{1}{3}.$$

(ii) $\frac{1}{1.2} = 1-\frac{1}{2}$; $\frac{1}{2.3} = \frac{1}{2}-\frac{1}{3}$; $\frac{1}{3.4} = \frac{1}{3}-\frac{1}{4}$; $\cdots \frac{1}{x(x+1)} = \frac{1}{x}-\frac{1}{x+1}.$

$\therefore \quad \frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\cdots+\frac{1}{x(x+1)}$

$$= \left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\cdots+\left(\frac{1}{x}-\frac{1}{x+1}\right) = 1-\frac{1}{x+1}$$

$\therefore$ Given limit $= \lim\limits_{x\to\infty} \left(1-\frac{1}{x+1}\right) = \lim\limits_{y\to 0+} \left(1-\frac{1}{\frac{1}{y}+1}\right)$

$\left[x=\frac{1}{y} \text{ (say)} \right]$

$$= \lim_{y\to 0+} \left(1-\frac{y}{1+y}\right) = \lim_{y\to 0+} \frac{1}{1+y} = \frac{1}{1+0} = \frac{1}{1} = 1.$$

Ex. 25. Show that the following limits do not exist

(i) $\lim\limits_{x\to\pi} \frac{1}{\pi-x}$ (ii) $\lim\limits_{x\to 1} \{x^2+\sqrt{x-1}\}$

(iii) $\lim\limits_{x\to 3} [x]$, when $[x]$ is the greatest integer less than or equal to x.

(i) Let $\pi - x = \frac{1}{y}$ $\qquad \therefore$ when $x \to \pi - 0$, then

$y = \frac{1}{\pi - x} \to +\infty$ and when $x \to \pi + 0$,

then $y = \frac{1}{\pi - x} \to -\infty$.

$$\therefore \lim_{x \to \pi - 0} \left(\frac{1}{\pi - x}\right) = \lim_{y \to +\infty} y = +\infty.$$

$$\lim_{x \to \pi + 0} \left(\frac{1}{\pi - x}\right) = \lim_{y \to -\infty} y = -\infty.$$

$\therefore \lim\limits_{x \to \pi} \frac{1}{\pi - x}$ does not exist.

(ii) When $x \to 1-$, then $\sqrt{x-1}$ is always imaginary,

$\therefore \lim\limits_{x \to 1-} \sqrt{x-1}$ does not exist and hence the given limit does not exist.

(iii) $\lim\limits_{x \to 3-} [x] = \lim\limits_{x \to 3-} 2 = 2.$

$\lim\limits_{x \to 3+} [x] = \lim\limits_{x \to 3+} 3 = 3.$

$\therefore \lim\limits_{x \to 3-} [x] \neq \lim\limits_{x \to 3+} [x]$

$\therefore \lim\limits_{x \to 3} [x]$ does not exist.

Ex. 26. Show that $\lim\limits_{x \to \infty} \frac{1+2+3+\cdots+x}{x}$ does not exist where x is a positive integer.

$$\lim_{x \to \infty} \frac{1+2+3+\cdots+x}{x}$$

$$= \lim_{x \to \infty} \frac{x(x+1)}{2x} = \lim_{x \to \infty} \frac{(x+1)}{2} = \infty$$

$\therefore$ the limit does not exist.

Ex. 27. A function $f(x)$ is defined as follows :

$f(x) = x^2$ when $x < 1$

$= 3{\cdot}3$ when $x = 1$

$= x^2 + 3$ when $x > 1$.

Examine the existence of $\underset{x\to 1}{\text{Lt}}\ f(x)$.

$\underset{x\to 1-}{\text{Lt}}\ f(x)=\underset{x\to 1-}{\text{Lt}}\ x^2=1.$

$\underset{x\to 1+}{\text{Lt}}\ f(x)=\underset{x\to 1+}{\text{Lt}}\ (x^2+3)=4.$

$\therefore\quad \underset{x\to 1-}{\text{Lt}}\ f(x)\neq \underset{x\to 1+}{\text{Lt}}\ f(x).$

$\therefore\quad \underset{x\to 1}{\text{Lt}}\ f(x)$ does not exist.

Ex. 28. A function $f(x)=1$, 0 or -1 according as $x>0$, $=0$ or <0.

Show that $\underset{x\to 0}{\text{Lt}}\ f(x)$ does not exist.

$\underset{x\to 0-}{\text{Lt}}\ f(x)=\underset{x\to 0-}{\text{Lt}}\ (-1)=-1$

$\underset{x\to 0+}{\text{Lt}}\ f(x)=\underset{x\to 0+}{\text{Lt}}\ (1)=1$

$\therefore\quad \underset{x\to 0-}{\text{Lt}}\ f(x)\neq \underset{x\to 0+}{\text{Lt}}\ f(x)$

Hence $\underset{x\to 0}{\text{Lt}}\ f(x)$ does not exist.

Ex. 29. (i) $\underset{x\to a}{\text{Lt}}\ \dfrac{x^2-a^2}{x-a}=\underset{x\to a}{\text{Lt}}\ \dfrac{x^2}{x-a}-\underset{x\to a}{\text{Lt}}\ \dfrac{a^2}{x-a}$

and (ii) $\underset{x\to a}{\text{Lt}}\ \dfrac{x^2-a^2}{x-a}=\underset{x\to a}{\text{Lt}}\ (x^2-a^2)\cdot\underset{x\to a}{\text{Lt}}\ \dfrac{1}{x-a}.$

Are the above two relations true ?

The above two relations are not true. For in (i) none of the limits on the right does exist and in (ii) $\underset{x\to a}{\text{Lt}}\ \dfrac{1}{x-a}$ in the right does not exist.

Note : In both cases, the limit on the left side exist and the limit is 2a.

Ex. 30. If $\underset{x\to a}{\text{Lt}}\ f(x)=\underset{x\to a}{\text{Lt}}\ g(x)$. Then $f(x)=g(x)$ always. Is the statement true ?

The statement is not true.

For $\underset{x\to 2}{\text{Lt}}\ x^2=4$ and $\underset{x\to 2}{\text{Lt}}\ 2x=4$

But $f(x)=x^2\neq g(x)=2x$ always.

Ex. 31. If $f(x)=\lim_{n\to\infty} \frac{1}{1+x^{2n}}$, show that

$$f(x)=1, \text{ when } |x|<1,$$
$$=\tfrac{1}{2}, \text{ when } |x|=1,$$
$$=0, \text{ when } |x|>1.$$

When $|x|<1$, $x^2<1$. $\therefore$ x^2 is a positive proper fraction, higher the power of x^2, lesser will be its value which will tend towards zero.

$\therefore$ $(x^2)^n=x^{2n}\to 0$, when $n\to\infty$.

$$\therefore \quad f(x)=\lim_{n\to\infty} \frac{1}{1+x^{2n}}=\frac{1}{1+\lim\limits_{n\to\infty} x^{2n}}=\frac{1}{1+0}=1.$$

when $|x|=1$, $x^2=1$ $\therefore$ $x^{2n}=1$.

$$\therefore \quad f(x)=\lim_{n\to\infty} \frac{1}{1+1}=\frac{1}{2}.$$

When $|x|>1$, $x^2>1$. $\therefore$ higher the power of x^2, more its value will increase. $\therefore$ $x^{2n}=(x^2)^n\to\infty$ as $n\to\infty$.

$$\therefore \quad f(x)=\lim_{n\to\infty} \frac{\frac{1}{x^{2n}}}{1+\frac{1}{x^{2n}}}=\frac{0}{1+0}=0.$$

Exercise 3C

1. Evaluate the limits

(i) $\lim\limits_{x\to -1} 3x$ (ii) $\lim\limits_{y\to 2} (2y+5)$

(iii) $\lim\limits_{x\to \cdot 1} (100x^2+10x+10)$ (iv) $\lim\limits_{x\to 4} x^3$ (v) $\lim\limits_{x\to -2} x^{-3}$

(vi) $\lim\limits_{x\to 1} \frac{1}{x^2}$ (vii) $\lim\limits_{x\to 2} \frac{x^2-4}{x-2}$ (viii) $\lim\limits_{x\to 1} (x^2+1)^2$

(ix) $\lim\limits_{x\to 0} \cos(x^2-x)$ (x) $\lim\limits_{x\to 0} e^{x^2+4x+2}$

(xi) $\lim\limits_{x\to 3} \frac{x^3-3x+4}{x^2+4x-8}$ (xii) $\lim\limits_{x\to 1} \frac{x^2+2x+1}{x^{-1}+1}$

2. Evaluate the limits

(i) $\lim\limits_{x\to 1} (2x^2+x+1)$ (ii) $\lim\limits_{x\to \frac{\pi}{2}} x \sin x$

(iii) $\lim\limits_{x\to 1} (x^2 - xe^x)$ (iv) $\lim\limits_{x\to 0} (x^2+4)(x^2-1)(2x+1)$

(v) $\lim\limits_{x\to 2} \dfrac{x^2+2x+1}{x^2-2x+1}$ (vi) $\lim\limits_{x\to \frac{\pi}{2}} \dfrac{\sin x}{x}$

(vii) $\lim\limits_{h\to 2} \left(2h^2 - 3h + 4 + \dfrac{5}{h} + \dfrac{6}{h^2}\right)$ (viii) $\lim\limits_{x\to 0} \dfrac{ax^2+bx+c}{dx^2+ex+f}$

$[f \neq 0]$

3. Evaluate

(i) $\lim\limits_{h\to 0} \dfrac{4h^2+h}{h}$ (ii) $\lim\limits_{u\to 4} \dfrac{u^2-4u}{u-4}$

(iii) $\lim\limits_{x\to \frac{1}{3}} \dfrac{2-6x}{1-3x}$ (iv) $\lim\limits_{h\to 0} \dfrac{4h^5-6h^4+3h^3+h^2}{6h^6-3h^3+2h^2}$

(v) $\lim\limits_{z\to 1} \dfrac{z^2-8z+7}{7z^2-6z-1}$ (vi) $\lim\limits_{x\to 0} \dfrac{x}{\sqrt{x+1}-1}$

(vii) $\lim\limits_{h\to 0} \dfrac{h}{\sqrt[3]{h+1}-1}$ (viii) $\lim\limits_{h\to 0} \dfrac{\sqrt{1+h^2}-\sqrt{1+h}}{\sqrt{1+h^4}-\sqrt{1+h}}$

(ix) $\lim\limits_{x\to 4} \dfrac{\sqrt{2x+1}-3}{\sqrt{x-2}-\sqrt{2}}$ (x) $\lim\limits_{x\to 2} \left[\dfrac{1}{x-2} - \dfrac{4}{x^2-4}\right]$

4. Show that

(i) $\lim\limits_{x\to 1} \dfrac{x^6-1}{x-1} = 6.$ (ii) $\lim\limits_{x\to 0} \dfrac{\sqrt{1+x}-1}{x} = \dfrac{1}{2}$

(iii) $\lim\limits_{x\to 1} \dfrac{x^2-1}{\sqrt{3x+1}-\sqrt{5x-1}} = -4$ (iv) $\lim\limits_{x\to 0} e^{\left(\frac{\sin x}{x}\right)} = e.$

[H. S. 1979]

5. Show that,

(i) $\lim\limits_{x\to 5} \dfrac{x^4-625}{x-5} = 500$ (ii) $\lim\limits_{x\to 4} \dfrac{x^3-64}{x^2-16} = 6.$

(iii) $\lim\limits_{x\to a} \dfrac{x^{\frac{3}{7}}-a^{\frac{3}{7}}}{x^{\frac{2}{5}}-a^{\frac{2}{5}}} = \frac{15}{14} a^{\frac{1}{35}}$ (iv) $\lim\limits_{x\to a} \dfrac{x^n-a^n}{x^m-a^m} = \dfrac{n}{m} a^{n-m}.$

(v) $\lim\limits_{x\to a} \dfrac{x^{-4}-a^{-4}}{x^{-7}-a^{-7}} = \frac{4}{7} a^3$ (vi) $\lim\limits_{x\to \sqrt{2}} \dfrac{x^{\frac{5}{2}}-2^{\frac{5}{4}}}{\sqrt{x}-2^{\frac{1}{4}}} = 10$

(vii) $\lim\limits_{x\to 2} \dfrac{(2x)^4-256}{2(x-2)} = 256$ (viii) $\lim\limits_{x\to -2} \dfrac{x^3+8}{x+2} = 12$

(ix) $\lim\limits_{x\to 1} \dfrac{x^{\frac{7}{2}}-1}{x^{\frac{5}{2}}-1}=\frac{7}{5}$ (x) $\lim\limits_{h\to 0} \dfrac{(2+h)^4-16}{h}=32.$

(xi) $\lim\limits_{x\to 0} \dfrac{(1+x)^5-1}{5x+2x^2}=1$ (xii) $\lim\limits_{x\to \frac{1}{2}} \dfrac{8x^3-1}{4x^2-1}=\frac{3}{2}.$

6. Evaluate the limits

(i) $\lim\limits_{x\to 0} \dfrac{3-\sqrt{9-x^2}}{x^2}$ (ii) $\lim\limits_{x\to 0} \dfrac{\sqrt{1+4x}-\sqrt{1-5x}}{9x}$

7. Prove that

(i) $\lim\limits_{x\to 0} \dfrac{\sin \frac{1}{2}x}{x}=\frac{1}{2}$ (ii) $\lim\limits_{x\to \frac{\pi}{2}} \dfrac{\cos x}{\frac{\pi}{2}-x}=1.$

(iii) $\lim\limits_{h\to 0} (h \operatorname{cosec} h)=1$ (iv) $\lim\limits_{h\to 0} \dfrac{\sin^2 h \cos h}{h^2}=1$

(v) $\lim\limits_{x\to 0} \dfrac{\tan 5x}{x}=5.$ (vi) $\lim\limits_{h\to 0} \dfrac{1-\cos h}{h^2}=\frac{1}{2}.$

(vii) $\lim\limits_{\theta\to 0} \dfrac{\tan m\theta}{n\theta}=\dfrac{m}{n}.$ (viii) $\lim\limits_{\theta\to 0} \dfrac{\tan \theta}{\theta}=1.$

(ix) $\lim\limits_{x\to 0} \dfrac{\tan \alpha x}{x}=\alpha.$ (x) $\lim\limits_{\theta\to 0} \dfrac{\tan \theta^c}{\theta}=\dfrac{\pi}{180}.$

(xi) $\lim\limits_{x\to \frac{\pi}{4}} \dfrac{1-\tan x}{1-\cot x}=-1.$

8. Evaluate the limits :

(i) $\lim\limits_{x\to 0} \dfrac{1-\cos 6x}{6x^2}$ (ii) $\lim\limits_{x\to 0} \dfrac{\cos 2x-\cos 3x}{x^2}$

(iii) $\lim\limits_{x\to 0} \dfrac{\cos ax-\cos bx}{x^2}$ (iv) $\lim\limits_{x\to 0} \dfrac{mx}{\tan nx}$

(v) $\lim\limits_{x\to 0} \dfrac{\sin 3x}{\sin 5x}$ (vi) $\lim\limits_{h\to 0} \dfrac{x \sin x}{1-\cos x}.$

(vii) $\lim\limits_{x\to 0} \dfrac{1-\cos x}{\sin^2 x}$ (viii) $\lim\limits_{x\to \frac{\pi}{4}} \dfrac{\cos 2x}{\frac{\pi}{4}-x}$ (ix) $\lim\limits_{x\to 0} \dfrac{2x}{\sin 3x}$

(x) $\lim\limits_{x\to a} \dfrac{x \sin a-a \sin x}{x-a}$ (xi) $\lim\limits_{x\to 0} \tan 3x \cot 4x$

(xii) $\lim\limits_{x\to 0} \tan \alpha x \cot \beta x$ (xiii) $\lim\limits_{x\to 0} \tan 2\alpha x \operatorname{cosec} \alpha x.$

(xiv) $\lim\limits_{x\to 0} \dfrac{x(\cos x+\cos 2x)}{\sin x}$

(xv) $\lim\limits_{x\to 0} \dfrac{(\sin^{-1}x)^2}{1-\sqrt{1-x^2}}$ (xvi) $\lim\limits_{x\to 0} (\tan 3x \operatorname{cosec} 3x)$

9. Prove that

(i) $\lim\limits_{h\to 0} \dfrac{e^{h^2}-1}{h}=0$ (ii) $\lim\limits_{x\to 0} \dfrac{e^{\tan x}-1}{x}=1.$

(iii) $\lim\limits_{h\to 0} \dfrac{e^{2+h}-e^2}{h}=e^2$ (iv) $\lim\limits_{h\to 0} \dfrac{e^{x+h}-e^x}{h}=e^x$

10. Show that

(i) $\lim\limits_{x\to 0} \dfrac{e^{mx}-1}{x}=m$ (ii) $\lim\limits_{h\to 0} \dfrac{e^{(x+h)^2}-e^{x^2}}{h}=2x.e^{x^2}$

(iii) $\lim\limits_{x\to 0} \dfrac{ae^x+be^{-x}}{e^x+e^{-x}}=\dfrac{a+b}{2}$

11. Show that

(i) $\lim\limits_{x\to 0} \dfrac{\log(1+ax)}{x}=a$ (ii) $\lim\limits_{x\to 0} \dfrac{\log(1+2x)}{\tan 3x}=\frac{2}{3}$

(iii) $\lim\limits_{x\to 0} (1+2x)^{\frac{1}{x}}=e^2$

12. Evaluate

(i) $\lim\limits_{x\to 0} \dfrac{\log(1+\sin x)}{x}$ (ii) $\lim\limits_{x\to 0} (1+ax)^{\frac{1}{x}}$

13. Show that

(i) $\lim\limits_{x\to 1} \dfrac{\log x}{x-1}=1.$ (ii) $\lim\limits_{x\to 0} \dfrac{e^x+\sin x-1}{\log(1+x)}=2$

(iii) $\lim\limits_{x\to 0} \dfrac{3^x-2^x}{x}=\log \frac{3}{2}$ (iv) $\lim\limits_{x\to e} \dfrac{\log x-1}{x-e}=\dfrac{1}{e}$

(v) $\lim\limits_{x\to 0} \dfrac{a^x-b^x}{\sin x}=\log \dfrac{a}{b}$ (vi) $\lim\limits_{x\to 0} \dfrac{\tan \log(1+x)}{x}=1$

(vii) $\lim\limits_{x\to 0} \dfrac{\tan \log(1+x)}{\log(1+\sin x)}=1$ (viii) $\lim\limits_{x\to 0} \dfrac{\log \cos x}{\sin^2 x}=-\frac{1}{2}$

14. Show that

(i) $\lim\limits_{h\to 0} \dfrac{\cos(x+h)-\cos x}{h}=-\sin x$

(ii) $\lim\limits_{h\to 0} \dfrac{\tan(x+h)-\tan x}{h}=\sec^2 x$

(iii) $\lim\limits_{h\to 0} \dfrac{\operatorname{cosec}(x+h)-\operatorname{cosec} x}{h}=\operatorname{cosec} x \cot x$

15. Find

$\lim\limits_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$ where

(i) $f(x)=x^2$ (ii) $f(x)=\cot x+2x+4$

(iii) $f(x)=\sec x$ (iv) $f(x)=\dfrac{1}{x^2}$

16. Evaluate

(i) $\lim\limits_{h\to 0} \dfrac{f(3+h)-f(3)}{h}$, if $f(x)=\dfrac{1}{x}$

(ii) $\lim\limits_{h\to 0} \dfrac{f(1+h)-f(1)}{h}$, if $f(x)=e^x$

(iii) $\lim\limits_{h\to 0} \dfrac{f(h)-f(0)}{h}$, if $f(x)=\dfrac{1}{x+1}+2x$

17. If $f(x)=2x+1$ when $x<1$

$=3$ when $x=1$

$=2x-1$ when $1<x<2$

$=3$ when $x=2$

$=x+1$ when $x>2$

then find $\lim\limits_{x\to 1} f(x)$ and $\lim\limits_{x\to 2} f(x)$

18. Show that

(i) $\lim\limits_{h\to 0} \dfrac{\sqrt{x+h}-\sqrt{x}}{h}=\dfrac{1}{2\sqrt{x}}$ (ii) $\lim\limits_{h\to 0} \dfrac{\sin 2h-2\sin h}{h^3}=-1$

(iii) $\lim\limits_{x\to 0} \dfrac{\sin(\alpha+\beta)x+\sin(\alpha-\beta)x+\sin 2\alpha x}{\sin\alpha x-\sin\beta x}=\dfrac{4\alpha}{\alpha-\beta}$

19. Evaluate

(i) $\lim\limits_{x\to\infty} \dfrac{1}{x^2}$ (ii) $\lim\limits_{x\to\infty} \dfrac{x^3-3x^2+1}{x^3+2x^2+x}$

(iii) $\lim\limits_{x\to\infty} \dfrac{x+1}{x^2+1}$ (iv) $\lim\limits_{x\to\infty} \left(2-\dfrac{3}{x^4}\right)$

(v) $\lim\limits_{x\to-\infty} \dfrac{3}{x^2}$ (vi) $\lim\limits_{x\to\infty} \dfrac{2x^3}{x^3+1}$ (vii) $\lim\limits_{x\to\infty} \sin\left(\dfrac{1}{x}\right)$

20. Show that

(i) $\lim\limits_{x\to\infty} \dfrac{1+2+\cdots+x}{x^2}=\frac{1}{2}$, [where x is a positive integer]

(ii) $\lim\limits_{x\to\infty} e^{\frac{1}{x^2}}=1$ (iii) $\lim\limits_{x\to\infty} \cos\left(\dfrac{x^2+2x+1}{x^3+x^2+1}\right)=1$

21. Show that

(i) $\lim\limits_{n\to\infty} \left(1+\frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^{n-1}}\right)=2.$

(ii) $\lim\limits_{n\to\infty} \frac{1}{n}\left\{\left(\frac{1}{n}\right)^3+\left(\frac{2}{n}\right)^3+\left(\frac{3}{n}\right)^3+\cdots+\left(\frac{n}{n}\right)^3\right\}=\frac{1}{4}$

(iii) $\lim\limits_{n\to\infty} \dfrac{1.2+2.3+3.4+\cdots+n(n+1)}{n^3}=\frac{1}{3}$

(iv) $\lim\limits_{n\to\infty} \left(1+\frac{1}{n}\right)^n=e$

22. Show that $\lim\limits_{x\to\infty} \dfrac{a_0x^n+a_1x^{n-1}+\cdots+a_{n-1}x+a_n}{b_0x^m+b_1x^{m-1}+\cdots+b_{m-1}x+b_m}$

$=\infty$, if $n>m$

$=\dfrac{a_0}{b_0}$ if $n=m$

$=0$ if $n<m$

23. Show that the following limits do not exist

(i) $\lim\limits_{x\to\infty} \dfrac{x^2+x+1}{2x+3}$ (ii) $\lim\limits_{x\to\infty} e^{x^2+1}$

(iii) $\lim\limits_{x\to\infty} x^2\left(\dfrac{1}{x+1}+\dfrac{1}{x-1}\right)$ (iv) $\lim\limits_{x\to 0} \dfrac{1+\cos^3 x}{\tan^2 x}$

24. Show that

(i) $\lim\limits_{x\to\infty} \dfrac{ae^x+be^{-x}}{e^x+e^{-x}}=a$ (ii) $\lim\limits_{x\to-\infty} \dfrac{ae^x+be^{-x}}{e^x+e^{-x}}=b$

(iii) $\lim\limits_{x\to\infty} \left\{\sqrt{x^2-x+1}-x\right\}=-\frac{1}{2}$

(iv) $\lim\limits_{x\to 0} \dfrac{\sqrt{1+x^2}-\sqrt{1-x^2}}{x^2}=1$ (v) $\lim\limits_{x\to\infty} \dfrac{\sqrt{1+x^2}-\sqrt{1-x^2}}{x^2}$

does not exist.

(vi) $\lim\limits_{x \to \infty} \dfrac{\sqrt{x^4+1}-2x^2-1}{x^2}=-1$ (vii) $\lim\limits_{x \to 0} \dfrac{\sqrt{x^4+1}-2x^2-1}{x^2}$
$=-2$

(viii) $\lim\limits_{x \to 0} \dfrac{ax^2+bx+c}{cx^2+dx+a}=\dfrac{c}{a}$ (ix) $\lim\limits_{x \to \infty} \dfrac{ax^2+bx+c}{cx^2+dx+a}=\dfrac{a}{c}$
$(a \neq 0)$ $(c \neq 0)$

25. Prove that $\lim\limits_{x \to \infty} \dfrac{1}{1+n \sin^2 \pi x}=0$, when x is not an integer.
$=1$, when x is an integer.

26. If $f(x)=\lim\limits_{n \to \infty} \dfrac{x^n g(x)+h(x)}{x^n+1}$ show that

$f(x)=h(x)$, when $0<x<1$
$=\frac{1}{2}\{g(x)+h(x)\}$ when $x=1$
$=g(x)$ when $x>1$.

CHAPTER FOUR

DERIVATIVE OF A FUNCTION

§ 4·1. **Increment :** If the value of a variable changes from x to x', then $(x'-x)$ is the increment of x. The increment of x is generally denoted by $\triangle x$. Hence if the value of x changes from x to x', then the increment of x is $\triangle x=x'-x$.

For example,

(i) If the value of x changes from 2 to 2·4 ; then the increment of x is given by

$$\triangle x=2{\cdot}4-2={\cdot}4.$$

(ii) If the value of x changes from 3 to 2·1, then the increment of x is $\triangle x=2{\cdot}1-3=-{\cdot}9$.

[Note. (i) If $\triangle x$ be the increment of x, then the changed value of x is $x'=x+\triangle x$.

(ii) $\triangle x$ is not the product of $\triangle$ and x i.e., $\triangle x \neq \triangle \times .x$. The symbol $\triangle x$ denotes one and only one number.

(iii) $\triangle x$ can be both positive and negative.

(iv) For convenience, h is frequently used for $\triangle x$.

(v) The increment can be written as

Increment = Final value − Initial value.]

Now, let y be a function of x and $y=f(x)$. If the value of x changes, then the value of y will also change. Let us now determine the increment $\triangle y$ of y corresponding to the increment $\triangle x$ of x.

If the value of x changes to $x+\triangle x$, then y will be changed to $f(x+\triangle x)$. Now as the increment of y

= Final value of y − initial value of y

$\therefore \quad \triangle y=f(x+\triangle x)-f(x)$, or, $\triangle y+f(x)=f(x+\triangle x)$.

Again, as $y=f(x)$, so $y+\triangle y=f(x)+\triangle y=f(x+\triangle x)$.

Hence by putting $x+\triangle x$ in place of x in $f(x)$ we can find the value of $y+\triangle y$ and subtracting the value of y from this changed value of $y+\triangle y$ we can get $\triangle y$.

Example (i) Let $y=f(x)=x^2$

$\therefore \quad y+\triangle y=f(x+\triangle x)=(x+\triangle x)^2=x^2+2x.\triangle x+(\triangle x)^2$

$\therefore \quad \triangle y=(y+\triangle y)-y=x^2+2x.\triangle x+(\triangle x)^2-x^2$

$$=\triangle x(2x+\triangle x)$$

(ii) Let $y=\dfrac{1}{\sqrt{x}}$.

$\therefore \quad \Delta y=f(x+\Delta x)-f(x)=\dfrac{1}{\sqrt{x+\Delta x}}-\dfrac{1}{\sqrt{x}}$

$$=\frac{\sqrt{x}-\sqrt{x+\Delta x}}{\sqrt{x}\sqrt{x+\Delta x}}=\frac{(\sqrt{x}-\sqrt{x+\Delta x})(\sqrt{x}+\sqrt{x+\Delta x})}{\sqrt{x}\sqrt{x+\Delta x}(\sqrt{x}+\sqrt{x+\Delta x})}$$

$$=-\frac{\Delta x}{\sqrt{x}\sqrt{x+\Delta x}(\sqrt{x}+\sqrt{x+\Delta x})}.$$

(iii) Let $y=a$ where a is a constant.

$\Delta y=f(x+\Delta x)-f(x)=a-a=0 \quad [y=a$, for all values of $x]$

If $f(x)$ is continuous $\lim\limits_{\Delta x\to 0} f(x+\Delta x)=f(x)$

$\therefore \quad \lim\limits_{\Delta x\to 0} \Delta y=\lim\limits_{\Delta x\to 0}[f(x+\Delta x)-f(x)]=\lim\limits_{\Delta x\to 0} f(x+\Delta x)-f(x)$

$=f(x)-f(x)=0$

$\therefore \quad$ If $y=f(x)$ is continuous, $\lim \Delta y=0$. Conversely if $\lim\limits_{\Delta x\to 0} \Delta y=0$, $y=f(x)$ is continuous

Δy is the increment in value of y corresponding to the increment Δx in the value of x. So, $\dfrac{\Delta y}{\Delta x}$ is the average rate of change of y for unit increment in the value of x.

Ex. 1. If the value of x changes from 2 to 2·1 find the change in value of $y=\dfrac{1}{x}$.

Here $\Delta x=2{\cdot}1-2={\cdot}1$

$\therefore$ change in value of y

$=\Delta y=f(2+\Delta x)-f(x)=f(2{\cdot}1)-f(2)$

$$=\frac{1}{2{\cdot}1}-\frac{1}{2}=\frac{2-2{\cdot}1}{2\,.\,2{\cdot}1}=-\frac{{\cdot}1}{4{\cdot}2}=-\frac{1}{42}.$$

Ex. 2. Find $\dfrac{\Delta y}{\Delta x}$ when $y=f(x)$ and

(i) $f(x)=\dfrac{x}{x+1}$ (ii) $f(x)=x^2-2x+\tan x$

(iii) $f(x)=x\sin x$.

(i) $$\frac{\Delta y}{\Delta x}=\frac{f(x+\Delta x)-f(x)}{\Delta x}=\frac{\dfrac{x+\Delta x}{x+\Delta x+1}-\dfrac{x}{x+1}}{\Delta x}$$

$$=\frac{(x+1)(x+\Delta x)-x(x+\Delta x+1)}{(x+1+\Delta x)(x+1).\ \Delta x}=\frac{1}{(x+1)(x+\Delta x+1)}.$$

(ii) $\Delta y=f(x+\Delta x)-f(x)$

$$=(x+\Delta x)^2-2(x+\Delta x)+\tan\,(x+\Delta x)-x^2+2x-\tan x$$

$$=(\Delta x)^2+\frac{\sin\,(x+\Delta x)}{\cos\,(x+\Delta x)}-\frac{\sin x}{\cos x}+2\Delta x(x-1)$$

$$=(\Delta x)^2+\frac{\sin\,\Delta x}{\cos\,(x+\Delta x)\,\cos x}+2\Delta x(x-1)$$

$$\therefore\quad \frac{\Delta y}{\Delta x}=\Delta x+\frac{\sin\,\Delta x}{\Delta x}.\ \frac{1}{\cos\,(x+\Delta x).\cos x}+2(x-1)$$

(iii) $\Delta y=f(x+\Delta x)-f(x)$

$$=(x+\Delta x)\sin\,(x+\Delta x)-x\sin x$$

$$=x\{\sin\,(x+\Delta x)-\sin x\}+\Delta x.\ \sin\,(x+\Delta x)$$

$$=x.\ 2\cos\left(x+\frac{\Delta x}{2}\right).\ \sin\frac{\Delta x}{2}+\Delta x.\ \sin\,(x+\Delta x)$$

$$\therefore\quad \frac{\Delta y}{\Delta x}=2x\cos\left(x+\frac{\Delta x}{2}\right).\frac{\sin\frac{\Delta x}{2}}{\Delta x}+\sin\,(x+\Delta x).$$

Ex. 3. If u and v both be functions of x, then show that

(i) $\Delta y=c\Delta u$, where $y=cu$ and c is a constant.

(ii) $\Delta y=\Delta u+\Delta v$, when $y=u+v$

(iii) $\Delta y=u\Delta v+v.\ \Delta u+\Delta u.\ \Delta v$, when $y=u.v$.

Let $u=f(x)$ and $v=g(x)$

$\therefore\quad \Delta u=f(x+\Delta x)-f(x)$ and $\Delta v=g(x+\Delta x)-g(x)$.

(i) $y=cu=cf(x)=\phi(x)$ (say)

$$\therefore\quad \Delta y=\phi(x+\Delta x)-\phi(x)=cf(x+\Delta x)-cf(x)$$
$$=c\{f(x+\Delta x)-f(x)\}=c.\ \Delta u$$

(ii) $y=u+v=f(x)+g(x)=h(x)$ (say)

$$\therefore\quad \Delta y=h(x+\Delta x)-h(x)=f(x+\Delta x)+g(x+\Delta x)-\{f(x)+g(x)\}$$
$$=\{f(x+\Delta x)-f(x)\}+\{g(x+\Delta x)-g(x)\}=\Delta u+\Delta v.$$

(iii) $y=u.v=f(x).g(x)=k(x)$ (say)

$$\therefore\quad \Delta y=k(x+\Delta x)-k(x)=f(x+\Delta x).g(x+\Delta x)-f(x).g(x)$$
$$=\{f(x)+\Delta u\}.\{g(x)+\Delta v\}-f(x)g(x)$$
$$=\Delta u.g(x)+\Delta v.f(x)+\Delta u.\Delta v=v\Delta u.+u.\Delta v.+\Delta u.\Delta v.$$

Exercise 4(A)

1. Find $\triangle y$ when

(i) $y=x^2-3x+4$, $\triangle x=\cdot 01$

(ii) $y=\dfrac{3}{x^2}$, $\triangle x=-\cdot 5$ (iii) $y=5$, $\triangle x=\cdot 3$.

2. Evaluate $\triangle x$, $\triangle y$ and $\dfrac{\triangle y}{\triangle x}$ when

(i) $y=x^2+4$ and x changes from 4 to 3·9.

(ii) $y=x^2-\dfrac{2}{x^2}$ and x changes from 1 to 2·1.

(iii) $y=-\dfrac{1}{x^3}$ and x increases from 3 to 5.

3. Find $\triangle y$ when

(i) $y=x^3+1$, (ii) $y=\dfrac{1}{x-1}$ (iii) $y=\dfrac{1}{(x-1)^2}$

(iv) $y=ax+\dfrac{b}{x}$ (v) $y=3x^2(x^2+1)$

(vi) $y=x\tan x+x^2$, (vii) $y=x^{-\frac{2}{3}}+4x$

4. If u and v both be functions of x prove that

(i) $\triangle y=\triangle u-\triangle v$ when $y=u-v$,

(ii) $\triangle y=\dfrac{v\triangle u-u\triangle v}{v(v+\triangle v)}$ when $y=\dfrac{u}{v}$,

(iii) $\triangle y=-\dfrac{\triangle u}{u(u+\triangle u)}$ when $y=\dfrac{1}{u}$.

§ 4·2. Derivative or differential coefficient of a function.

Let $y=f(x)$ be a function of x and as x changes to $x+\triangle x$ ($\triangle x$ may be positive or negative but not zero) then the increment of y be $\triangle y$.

Now $\dfrac{\triangle y}{\triangle x}$ is the average rate of change of y for the values of x from x to $x+\triangle x$. If we are to find the instantaneous rate of change of y at the point x, then $\triangle x$ is to be zero. But if $\triangle x=0$, then $\dfrac{\triangle y}{\triangle x}$ takes the form $\dfrac{0}{0}$ which is indeterminate. Hence

taking $\triangle x$ to be zero, we cannot get the instantaneous rate of change of y at x. So, the instantaneons rate of change of y at x is determined by making $\triangle x$ tending to zero instead of taking $\triangle x$ as exactly zero. Now as $\triangle x$ will tend to zero, $x+\triangle x$ will approach x and hence $\frac{\triangle y}{\triangle x}$, the average rate of change of y for the change in value of x from x to $x+\triangle x$ will tend to the instantaneous rate of change of y at x. The knowledge of this instantaneous rate of change of y at x for change of x is very important in different branches of mathematics. This rate of change is called the derivative or differential coefficient of $y=f(x)$ with respect to x and is denoted by the symbols $f'(x)$ or $\frac{dy}{dx}$ or y_1. Following is the formal definition of derivative of a function with respect to a variable.

Def : If x be a point in the domain of definition of a function $f(x)$ and $\triangle y=f(x+\triangle x)-f(x)$ be the increment of y or $f(x)$ corresponding to an increment $\triangle x$ (positive or negative) of x, then $\lim\limits_{\triangle x\to 0}\frac{\triangle y}{\triangle x}$. i.e., $\lim\limits_{\triangle x\to 0}\frac{f(x+\triangle x)-f(x)}{\triangle x}$ when it exists is called the derivative of $y=f(x)$ at the point x with respeet to x.

$$\therefore\ \frac{dy}{dx}=f'(x)=\lim_{\triangle x\to 0}\frac{f(x+\triangle x)-f(x)}{\triangle x}.$$

The derivative of $f(x)$ at the point $x=a$ is denoted by the symbol $f'(a)$ or $\left(\frac{dy}{dx}\right)_{x=a}$.

$$\therefore\ \left(\frac{dy}{dx}\right)_{x=a}=f'(a)=\lim_{\triangle x\to 0}\frac{f(a+\triangle x)-f(a)}{\triangle x}.$$

For convenience, h is written instead of $\triangle x$.

$$\therefore\ f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}.$$

Note (1) If $h=0$, then $\frac{f(x+h)-f(x)}{h}$ is of the form $\frac{0}{0}$.

So, $\lim\limits_{h\to 0}\frac{f(x+h)-f(x)}{h}$ may not exist.

If $\lim\limits_{h \to 0} \dfrac{f(a+h)-f(a)}{h}$ does not exist, then the derivative of $f(x)$ at the point $x=a$ does not exist.

(2) h may approach 0 from both right and left, i.e., $h \to 0+$, $h \to 0-$ both are to be considered. When $h \to 0+$, then the limit of $\dfrac{f(a+h)-f(a)}{h}$, when it exists, is called the right-hand derivative of $f(x)$ at the point a. Similarly, the left-hand derivative of $f(x)$ at $x=a$ is $\lim\limits_{h \to 0-} \dfrac{f(a+h)-f(a)}{h}$ when it exists. The right-hand and left-hand derivatives of $f(x)$ at $x=a$ are denoted by the symbols $f'(a+)$ and $f'(a-)$ respectively.

If $f(x)$ possesses a derivative at $x=a$, then $f'(a)=f'(a+)=f'(a-)$.

(3) If the derivatives of a function $f(x)$ exist at every point of an interval (a, b), then $f(x)$ is said to be differentiable in the interval. At the extreme points $f'(a+)$ and $f'(b-)$ only are to be considered.

(4) The derivative of $y=f(x)$ at the point x is denoted by the symbols $\dfrac{dy}{dx}$, $f'(x)$, $\dfrac{d}{dx}(y)$, $\dfrac{d}{dx}f(x)$, y_1, y', $\mathrm{D}y$, $\mathrm{D}(y)$. The symbol $\mathrm{D}y$ or $\mathrm{D}(y)$ are used in differential equations. In differential calculus we shall use the two symbols $\dfrac{dy}{dx}$ and $f(x)$.

(5) In the symbol $\dfrac{dy}{dx}$, dy and dx are not two different quantities. The meaning of $\dfrac{dy}{dx}$ is $\dfrac{d}{dx}(y)$, i.e., the operation $\dfrac{d}{dx}$ has been applied on. It is for this reason using D for $\dfrac{d}{dx}$, $\dfrac{dy}{dx}$ is expressed as $\mathrm{D}y$. In a later article we shall discuss about the concept of differential and there we shall show that the derivative $\dfrac{dy}{dx}$ is the quotient of the two differentials dy and dx. $\dfrac{dy}{dx}$ as derivative is to be considered as a single quantity and it means that $\dfrac{d}{dx}$ has been operated on y.

The process of determination of the derivative of a function is called the process of differentiation.

(6) $\because\ f(x+h)=\dfrac{f(x+h)-f(x)}{h}.h+f(x)$

$$\therefore\ \lim_{h\to 0} f(x+h)=\lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{h}.h+f(x)\right]$$

$$=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}.h+f(x),$$

[$\because$ as $h\to 0$, $f(x)$ remains unchanged]

$$=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}.\lim_{h\to 0}h+f(x)$$

$\left[\text{ assuming } \lim_{h\to 0}\dfrac{f(x+h)-f(x)}{h}=f'(x) \text{ exists }\right]$

$=f(x)\ 0+f(x)=f(x).$

Now, if $\lim_{h\to 0} f(x+h)=f(x)$, $f(x)$ is continuous.

Therefore, if $f(x)$ exists i.e., $f(x)$, is differentiable at a point, it is continuous at that point.

(7) To find the derivative of a function $f(x)$ from the definition, follow the following steps :

(*i*) Determine the value of $f(x+\Delta x)$ or $f(x+h)$ putting $x+\Delta x$ or $x+h$ for x.

(*ii*) Determine the increment $f(x+\Delta x)-f(x)$ of y.

(*iii*) Divide the increment of y by the increment Δx of x i.e., determine $\dfrac{f(x+\Delta x)-f(x)}{\Delta x}$.

(*iv*) Make $\Delta x\to 0$ and find the limiting value of $\dfrac{\text{increment of } y}{\text{increment of } x}$. i.e., $\dfrac{f(x+\Delta x)-f(x)}{\Delta x}$.

§ 4·3. Derivatives of elementary functions :

I. Derivative of x^n.

If $y=x^n$, then $\dfrac{dy}{dx}=nx^{n-1}$, or, $\dfrac{d}{dx}(x^n)=nx^{n-1}$, where n is any real number.

Let Δy be the increment of y for the increment Δx of x.

$$\therefore\ y+\Delta y=(x+\Delta x)^n,\quad \text{or,}\quad \Delta y=(x+\Delta x)^n-x^n$$

$$\therefore\ \frac{\Delta y}{\Delta x}=\frac{(x+\Delta x)^n-x^n}{\Delta x}=\frac{(x+\Delta x)^n-x^n}{(x+\Delta x)-x}$$

$$\therefore \quad \frac{dy}{dx}=\lim_{\triangle x \to 0} \frac{\triangle y}{\triangle x}=\lim_{\triangle x \to 0} \frac{(x+\triangle x)^n - x^n}{(x+\triangle x)-x}$$

Now putting $u=x+\triangle x$, as $\triangle x \to 0$, then $u \to x$

$$\therefore \quad \frac{dy}{dx}=\lim_{\triangle x \to 0} \frac{(x+\triangle x)^n - x^n}{(x+\triangle x)-x}=\lim_{u \to x} \frac{u^n - x^n}{u-x}=nx^{n-1}$$

[By Limit theorem]

$$\therefore \quad \frac{d}{dx}(x^n)=nx^{n-1}$$

Note : Remember the formula as follows :

$$\frac{d}{dx}\left(x^{\text{exponent}}\right)=\text{exponent}\times\left(x^{\text{exponent}-1}\right)$$

Ex. (i) $\frac{d}{dx}(x^7)=7.x^{7-1}=7x^6$ for here exponent$=7$.

(ii) $\frac{d}{dx}(x)=1$, for $\frac{d}{dx}(x)=\frac{d}{dx}(x^1)=1.x^{1-1}=1\ x^0=1$.

Examples

$$\frac{d}{dx}(x^4)=4x^{4-1}=4x^3,\ \frac{d}{dx}(x^{\frac{1}{2}})=\tfrac{1}{2}x^{\frac{1}{2}-1}=\tfrac{1}{2}x^{-\frac{1}{2}}=\frac{1}{2\sqrt{x}}.$$

$$\frac{d}{dx}\left(x^{\frac{1}{3}}\right)=\tfrac{1}{3}x^{\frac{1}{3}-1}=\tfrac{1}{3}x^{-\frac{2}{3}},\ \frac{d}{dx}(\sqrt[4]{x})=\frac{d}{dx}\left(x^{\frac{1}{4}}\right)=\tfrac{1}{4}x^{\frac{1}{4}-1}=\tfrac{1}{4}x^{-\frac{3}{4}}$$

$$\frac{d}{dx}\left(\frac{d}{x^2}\right)=\frac{d}{dx}(x^{-2})=(-2)x^{-2-1}=-2x^{-3}=-\frac{2}{x^3}$$

$$\frac{d}{dx}\left(\frac{1}{x^n}\right)=(x^{-n})=-nx^{-n-1}=-\frac{n}{x^{n+1}}$$

II. Derivative of a constant : If $y=f(x)=c$,

$f(x+\triangle x)=c \quad \therefore \quad \triangle y=f(x+\triangle x)-f(x)=c-c=0$.

$$\therefore \quad \frac{dy}{dx}=\lim_{\triangle x \to 0} \frac{\triangle y}{\triangle x}=\lim_{\triangle x \to 0} \frac{0}{\triangle x}=0$$

III. If $y=e^x$, then $\frac{dy}{dx}=e^x$, or, $\frac{d}{dx}(e^x)=e^x$.

Here $y=f(x)=e^x$. $\therefore \quad y+\triangle y=f(x+\triangle x)=e^{x+\triangle x}$

$$\therefore \quad \frac{dy}{dx}=\lim_{\triangle x \to 0} \frac{\triangle y}{\triangle x}=\lim_{\triangle x \to 0} \frac{e^{x+\triangle x}-e^x}{\triangle x}=\lim_{\triangle x \to 0} \frac{e^x(e^{\triangle x}-1)}{\triangle x}$$

$$=e^x \lim_{\triangle x \to 0} \frac{e^{\triangle x}-1}{\triangle x}=e^x.1=e^x. \quad \therefore \quad \frac{d}{dx}(e^x)=e^x.$$

IV. If $y=a^x$, then $\frac{dy}{dx}=a^x \log a$

Here $y=a^x=e^{x \log a}$, $y+\Delta y=a^{x+\Delta x}=e^{(x+\Delta x)\log a}$

$$\therefore \quad \frac{dy}{dx}=\lim_{\Delta x\to 0}\frac{e^{(x+\Delta x)\log a}-e^{x\log a}}{\Delta x}$$

$$=\lim_{\Delta x\to 0}\left\{\frac{e^{x\log a}(e^{\Delta x\log a}-1)}{\Delta x\log a}\log a\right\}$$

$$=e^{x\log a}\log a\lim_{\Delta x\to 0}\frac{e^{\Delta x\log a}-1}{\Delta x\log a}$$

$=a^x.\log_e a.1$ [$\because$ as $\Delta x\to 0$, $\Delta x\log a\to 0$]
$=a^x \log_e a.$

V. If $y=\log x$, then $\frac{dy}{dx}=\frac{1}{x}$, or, $\frac{d}{dx}\left[\log x\right]=\frac{1}{x}$

Here $y=\log x \quad \therefore \quad y+\Delta y=\log(x+\Delta x)$

$$\therefore \quad \frac{dy}{dx}=\lim_{\Delta x\to 0}\frac{\log(x+\Delta x)-\log x}{\Delta x}$$

$$=\lim_{\Delta x\to 0}\frac{\log\left(\frac{x+\Delta x}{x}\right)}{\Delta x}=\lim_{\Delta x\to 0}\frac{\log\left(1+\frac{\Delta x}{x}\right)}{\frac{\Delta x}{x}}.\frac{1}{x}$$

$$=\frac{1}{x}\lim_{k\to 0}\frac{\log(1+k)}{k}, \text{ where } k=\frac{\Delta x}{x}\to 0 \text{ as } \Delta x\to 0$$

$$=\frac{1}{x}.1=\frac{1}{x}.$$

VI. If $y=\sin x$, then $\frac{dy}{dx}=\cos x$, or, $\frac{d}{dx}(\sin x)=\cos x$

$$\frac{dy}{dx}=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h}$$

$$=\lim_{h\to 0}\frac{2\cos(x+\frac{h}{2})\sin\frac{h}{2}}{h}$$

$$=\lim_{h\to 0}\frac{\sin\frac{1}{2}h}{\frac{1}{2}h}.\cos(x+\tfrac{1}{2}h)=1.\cos x=\cos x.$$

$$\therefore \quad \frac{d}{dx}(\sin x)=\cos x.$$

VII. If $y=\cos x$, then $\dfrac{dy}{dx}=-\sin x$,

or, $\dfrac{d}{dx}(\cos x)=-\sin x$

$$\frac{dy}{dx}=\lim_{h\to 0}\frac{\cos(x+h)-\cos x}{h}=\lim_{h\to 0}\frac{2\sin\frac{x+h+x}{2}\ \sin\frac{x-(x+h)}{2}}{h}$$

$$=\lim_{h\to 0}.\frac{\sin\frac{1}{2}h}{\frac{1}{2}h}\sin(x+\tfrac{1}{2}h)=-1.\sin x=-\sin x.$$

VIII. If $y=\tan x$ then $\dfrac{dy}{dx}=\sec^2 x$, or, $\dfrac{d}{dx}(\tan x)=\sec^2 x$.

$$\frac{dy}{dx}=\lim_{h\to 0}\frac{\tan(x+h)-\tan x}{h}$$

$$=\lim_{h\to 0}\frac{\sin(x+h)\cos x-\cos(x+h)\sin x}{h\cos(x+h)\cos x}$$

$$=\lim_{h\to 0}\frac{\sin(x+h-x)}{h\cos(x+h)\cos x}=\lim_{h\to 0}\left\{\frac{\sin h}{h}.\ \frac{1}{\cos(x+h)\cos x}\right\}$$

$$=1.\frac{1}{\cos x.\cos x}=\frac{1}{\cos^2 x}=\sec^2 x.$$

$$\therefore\quad \frac{d}{dx}(\tan x)=\sec^2 x,\ \left[\ x\neq(2n+1)\frac{\pi}{2}\ \right]$$

IX. If $y=\cot x$, then $\dfrac{dy}{dx}=-\operatorname{cosec}^2 x$

or, $\dfrac{d}{dx}(\cot x)=-\operatorname{cosec}^2 x$.

$$\frac{dy}{dx}=\lim_{h\to 0}\frac{\cot(x+h)-\cot x}{h}$$

$$=\lim_{h\to 0}\frac{\cos(x+h).\sin x-\sin(x+h)\cos x}{h.\sin(x+h)\sin x}$$

$$=\lim_{h\to 0}\frac{\sin(x-x-h)}{h.\sin(x+h)\sin x}=-\lim_{h\to 0}\frac{\sin h}{h}.\ \operatorname*{Lim}_{h\to 0}\frac{1}{\sin(x+h)\sin x}$$

$$=-1.\frac{1}{\sin x\ \sin x}=-\frac{1}{\sin^2 x}=-\operatorname{cosec}^2 x$$

$$\therefore\quad \frac{d}{dx}(\cot x)=-\operatorname{cosec}^2 x,\ [\ x\neq n\pi\ ]$$

X. If $y=\sec x$, then $\dfrac{dy}{dx}=\sec x \cdot \tan x$,

or, $\dfrac{d}{dx}(\sec x)=\sec x, \tan x$.

$$\frac{dy}{dx}=\lim_{h\to 0}\frac{\sec(x+h)-\sec x}{h}=\lim_{h\to 0}\frac{\cos x-\cos(x+h)}{h\cos(x+h).\cos x}$$

$$=\lim_{h\to 0}\frac{2\sin\frac{1}{2}h\sin(x+\frac{1}{2}h)}{h\cos(x+h).\cos x}$$

$$=\lim_{h\to 0}\frac{\sin\frac{1}{2}h}{\frac{1}{2}h}\cdot \underset{h\to 0}{\text{Lt}}\frac{\sin(x+\frac{1}{2}h)}{\cos(x+h).\cos x}$$

$$=1.\frac{\sin x}{\cos x.\cos x}=\frac{\sin x}{\cos^2 x}=\sec x.\tan x.$$

$$\therefore \frac{d}{dx}(\sec x)=\sec x.\tan x. \quad \left[x\neq(2n+1)\frac{\pi}{2}\right]$$

XI. If $y=\text{cosec } x$, then $\dfrac{dy}{dx}=-\text{cosec } x.\cot x$

or, $\dfrac{d}{dx}(\text{cosec } x)=-\text{cosec } x.\cot x$

$$\frac{dy}{dx}=\lim_{h\to 0}\frac{\text{cosec}(x+h)-\text{cosec } x}{h}=\lim_{h\to 0}\frac{\sin x-\sin(x+h)}{h\sin(x+h).\sin x}$$

$$=\lim_{h\to 0}\frac{-2\sin\frac{1}{2}h\cos(x+\frac{1}{2}h)}{h\sin(x+h).\sin x}=-\lim_{h\to 0}\frac{\sin\frac{1}{2}h}{\frac{1}{2}h}\cdot\lim_{h\to 0}\frac{\cos(x+\frac{1}{2}h)}{\sin(x+h)\sin x}$$

$$=-1.\frac{\cos x}{\sin x.\sin x}=-\text{cosec } x.\cot x$$

$$\therefore \frac{d}{dx}(\text{cosec } x)=-\text{cosec } x.\cot x. \quad [x\neq n\pi]$$

Note : (i) The derivatives of the above functions must be remembered as formulas. Some other standard formulas will be established in later articles.

(ii) The derivatives of trigonometric functions prefixed with Co are negative, others are positive. For examples, $\dfrac{d}{dx}\Big(\sin x\Big)=-\cos x$ and $\dfrac{d}{dx}\Big(\cos x\Big)=-\sin x \cos x$ begins with Co.

§ 4.4. **General formulas for determination of differential Coefficients (derivatives of sum, difference, product and quotient of functions.)**

In this article we shall establish formulas expressing derivatives of sum, difference, product and quotients of functions in terms of the derivatives of the individual functions. With the help of these formulas derivatives of sum, difference, product and quotients of elementary functions can be derived easily. You must remember these formulas carefully.

Rule I. If c be a constant, then the derivative of $cf(x)$ is $cf'(x)$, *i.e.* $\frac{d}{dx}\left\{cf(x)\right\}=c\ \frac{d}{dx}\left\{\ (x)\ \right\}$.

Proof : $\frac{d}{dx}\left\{cf(x)\right\}=\lim_{h\to0}\frac{cf(x+h)-cf(x)}{h}$

$$=c\lim_{h\to0}\frac{f(x+h)-f(x)}{h}=cf'(x)=c\ \frac{d}{dx}\left\{f(x)\right\}$$

Examples : $\frac{d}{dx}(3x^2)=3\frac{d}{dx}(x^2)=3.2x^{2-1}=6x.$

$$\frac{d}{dx}\left(\frac{2}{x^2}\right)=\frac{d}{dx}(2x^{-2})=2\left(\frac{d}{dx}(x^{-2})\right)=2.(-2)x^{-2-1}=-\frac{4}{x^3}.$$

$$\frac{d}{dx}(2\tan x)=2\frac{d}{dx}(\tan x)=2\sec^2x.$$

$$\frac{d}{dx}(3\sqrt[3]{x}=3)\ \ \frac{d}{dx}\left(x^{\frac{1}{3}}\right)=3.\tfrac{1}{3}.x^{\frac{1}{3}-1}=x^{-\frac{2}{3}}=\frac{1}{\sqrt[3]{x^2}}.$$

Rule II. If u and v be two functions of x, then

$$\frac{d}{dx}\left(u+v\right)=\frac{du}{dx}+\frac{dv}{dx}\text{ and }\frac{d}{dx}\left(u-v\right)=\frac{du}{dx}-\frac{dv}{dx}.$$

Proof : Let $y=u+v$ and corresponding to the increment Δx of x, increments of y, u and v be respectively Δy, Δu and Δv.

$$\therefore\ \ \Delta y=\Delta u+\Delta v,\text{ or, }\ \frac{\Delta y}{\Delta x}=\frac{\Delta u}{\Delta x}+\frac{\Delta v}{\Delta x}$$

$$\therefore\ \ \frac{dy}{dx}=\lim_{\Delta x\to0}\frac{\Delta y}{\Delta x}=\lim_{\Delta x\to0}\left(\frac{\Delta u}{\Delta x}+\frac{\Delta v}{\Delta x}\right)$$

$$=\lim_{\Delta x\to0}\frac{\Delta u}{\Delta x}+\lim_{\Delta x\to0}\frac{\Delta v}{\Delta x}=\frac{du}{dx}+\frac{dv}{dx}$$

$$\therefore\ \ \frac{d}{dx}\left(u+v\right)=\frac{du}{dx}+\frac{dv}{dx}.$$

Similarly, $\frac{d}{dx}\left(u-v\right)=\frac{du}{dx}-\frac{dv}{dx}.$

In general if $u_1, u_2, u_3, \cdots$ be functions of x and $y = u_1 \pm u_2 \pm u_3 \pm \cdots$, then

$$\frac{dy}{dx} = \frac{du_1}{dx} \pm \frac{du_2}{dx} \pm \frac{du_3}{dx} \pm \cdots$$

i.e., the derivative of the sum or difference of several functions of a variable is equal to the sum or difference of the derivatives of those functions with respect to the variable if each of the derivatives exists.

Ex. $\frac{d}{dx}(x^3 + x^2) = \frac{d}{dx}(x^3) + \frac{d}{dx}(x^2)$

[Here $u = x^3$ and $v = x^2$ and so $y = u + v$]

$$= 3x^{3-1} + 2x^{2-1} = 3x^2 + 2x$$

$$\frac{d}{dx}(x^4 + \sec x) = \frac{d}{dx}(x^4) + \frac{d}{dx}(\sec x) = \{4x^3 + \sec x \tan x\}$$

$$\frac{d}{dx}\left(\frac{x^3 + 1}{x^2}\right) = \frac{d}{dx}(x + x^{-2}) = \frac{d}{dx}(x) + \frac{d}{dx}(x^{-2}) = 1 - 2x^{-3} = 1 - \frac{2}{x^3}$$

$$\frac{d}{dx}\left(8x^5 - \frac{3}{x^2} + 5 + 4\cos x\right) = \frac{d}{dx}\left(8x^5\right) - \frac{d}{dx}\left(\frac{3}{x^2}\right) + \frac{d}{dx}(5) + \frac{d}{dx}(4\cos x)$$

$$= 8\frac{d}{dx}(x^5) - 3\frac{d}{dx}(x^{-2}) + \frac{d}{dx}(5) + 4\frac{d}{dx}(\cos x)$$

$$= 8.5x^4 - 3(-2)x^{-2-1} + 0 + 4(-\sin x)$$

$$= 40x^4 + \frac{6}{x^3} - 4\sin x.$$

$$\frac{d}{dx}\left(x - \frac{1}{x}\right)^3 = \frac{d}{dx}\left(x^3 - 3x + \frac{3}{x} - \frac{1}{x^3}\right)$$

$$= \frac{d}{dx}(x^3) - \frac{d}{dx}(3x) + \frac{d}{dx}\left(\frac{3}{x}\right) - \frac{d}{dx}(x^{-3})$$

$$= 3x^2 - 3.1 + 3.(-1)x^{-2} - (-3)x^{-4}$$

$$= 3x^2 - 3 - \frac{3}{x^2} + \frac{3}{x^4}.$$

Rule III. If $y = uv$ and u and v are both functions of x, then $\frac{dy}{dx} = u.\frac{dv}{dx} + v.\frac{du}{dx}$.

Proof: Let the increments of y, u, v corresponding to the increments $\triangle x$ of x be $\triangle y$, $\triangle u$ and $\triangle v$ respectively.

$$\therefore \quad y + \triangle y = (u + \triangle u).(v + \triangle v)$$

$$\therefore \quad \Delta y=(u+\Delta u)(v+\Delta v)-uv=u\Delta v+v\Delta u+\Delta u\Delta v.$$

$$\therefore \quad \frac{\Delta y}{\Delta x}=u.\ \frac{\Delta v}{\Delta x}=\text{v}.\ \frac{\Delta u}{\Delta x}.+\frac{\Delta u}{\Delta x}.\frac{\Delta v}{\Delta x}.\Delta x.$$

$$\therefore \quad \frac{dy}{dx}=\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\lim_{\Delta x\to 0} u.\frac{\Delta v}{\Delta x}+\lim_{\Delta x\to 0} v.\frac{\Delta u}{\Delta x}$$

$$+\lim_{\Delta x\to 0}\left(\frac{\Delta u}{\Delta x}.\frac{\Delta v}{\Delta x}.\Delta x\right)$$

$$=u.\frac{dv}{dx}+v.\frac{du}{dx}+\frac{du}{dx}.\frac{dv}{dx}.0=u.\ \frac{dv}{dx}+v.\ \frac{du}{dx}.$$

Note. (1) Remember the formula in the following way :

The derivative of the product of two functions.

= first function × derivative of the second + second function × derivative of the first.

(2) Dividing both sides of

$\frac{dy}{dx}=u\frac{dv}{dx}+v\frac{du}{dx}$ by y we get

$$\frac{1}{v}.\frac{dy}{dx}=\frac{1}{u}.\frac{du}{dx}+\frac{1}{v}.\frac{dv}{dx},\text{ or, }\frac{y'}{y}=\frac{u'}{u}+\frac{v'}{v},$$

where, $y'=\frac{dy}{dx}$ etc.

(3) If y be the product of three functions u, v, w, *i.e.*, if $y=u, v, w$, then

$$\frac{dy}{dx}=\frac{d}{dx}(uvw)=u\frac{d}{dx}(vw)+v.w\frac{du}{dx}$$

$$=u\left(v\frac{dw}{dx}+w\ \frac{dv}{dx}\right)+vw\frac{du}{dx}$$

$$=uv\frac{dw}{dx}+uw\frac{dv}{dx}+vw\frac{du}{dx}$$

Dividing both sides by y or u, v, w we get

$$\frac{y'}{y}=\frac{u'}{u}+\frac{v'}{v}+\frac{w'}{w}$$

In general, if $y=u_1, u_2\cdots u_n$, then

$$\frac{dy}{dx}=\frac{du_1}{dx}(u_2\cdots u_n)+\left(u_1\frac{du_2}{dx}\ u_3\cdots u_n\right)$$

$$+\cdots+u_1u_2\cdots u_{n-1}\ \frac{du_n}{dx}$$

$$\text{or, }\frac{y'}{y}=\frac{u'_1}{u_1}+\frac{u'_2}{u_2}+\cdots+\frac{u'_n}{u_n}.$$

Rule IV. If $y=\frac{u}{v}$, where u and v are functions of x

$$\text{then } \frac{dy}{dx}=\frac{v\frac{du}{dx}-u.\frac{dv}{dx}}{v^2}$$

Proof : Here $y=\frac{u}{v}$ $\therefore$ $y+\triangle y=\frac{u+\triangle u}{v+\triangle v}$.

$$\therefore \quad \triangle y=\frac{u+\triangle u}{v+\triangle v}-\frac{u}{v}=\frac{v\triangle u-u\triangle v}{v(v+\triangle v)}$$

$$\therefore \quad \frac{dy}{dx}=\lim_{\triangle x\to 0}\frac{\triangle y}{\triangle x}=\lim_{\triangle x\to 0}\frac{v.\frac{\triangle u}{\triangle x}-u.\frac{\triangle v}{\triangle x}}{v(v+\triangle v)}$$

$$=\frac{\lim_{\triangle x\to 0} v.\frac{\triangle u}{\triangle x}-\lim_{\triangle x\to 0} u.\frac{\triangle v}{\triangle x}}{\lim_{\triangle x\to 0} v(v+\triangle v)}$$

$$=\frac{v.\frac{du}{dx}-u.\frac{dv}{dx}}{v^2}. \quad [\because \text{ When } \triangle x\to 0, \text{ then } \triangle v\to 0]$$

Remember the formula in the following way :

The derivative of the quotient of two functions=

$$\frac{\text{Denominator}\times\text{derivative of the numerator}-\text{Numerator derivative of the denominator}}{\text{square of the denominator}}$$

Cor. Putting $u=1$ in the above formula,

$$\text{we have } \frac{d}{dx}\left(\frac{l}{v}\right)=\frac{v\frac{d}{dx}1-1\frac{d}{dx}(v)}{v^2}=\frac{v0-\frac{dv}{dx}}{v^2}=-\frac{\frac{dv}{dx}}{v^2}$$

$$=-\frac{1}{v^2}\cdot\frac{dv}{dx}.$$

Examples 4A

Ex. 1. Differentiate $y=x^4$ with respect to x from definition.
Here $y=x^4$; $\therefore$ $y+\triangle y=(x+\triangle x)^4$

$$\therefore \quad \frac{dy}{dx}=\underset{\triangle x\to 0}{\text{Lt}}\frac{\triangle y}{\triangle x}=\underset{\triangle x\to 0}{\text{Lt}}\frac{(x+\triangle x)^4-x^4}{\triangle x}$$

$$=\underset{\Delta x\to 0}{\text{Lt}}\ \frac{x^4+4x^3\Delta x+6x^2(\Delta x)^2+4x(\Delta x)^3+(\Delta x)^4-x^4}{\Delta x}$$

$$=\underset{\Delta x\to 0}{\text{Lt}}\ \frac{4x^3\Delta x+6x^2(\Delta x)^2+4x(\Delta x)^3+(\Delta x)^4}{\Delta x}$$

$$=\underset{\Delta x\to 0}{\text{Lt}}\ \frac{\Delta x\{4x^3+6x^2\Delta x+4x(\Delta x)^2+(\Delta x)^3\}}{\Delta x}$$

$$=\underset{\Delta x\to 0}{\text{Lt}}\ \{4x^3+6x^2\Delta x+4x(\Delta x)^2+(\Delta x)^3\}$$

$$=4x^3.$$

Ex. 2. From definition of derivative, show that

$$\frac{d}{dx}\left(\frac{1}{2x+1}\right)=-\frac{2}{(2x+1)^2}.$$

Let $f(x)=\dfrac{1}{2x+1}$

$$\therefore\quad \frac{d}{dx}\left(\frac{1}{2x+1}\right)=\frac{d}{dx}\{f(x)\}$$

$$\therefore\quad \underset{h\to 0}{\text{Lt}}\ \frac{f(x+h)-f(x)}{h}$$

$$=\underset{h\to 0}{\text{Lt}}\ \frac{\dfrac{1}{2(x+h)+1}-\dfrac{1}{2x+1}}{h}$$

$$=\underset{h\to 0}{\text{Lt}}\ \frac{2x+1-2x-2h-1}{h\{2(x+h)+1\}\{2x+1\}}=\underset{h\to 0}{\text{Lt}}\ \frac{-2h}{h\{2(x+h)+1\}\{2x+1\}}$$

$$=\underset{h\to 0}{\text{Lt}}\ \frac{-2}{\{2(x+h)+1\}\{2x+1\}}=-\frac{2}{(2x+1)^2}$$

Ex. 3. Find from the first principle the derivative with respect to x of $f(x)=x^2+2$ at $x=2$.

$$f'(2)=\underset{h\to 0}{\text{Lt}}\ \frac{f(2+h)-f(2)}{h}$$

$$=\underset{h\to 0}{\text{Lt}}\ \frac{(2+h)^2+2-2^2-2}{h}$$

$$=\underset{h\to 0}{\text{Lt}}\ \frac{4+h^2+4h+2-4-2}{h}$$

$$=\underset{h\to 0}{\text{Lt}}\ \frac{h^2+4h}{h}=\underset{h\to 0}{\text{Lt}}\ (h+4)=4.$$

Ex. 4. If $y=\frac{1}{(x+1)^2}$, then find from definition $\left[\frac{dy}{dx}\right]_{x=1}$.

Let $y=f(x)$.

$$\therefore \quad \left[\frac{dy}{dx}\right]_{x=1}=\left[\frac{d}{dx}\{f(x)\}\right]_{x=1}=\underset{h\to 0}{\text{Lt}}\ \frac{f(1+h)-f(1)}{h}$$

$$=\underset{h\to 0}{\text{Lt}}\ \frac{\frac{1}{(1+h+1)^2}-\frac{1}{(1+1)^2}}{h}$$

$$=\underset{h\to 0}{\text{Lt}}\ \frac{\frac{1}{(2+h)^2}-\frac{1}{4}}{h}=\underset{h\to 0}{\text{Lt}}\ \frac{4-(2+h)^2}{h(2+h)^2.4}$$

$$=\underset{h\to 0}{\text{Lt}}\ \frac{4-4-4h-h^2}{h(2+h)^2.4}=\underset{h\to 0}{\text{Lt}}\ \frac{-4h-h^2}{h(2+h)^2.4}$$

$$=\underset{h\to 0}{\text{Lt}}\ \frac{-h(4+h)}{h(2+h)^2.4}=\underset{h\to 0}{\text{Lt}}\ \frac{-(4+h)}{(2+h)^2.4}$$

$$=\frac{-4}{(2)^2.4}=-\tfrac{1}{4}$$

Ex. 5. Find from definition $\frac{dy}{dx}$ if

(i) $y=\frac{1}{x^2}$ (ii) $y=\frac{1}{\sqrt{x}}$

(i) Let $y=f(x)=\frac{1}{x^2}$.

$$\therefore \quad \frac{dy}{dx}=f'(x)=\underset{h\to 0}{\text{Lt}}\ \frac{\frac{1}{(x+h)^2}-\frac{1}{x^2}}{h}$$

$$=\underset{h\to 0}{\text{Lt}}\ \frac{x^2-(x+h)^2}{h(x+h)^2.x^2}$$

$$=\underset{h\to 0}{\text{Lt}}\ \frac{x^2-x^2-2hx-h^2}{h(x+h)^2.x^2}$$

$$=\underset{h\to 0}{\text{Lt}}\ \frac{-h(2x+h)}{h(x+h)^2.x^2}=\underset{h\to 0}{\text{Lt}}\ \frac{-(2x+h)}{(x+h)^2.x^2}$$

$$=\frac{-2x}{x^2\ .\ x^2}=-\frac{2}{x^3}.$$

(ii) Let $y=f(x)=\frac{1}{\sqrt{x}}$

$$\therefore \quad \frac{dy}{dx}=f'(x)=\underset{h\to 0}{\text{Lt}}\ \frac{f(x+h)-f(x)}{h}$$

$$=\underset{h\to 0}{\mathrm{Lt}}\ \frac{\frac{1}{\sqrt{x+h}}-\frac{1}{\sqrt{x}}}{h}=\underset{h\to 0}{\mathrm{Lt}}\ \frac{\sqrt{x}-\sqrt{x+h}}{h\sqrt{x+h}\ \sqrt{x}}$$

$$=\underset{h\to 0}{\mathrm{Lt}}\ \frac{(\sqrt{x}-\sqrt{x+h})(\sqrt{x}+\sqrt{x+h})}{h\sqrt{x+h}\sqrt{x}(\sqrt{x}+\sqrt{x+h})}$$

$$=\underset{h\to 0}{\mathrm{Lt}}\ \frac{x-x-h}{h\sqrt{x+h}\sqrt{x}(\sqrt{x}+\sqrt{x+h})}$$

$$=\underset{h\to 0}{\mathrm{Lt}}\ \frac{-h}{h\sqrt{x+h}\sqrt{x}(\sqrt{x+h}+\sqrt{x})}$$

$$=\underset{h\to 0}{\mathrm{Lt}}\ -\frac{1}{\sqrt{x+h}\sqrt{x}(\sqrt{x}+\sqrt{x+h})}$$

$$=-\frac{1}{\sqrt{x}.\ \sqrt{x}.2\sqrt{x}}=-\frac{1}{2x\sqrt{x}}.$$

Ex. 6. If $f(x)=\frac{1-x^2}{1+x^2}$, then find from definition,

(i) $f'(0)$; (ii) $f'(1)$; (iii) $f'(a)$

(i) $$f'(0)=\underset{h\to 0}{\mathrm{Lt}}\ \frac{f(0+h)-f(0)}{h}=\underset{h\to 0}{\mathrm{Lt}}\ \frac{f(h)-f(0)}{h}$$

$$=\underset{h\to 0}{\mathrm{Lt}}\ \frac{\frac{1-h^2}{1+h^2}-1}{h}=\underset{h\to 0}{\mathrm{Lt}}\ \frac{1-h^2-1-h^2}{h(1+h^2)}$$

$$=\underset{h\to 0}{\mathrm{Lt}}\ \frac{-2h^3}{h(1+h^2)}=\underset{h\to 0}{\mathrm{Lt}}\ \frac{-2h}{(1+h^2)}=0.$$

(ii) $$f'(1)=\underset{h\to 0}{\mathrm{Lt}}\ \frac{f(1+h)-f(1)}{h}$$

$$=\underset{h\to 0}{\mathrm{Lt}}\ \frac{\frac{1-(1+h)^2}{1+(1+h)^2}-\frac{1-1^2}{1+1^2}}{h}=\underset{h\to 0}{\mathrm{Lt}}\ \frac{1-1-2h-h^2}{h\{1+(1+h)^2\}}$$

$$=\underset{h\to 0}{\mathrm{Lt}}\ \frac{-h(2+h)}{h\{1+(1+h)^2\}}=\underset{h\to 0}{\mathrm{Lt}}.\ \frac{-(2+h)}{1+(1+h)^2}$$

$$=\frac{-2}{1+1}=-1.$$

(iii) $$f'(a)=\underset{h\to 0}{\mathrm{Lt}}\ \frac{f(a+h)-f(a)}{h}$$

$$=\underset{h\to 0}{\mathrm{Lt}}\ \frac{\frac{1-(a+h)^2}{1+(a+h)^2}-\frac{1-a^2}{1+a^2}}{h}$$

$$=\underset{h\to 0}{\text{Lt}}\left[\frac{1}{h}\frac{(1+a^2)\{1-(a+h)^2\}-(1-a^2)\{1+(a+h)^2\}}{\{1+(a+h)^2\}\{1+a^2\}}\right]$$

$$=\underset{h\to 0}{\text{Lt}}\left[\frac{1}{h}\frac{2a^2-2(a+h)^2}{\{1+(a+h)^2\}\{1+a^2\}}\right]=\underset{h\to 0}{\text{Lt}}\left[\frac{1}{h}\frac{-4ah-4h^2}{\{1+(a+h)^2\}\{1+a^2\}}\right]$$

$$=\underset{h\to 0}{\text{Lt}}\left[\frac{-4a-4h}{\{1+(a+h)^2\}\{1+a^2\}}\right]=\frac{-4a}{(1+a^2)^2}.$$

Ex. 7. If $f(x)=e^{\sin x}$, then find from definition $f'(x)$.

$$f'(x)=\underset{h\to 0}{\text{Lt}}\frac{f(x+h)-f(x)}{h}$$

$$=\underset{h\to 0}{\text{Lt}}\frac{e^{\sin (x+h)}-e^{\sin x}}{h}$$

$$=\underset{h\to 0}{\text{Lt}}\left[e^{\sin x}\left\{\frac{e^{\sin (x+h)-\sin x}-1}{h}\right\}\right]$$

$$=e^{\sin x}.\underset{h\to 0}{\text{Lim}}\frac{e^k-1}{k}\text{ where } k=\sin (x+h)-\sin x.$$

$$=e^{\sin x}\underset{h\to 0}{\text{Lim}}\left\{\frac{e^k-1}{k}.\frac{k}{h}\right\}$$

$$=e^{\sin x}\underset{h\to 0}{\text{Lim}}\frac{e^k-1}{k}.\underset{h\to 0}{\text{Lim}}\frac{k}{h}.$$

$$=e^{\sin x}\underset{k\to 0}{\text{Lim}}\frac{e^k-1}{k},\underset{h\to 0}{\text{Lim}}\frac{\sin (x+h)-\sin x}{h}$$

[when $h\to 0$, then $k=\{\sin (x+h)-\sin x\}\to 0$].

$$=e^{\sin x}.1.\underset{h\to 0}{\text{Lim}}\frac{2\cos (x+\frac{h}{2})\sin\frac{h}{2}}{h}$$

$$=e^{\sin x}\underset{h\to 0}{\text{Lim}}\cos (x+\tfrac{h}{2})\underset{h\to 0}{\text{Lim}}\frac{\sin\frac{h}{2}}{\frac{h}{2}}$$

$$=e^{\sin x}.\cos x.1=e^{\sin x}.\cos x.$$

Ex. 8. Find from definition $\frac{d}{dx}(\tan^{-1}x)$.

Let $y=\tan^{-1}x$ and $y+k=\tan^{-1}(x+h)$

$\therefore\ \tan y=x$ and $\tan (y+k)=x+h$

Also $\tan^{-1}(x+h)-\tan^{-1}x=k$

$$\therefore\ \frac{\tan^{-1}(x+h)-\tan^{-1}x}{h}=\frac{k}{h}$$

$$\therefore \quad \frac{dy}{dx}=\underset{h\to 0}{\text{Lt}}\ \frac{k}{h}=\underset{h\to 0}{\text{Lt}}\ \frac{k}{\tan(y+k)-\tan y}$$

$$=\underset{k\to 0}{\text{Lt}}\ \frac{k}{\dfrac{\sin(y+k)}{\cos(y+k)}-\dfrac{\sin\ y}{\cos\ y}}$$

[when $h\to 0$, then $k=\tan^{-1}(x+h)-\tan^{-1}x\to 0$. (assuming that $\tan^{-1}x$ is continuous)].

$$=\underset{k\to 0}{\text{Lt}}\ \frac{\cos(y+k)\cos y.k}{\sin(y+k)\cos y-\cos(y+k)\sin y}$$

$$=\underset{k\to 0}{\text{Lt}}\ \frac{\cos(y+k).\cos y.k}{\sin(y+k-y)}$$

$$=\underset{k\to 0}{\text{Lt}}\ \frac{\cos(y+k).\cos y}{\dfrac{\sin k}{k}}$$

$$=\frac{\underset{k\to 0}{\text{Lt}}\ \cos(y+k).\ \cos y}{\underset{k\to 0}{\text{Lt}}\ \dfrac{\sin k}{k}}=\frac{\cos^2 y}{1}=\cos^2 y$$

$$=\frac{1}{\sec^2 y}=\frac{1}{1+\tan^2 y}=\frac{1}{1+x^2}.$$

Ex. 9. Find from definition the derivative of x^x with respect to x.

Let $y=x^x=e^{\log x^x}=e^{x\log x}$.

$$\therefore \quad \frac{dy}{dx}=\underset{h\to 0}{\text{Lt}}\ \frac{e^{(x+h)\log(x+h)}-e^{x\log x}}{h}$$

$$=\underset{h\to 0}{\text{Lt}}\left\{e^{x\log x}\frac{e^{(x+h)\log(x+h)-x\log x}-1}{h}\right\}$$

$$=e^{x\log x}\cdot\underset{h\to 0}{\text{Lt}}\ \frac{e^k-1}{h},$$

[where $k=(x+h)\log(x+h)-x\log x$.
Also as $h\to 0$, then $k\to 0$ assuming $\log x$ to be continuous].

$$=e^{x\log x}\ \underset{h\to 0}{\text{Lt}}\left\{\frac{e^k-1}{k}\cdot\frac{k}{h}\right\}$$

$$=e^{\log x^x}\ \underset{k\to 0}{\text{Lt}}\ \frac{e^k-1}{k}\cdot\underset{h\to 0}{\text{Lt}}\ \frac{(x+h)\log(x+h)-x\log x}{h}$$

$$=x^x.1.\lim_{h\to 0}\frac{x\{\log(x+h)-\log x\}+h\log(x+h)}{h}$$

$$=x^x.1\left[\lim_{h\to 0}x\frac{\log(x+h)-\log(x)}{h}+\lim_{h\to 0}\log(x+h)\right]$$

$$=x^x\left[\lim_{h\to 0}x\;\frac{\log\left(\frac{x+h}{x}\right)}{h}+\log x\right]$$

$$=x^x.\left[\lim_{h\to 0}\frac{\log\left(1+\frac{h}{x}\right)}{\frac{h}{x}}+\log x\right]=x^x(1+\log x).$$

Ex. **10.** If $y=\log\sin x$, find from definition $\frac{dy}{dx}$.

Let $z=\sin x$ and $\sin(x+h)=z+k$

$\therefore\;\; k=\sin(x+h)-\sin x\to 0$ as $h\to 0$.

Now, $$\frac{dy}{dx}=\lim_{h\to 0}\frac{\log\sin(x+h)-\log x}{h}$$

$$=\lim_{h\to 0}\frac{\log(z+k)-\log z}{h}$$

$$=\lim_{h\to 0}\frac{\log\frac{z+k}{z}}{h}$$

$$=\lim_{h\to 0}\left\{\frac{\log\left(1+\frac{k}{z}\right)}{k}\cdot\frac{k}{h}\right\}$$

$$=\lim_{h\to 0}\left\{\frac{\log\left(1+\frac{k}{z}\right)}{\frac{k}{z}}\cdot\frac{1}{z}\cdot\frac{k}{h}\right\}$$

$$=\lim_{k\to 0}\frac{\log\left(1+\frac{k}{z}\right)}{\frac{k}{z}}\cdot\lim_{k\to 0}\frac{1}{\sin x}\cdot\lim_{k\to 0}\frac{\sin(x+h)-\sin x}{h}$$

$$=1.\frac{1}{\sin x}\lim_{h\to 0}\frac{2\cos(x+\frac{h}{2})\sin\frac{h}{2}}{h}$$

$$=\frac{1}{\sin x}\cdot\lim_{h\to 0}\cos(x+\tfrac{h}{2}).\lim_{h\to 0}\frac{\sin\frac{h}{2}}{\frac{h}{2}}$$

$$=\frac{1}{\sin x}.\cos x.1=\cot x.$$

Ex. 11. If $s=ut+\frac{1}{2}ft^2$, where u and f are constants, find $\frac{ds}{dt}$.

Here $s=\phi(t)=ut+\frac{1}{2}ft^2$.

$$\therefore \quad \frac{ds}{dt}=\underset{h\to 0}{\text{Lt}}\ \frac{\phi(t+h)-\phi(t)}{h}$$

$$=\underset{h\to 0}{\text{Lt}}\ \frac{u(t+h)+\frac{1}{2}f(t+h)^2-ut-\frac{1}{2}ft^2}{h}$$

$$=\underset{h\to 0}{\text{Lt}}\ \frac{uh+fth+\frac{1}{2}h^2}{h}=\underset{h\to 0}{\text{Lt}}\left(u+ft+\tfrac{1}{2}h\right)$$

$$=u+ft.$$

Ex. 12. If $f(x)=|\,x\,|$, show that $f'(0)$ does not exist.

$$f'(0+)=\underset{h\to 0+}{\text{Lt}}\ \frac{f(0+h)-f(0)}{h}=\underset{h\to 0+}{\text{Lt}}\ \frac{f(h)-f(0)}{h}$$

$$=\underset{h\to 0}{\text{Lt}}\ \frac{|\,h\,|-0}{h}=\underset{h\to 0}{\text{Lt}}\ \frac{|\,h\,|}{h}=\underset{h\to 0}{\text{Lt}}\ \frac{h}{h}$$

$$=\underset{h\to 0}{\text{Lt}}\ (1)=1.$$

$$f'(0-)=\underset{h\to 0-}{\text{Lt}}\ \frac{f(0+h)-f(0)}{h}=\underset{h\to 0-}{\text{Lt}}\ \frac{f(h)-f(0)}{h}$$

$$=\underset{h\to 0-}{\text{Lt}}\ \frac{-h-0}{h} \qquad \left[\text{ as } h\to 0-,\text{ so } h<0 \quad \therefore\ |\,h\,|=-h\ \right]$$

$$=\underset{h\to 0-}{\text{Lt}}\ (-1)=-1$$

$\therefore \quad f'(0+)\neq f'(0-)$

$\therefore \quad f'(0)$ does not exist.

Ex. 13. $f(x)=x$ when $x\geqslant 1$

$\qquad\qquad =2-x$ when $x<1$

show that $f(x)$ is continuous at $x=1$ but not differentiable at $x=1$.

$$\underset{x\to 1+}{\text{Lt}}\ f(x)=\underset{x\to 1+}{\text{Lt}}\ (x)=1$$

$$\underset{x\to 1-}{\text{Lt}}\ f(x)=\underset{x\to 1-}{\text{Lt}}\ (2-x)=1$$

$$\therefore \quad \underset{x\to 1+}{\text{Lt}}\ f(x)=\underset{x\to 1-}{\text{Lt}}\ f(x)=1$$

$$\therefore \quad \underset{x\to 1}{\text{Lt}}\ f(x)=1.$$

Also $f(1)=1$

$$\therefore \quad \underset{x\to 1}{\text{Lt}}\ f(x)=f(1).$$

So, $f(x)$ is continuous at $x=1$

Now, $f'(1+)=\underset{h\to0+}{\text{Lt}}\ \dfrac{f(1+h)-f(1)}{h}$

$=\underset{h\to0+}{\text{Lt}}\ \dfrac{1+h-1}{h}=\underset{h\to0+}{\text{Lt}}\ \dfrac{h}{h}=\underset{h\to0+}{\text{Lt}}\ (1)=1$

$f'(1-)=\underset{h\to0-}{\text{Lt}}\ \dfrac{f(1+h)-f(1)}{h}$

$=\underset{h\to0-}{\text{Lt}}\ \dfrac{2-1-h-1}{h}=\underset{h\to0-}{\text{Lt}}\ \dfrac{-h}{h}=\underset{h\to0-}{\text{Lt}}\ (-1)=-1$

$\therefore\ f'(1+)\neq f'(1-)$

$\therefore\ f'(1)$ does not exist.

Ex. 14. Differentiate the following functions with respect to x :

(i) x^3+x^2 ; (ii) $x^4+\sec x$; (iii) $\left(\dfrac{x^3+1}{x^2}\right)$;

(iv) $\left(8x^5-\dfrac{3}{x^2}+5+4\cos x\right)$; (v) $\left(x-\dfrac{1}{x}\right)^3$.

(i) $\dfrac{d}{dx}(x^3+x^2)=\dfrac{d}{dx}(x^3)+\dfrac{d}{dx}(x^2)=3x^2+2x.$

(ii) $\dfrac{d}{dx}(x^4+\sec x)=\dfrac{d}{dx}(x^4)+\dfrac{d}{dx}(\sec x)$

$=4x^3+\sec x\tan x.$

(iii) $\dfrac{d}{dx}\left(\dfrac{x^3+1}{x^2}\right)=\dfrac{d}{dx}\left(x+\dfrac{1}{x^2}\right)=\dfrac{d}{dx}(x)+\dfrac{d}{dx}\left(\dfrac{1}{x^2}\right)$

$=1+\dfrac{d}{dx}(x^{-2})=1+\{-2.x^{-2-1}\}=1-2x^{-3}$

$=1-\dfrac{2}{x^3}.$

(iv) $\dfrac{d}{dx}\left\{8x^5-\dfrac{3}{x^2}+5+4\cos x\right\}$

$=\dfrac{d}{dx}\{8x^5\}-\dfrac{d}{dx}\left(\dfrac{3}{x^2}\right)+\dfrac{d}{dx}(5)+\dfrac{d}{dx}(4\cos x)$

$=8\dfrac{d}{dx}(x^5)-3\dfrac{d}{dx}(x^{-2})+0+4\dfrac{d}{dx}(\cos x)$

$=8.5x^4-3.(-2x^{-3})+4(-\sin x)$

$=40x^4+\dfrac{6}{x^3}-4\sin x.$

(v) $\frac{d}{dx}\left(x-\frac{1}{x}\right)^3=\frac{d}{dx}\left(x^3-3x+\frac{3}{x}-\frac{1}{x^3}\right)$

$$=\frac{d}{dx}(x^3)-3\frac{d}{dx}(x)+3\frac{d}{dx}(x^{-1})-\frac{d}{dx}(x^{-3})$$

$$=3x^2-3.1+3(-x^{-2})-(-3x^{-4})$$

$$=3x^2-3-\frac{3}{x^2}+\frac{3}{x^4}.$$

Ex. 15. Find the derivatives of the following functions with respect to x :

(i) $x^2 \sin x$; (ii) $(2x+1)^2e^x$; (iii) $\sin x \cos x$; (iv) $\sin^2 x$;
(v) $(x^2+1)(3x^3+1)$; (vi) $x^2e^x \tan x$; (vii) $x \tan x-e^x\dfrac{x^2+1}{\sqrt{x}}$.

(i) $\frac{d}{dx}(x^2 \sin x)=\sin x\frac{d}{dx}(x^2)+x^2 \frac{d}{dx}(\sin x)=2x \sin x+x^2 \cos x.$

(ii) $\frac{d}{dx}\{(2x+1)^2e^x\}$

$$=\frac{d}{dx}\{(4x^2+4x+1)e^x\}$$

$$=e^x\frac{d}{dx}(4x^2+4x+1)+(4x^2+4x+1)\frac{d}{dx}(e^x)$$

$$=e^x\left\{\frac{d}{dx}(4x^2)+\frac{d}{dx}(4x)+\frac{d}{dx}(1)\right\}+(4x^2+4x+1)e^x$$

$$=e^x(8x+4)+(4x^2+4x+1)e^x$$

$$=e^x(4x^2+12x+5).$$

(iii) $\frac{d}{dx}(\sin x \cos x)=\cos x\frac{d}{dx}(\sin x)+\sin x\frac{d}{dx}(\cos x)$

$$=\cos x.\cos x+\sin x(-\sin x)=\cos^2 x-\sin^2 x=\cos 2x.$$

(iv) $\frac{d}{dx}(\sin^2 x)=\frac{d}{dx}(\sin x.\sin x)$

$$=\sin x\frac{d}{dx}(\sin x)+\sin x\frac{d}{dx}(\sin x)=\sin x.\cos x+\sin x.\cos x$$

$$=2\sin x\cos x=\sin 2x.$$

(v) $\frac{d}{dx}\{(x^2+1)(3x^3+1)\}=(3x^3+1)\frac{d}{dx}(x^2+1)+(x^2+1)\frac{d}{dx}(3x^3+1)=(3x^3+1).2x+(x^2+1).9x^2$

$$=15x^4+9x^2+2x.$$

(vi) $\frac{d}{dx}(x^2 e^x \tan x)$

$=\tan x \frac{d}{dx}(x^2 e^x)+x^2 e^x \frac{d}{dx}(\tan x)$

$=\tan x\left\{e^x \frac{d}{dx}(x^2)+x^2 \frac{d}{dx}(e^x)\right\}+x^2 e^x.\sec^2 x$

$=\tan x\{e^x.2x+x^2.e^x\}+x^2 e^x.\sec^2 x$

$=e^x(2x \tan x+x^2 \tan x+x^2 \sec^2 x)$

(vii) $\frac{d}{dx}\left\{x \tan x-e^x \frac{x^2+1}{\sqrt{x}}\right\}$

$=\frac{d}{dx}(x \tan x)-\frac{d}{dx}\{e^x(x^{\frac{3}{2}}+x^{-\frac{1}{2}})\}$

$=x\frac{d}{dx}(\tan x)+\tan x\frac{d}{dx}(x)-e^x \frac{d}{dx}(x^{\frac{3}{2}}+x^{-\frac{1}{2}})+(x^{\frac{3}{2}}+x^{-\frac{1}{2}})\frac{d}{dx}(e^x)$

$=x \sec^2 x+\tan x . 1-e^x(\frac{3}{2}x^{\frac{1}{2}}-\frac{1}{2}x^{-\frac{3}{2}})+(x^{\frac{3}{2}}+x^{-\frac{1}{2}})e^x$

$=x \sec^2 x+\tan x-e^x\left(x^{\frac{3}{2}}+\frac{1}{\sqrt{x}}+\frac{3}{2}\sqrt{x}-\frac{1}{2}\frac{1}{x\sqrt{x}}\right)$

Ex. 16. Find the derivatives of the following functions with respect to x :

(i) $\frac{x^2}{\sin x}$; (ii) $\left(\frac{x^2+1}{x^2-x+1}\right)$; (iii) $\frac{x \tan x}{x^2+1}$; (iv) $\frac{1}{x+1}$;

(v) $\frac{1}{ax^2+b}$; (vi) $\frac{1}{(x+1)(x^2+1)}$.

(i) $\frac{d}{dx}\left\{\frac{x^2}{\sin x}\right\}=\frac{\sin x\frac{d}{dx}(x^2)-x^2\frac{d}{dx}(\sin x)}{\sin^2 x}$

$=\frac{\sin x . 2x-x^2 \cos x}{\sin^2 x}=\frac{2x \sin x-x^2 \cos x}{\sin^2 x}$

(ii) $\frac{d}{dx}\left(\frac{x^2+1}{x^2-x+1}\right)=\frac{(x^2-x+1) \frac{d}{dx}(x^2+1)-(x^2+1)\frac{d}{dx}(x^2-x+1)}{(x^2-x+1)^2}$

$=\frac{(x^2-x+1).2x-(x^2+1)(2x-1)}{(x^2-x+1)^2}$

$=\frac{2x^3-2x^2+2x-2x^3-2x+x^2+1}{(x^2-x+1)^2}=\frac{1-x^2}{(x^2-x+1)^2}$

(iii) $\frac{d}{dx}\left(\frac{x \tan x}{x^2+1}\right)=\frac{(x^2+1)\frac{d}{dx}(x \tan x)-x \tan x \frac{d}{dx}(x^2+1)}{(x^2+1)^2}$

$$=\frac{(x^2+1)\left\{x\frac{d}{dx}(\tan x)+\tan x\frac{d}{dx}(x)-x\tan x\,(2x)\right.}{(x^2+1)^2}$$

$$=\frac{(x^2+1)(x\sec^2 x+\tan x)-2x^2\tan x}{(x^2+1)^2}$$

$$=\frac{x\sec^2 x(x^2+1)+\tan x(1-x^2)}{(x^2+1)^2}$$

(iv) $\frac{d}{dx}\left(\frac{1}{x+1}\right)=\frac{d}{dx}\left(\frac{1}{v}\right)$ [$v=x+1$ (say)]

$=-\frac{1}{v^2}\frac{dv}{dx}$ [Cor. Formula (iv)]

$$=-\frac{1}{(x+1)^2}\frac{d}{dx}(x+1)=-\frac{1}{(x+1)^2}.1=-\frac{1}{(x+1)^2}.$$

(v) $\frac{d}{dx}\left(\frac{1}{ax^2+b}\right)=\frac{d}{dx}\left(\frac{1}{v}\right)$ [$v=ax^2+b$ (say)]

$$=-\frac{1}{v^2}\frac{dv}{dx}=-\frac{1}{v^2}\frac{d}{dx}(ax^2+b)$$

$$=-\frac{1}{(ax^2+b)^2}(2ax)=-\frac{2ax}{(ax^2+b)^2}.$$

(vi) $\frac{d}{dx}\left\{\frac{1}{(x+1)(x^2+1)}\right\}$

$$=-\frac{1}{(x+1)^2(x^2+1)^2}\frac{d}{dx}\{(x+1)(x^2+1)\}$$

$$=-\frac{1}{(x+1)^2(x^2+1)^2}\frac{d}{dx}(x^3+x^2+x+1)$$

$$=-\frac{1}{(x+1)^2(x^2+1)^2}(3x^2+2x+1)$$

$$=-\frac{3x^2+2x+1}{(x+1)^2(x^2+1)^2}.$$

Ex. 17. Express $\tan x$ as $\frac{\sin x}{\cos x}$ and hence show that $\frac{d}{dx}(\tan x)$ $=\sec^2 x.$

$$\frac{d}{dx}(\tan x)=\frac{d}{dx}\left(\frac{\sin x}{\cos x}\right)=\frac{\cos x\frac{d}{dx}(\sin x)-\sin x\frac{d}{dx}(\cos x)}{\cos^2 x}$$

$$=\frac{\cos x.\cos x-\sin x\,(-\sin x)}{\cos^2 x}=\frac{\cos^2 x+\sin^2 x}{\cos^2 x}=\frac{1}{\cos^2 x}=\sec^2 x.$$

Ex. 18. If $y=\{x^2-e^x \tan x\}/\{(x+1)e^x+4x^2\}$, find $\frac{dy}{dx}$.

$$\frac{dy}{dx}=\frac{d}{dx}\left\{\frac{x^2-e^x \tan x}{(x+1)e^x+4x^2}\right\}$$

$$=\frac{\{(x+1)e^x+4x^2\}\frac{d}{dx}(x^2-e^x \tan x)-(x^2-e^x \tan x)\frac{d}{dx}\{(x+1)e^x+4x^2\}}{\{(x+1)e^x+4x^2\}^2}$$

$$=\frac{\{(x+1)e^x+4x^2\}\{2x-e^x \tan x-e^x \sec^2 x\} -(x^2-e^x \tan x)\{(x+1)e^x+e^x+8x\}}{\{(x+1)e^x+4x^2\}^2}$$

Exercise 4A

Find from the first principle the derivatives of the following functions with respect to x(1 – 7).

1. (i) $2x$; (ii) $4x^2+2$; (iii) $\frac{1}{3}x^3$; (iv) ax^2+b ;
(v) $\frac{1}{x-1}$; (vi) $\frac{1}{ax+b}$ (a and b constants) ;
(vii) $\frac{1}{x^2}$; (vii) x^3+x ; (ix) $\sqrt{x}$; (x) $\sin 2x$, when $x=4$.

2. (i) If $y=\sqrt{x+1}$; find $\frac{dy}{dx}$ when $x=3$

(ii) If $y=\frac{2}{x}$, find $\frac{dy}{dx}$ when $x=-1$.

3. (i) If $y=x^2+1$, then find $\left(\frac{dy}{dx}\right)_{x=1}$

(ii) If $y=x+\frac{1}{x}$, then find $\left(\frac{dy}{dx}\right)_{x=1}$

(iii) If $y=\frac{x-1}{x+1}$, then find $\left(\frac{dy}{dx}\right)_{x=0}$.

4. If $f(x)=\frac{2x}{x^2-1}$, then find the values of
(i) $f'(0)$; (ii) $f'(2)$; (iii) $f'(a)$.

5. If $y=e^{x^2}$, then find $\frac{dy}{dx}$.

6. If $y=\sqrt{\frac{1+x}{1-x}}$, then find $\frac{dy}{dx}$.

7. Find $\frac{dy}{dx}$ if

(i) $y=ax^2+bx+c$; (ii) $y=\sqrt{x^2+1}$; (iii) $y=\frac{1}{x^2}$;

(iv) $y=(2x+3)^2$; (v) $y=\frac{1}{x+1}+e^x$;

(vi) $y=\sqrt{ax^2+b}$.

8. If $s=3t^2+4t$, then find $\frac{ds}{dt}$.

9. If $x=\sin t+t^2$, find form the first principle $\frac{dx}{dt}$.

10. (*a*) Examine the continuity and differentiability of the function $f(x)$ at $x=\frac{1}{2}$.

$$\begin{aligned} f(x)&=0 \text{ when } 0\leqslant x<\tfrac{1}{2} \\ &=1 \text{ when } x=\tfrac{1}{2} \\ &=2 \text{ when } \tfrac{1}{2}<x\leqslant 1. \end{aligned}$$

(*b*) Examine the continuity and differentiability of the function $f(x)$ at $x=0$.

$$\begin{aligned} f(x)&=2+x \text{ when } x\geqslant 0 \\ &=2-x \text{ when } x\leqslant 0. \end{aligned}$$

11. Examine the continuity and differentiability of the function $f(x)$ at $x=a$.

$$\begin{aligned} f(x)&=\frac{x^2}{a^2}-a \text{ when } 0<x<a \\ &=a \text{ when } x=a, \\ &=a-\frac{x^2}{a} \text{ when } x>a. \end{aligned}$$

12. If $f(x)=x$ when $0<x<1$

$$\begin{aligned} &=2-x \text{ when } 1\leqslant x\leqslant 2 \\ &=3x-x^2-2 \text{ when } x>2, \end{aligned}$$

then show that at $x=1$, $f(x)$ is continuous but not differentiable and at $x=2$, $f(x)$ is both continuous and differentiable.

13. If $f(x)=\frac{1}{2}(b^2-a^2)$ when $0\leqslant x\leqslant a$

$$\begin{aligned} &=\tfrac{1}{2}b^2-\tfrac{1}{6}x^2-\tfrac{1}{3}\frac{a^3}{x} \text{ when } a<x\leqslant b \\ &=\tfrac{1}{3}\frac{b^3-a^3}{x^3} \text{ when } x>b. \end{aligned}$$

Then find $f'(x)$ and show that $f'(x)$ is continuous everywhere $(x\geqslant 0)$.

14. $f(x)=x^2 \sin\left(\frac{1}{x}\right)$ when $x\neq 0$

$=0$ when $x=0$

Show that $f'(0)=0$.

15. Find $\frac{dy}{dx}$ if $y=$

(i) x^2 (ii) x^{99} (iii) x^{1000} (iv) $x^{2\cdot 5}$ (v) $x^{1\cdot 5}$

(vi) x^{-3} (vii) $x^{-1\cdot 5}$ (viii) $\frac{1}{x^3}$ (ix) $\frac{1}{x^{26}}$ (x) $\frac{1}{x^{2\cdot 2}}$

(xi) $\sqrt{x}$ (xii) $\sqrt{x^3}$ (xiii) $\sqrt[n]{x}$ (xiv) $\frac{1}{\sqrt[3]{x^4}}$ (xv) $\frac{x^2}{\sqrt[3]{x}}$

(xvi) x^e.

16. Find the derivatives of the following functions with respect to x :

(i) $4x^3$ (ii) $2\sqrt{x}$ (iii) $5 \cos x$ (iv) $\frac{3}{4x^3}$

(v) $\frac{2x^2}{3\sqrt{x}}$ (vi) $\frac{2(x^2-1)}{x^4-x^2}$ (vii) 2^{x+2}.

17. Differentiate the following functions with respect to x :

(i) $4x^2+5x$ (ii) $\frac{3}{x}+2x^2$ (iii) $3x^2+x-1$

(iv) ax^2+bx+c where a, b, c are constants.

(v) $x\sqrt{x}+\frac{1}{x\sqrt{x}}-\sec x$ (vi) $4\sqrt{x}+\sqrt{8}+\frac{1}{2}\cot x$

(vii) $(x+1)(x+2)$ (viii) $x^{2n}-nx^2+6^n$

(ix) $\frac{1-x}{\sqrt{x}}$ (x) $\frac{x+x^{\frac{1}{3}}}{x^{\frac{1}{6}}}$ (xi) $\frac{x^2+x+1}{\sqrt{x}}$

(xii) $\sin x+2\cos x+3\tan x+4\cot x+5\sec x+6\operatorname{cosec} x-e^x$

(xiii) $(\sqrt{x}-\sqrt{a})^2$ (xiv) $\frac{(1-\sqrt{x})^3}{x}+\operatorname{cosec} x$.

18. Differentiate the following functions with respect to x :

(i) $(2x+1)(3-2x^2)$ (ii) $(5-3x^2)(2-3x^2)$

(iii) $(x^2+x+1)(x^2-x+1)$ (iv) $x^2 \sec x$

(v) $\frac{x^2+3x+1}{x}$ (vi) $e^x \sin x-\cos x+4$

(vii) $\sec x \tan x$ (viii) $\frac{1}{\sin x \cos x}$ (ix) $\tan^2 x$

(x) $x^4(1+\sqrt{x})$ (xi) $x \operatorname{cosec} x$ (xii) $x^n \cot x$

(xiii) $x \operatorname{cosec}^2 x$ (xiv) $x^2 e^x \cos x$

(xv) $(x^3-2x\cos x+4)(x^2+4x-e^x)$

(xvi) $\dfrac{x}{\sin x}$ (xvii) $\dfrac{\cos x}{x}$ (xviii) $\dfrac{x^2}{\tan x}$.

19. Express $\tan x = \sin x \sec x$ and hence use the product formula of finding the derivative of a function to determine $\dfrac{d}{dx}(\tan x)$.

20. Find the derivatives of the following functions with respect to x :

(i) $\dfrac{\tan x}{x}$ (ii) $\dfrac{x-1}{x+1}$ (iii) $\dfrac{x^2}{x-4}$ (iv) $\dfrac{\sqrt{x}-1}{\sqrt{x}+1}$

(v) $\dfrac{x^{\frac{1}{2}}+2}{x^{\frac{3}{2}}+1}$ (vi) $\dfrac{ax+b}{cx+d}$; a, b, c, d constants.

(vii) $\dfrac{ax^2+2bx+c}{ax^2-2bx+c}$ (viii) $\dfrac{1}{2x+1}$ (ix) $\dfrac{1}{ax+b}$

(x) $\dfrac{1}{x^3-1}$ (xi) $\dfrac{e^x+\cos x}{xe^x+1}$ (xii) $\dfrac{x^3-1}{x^2+1}$

(xiii) $\dfrac{(x+1)(x+2)+\cot x}{(x^2-x+1)e^x}$ (xiv) $\dfrac{1}{x-1}+\dfrac{1}{x+1}$

(xv) $\sqrt{x}\left(\dfrac{1}{x}+\sin x\right)\dfrac{1}{x^2+1}$.

§ 4·5. Method of differentiation of function of a function.

Let $z=\phi(x)$ be a function of x and $y=f(z)$ be a function of z. Now substituting the value of z, $y=f(z)=f\{\phi(x)\}$. So, y is a function of x. y is called a function of a function of x.

Now to find the derivative of y with respect to x, let the increments of z and y corresponding to the increment Δx of x be Δz and Δy respectively.

Hence $z+\Delta z=\phi(x+\Delta x)$ and

$$y+\Delta y=f\{\phi(x+\Delta x)\}=f(z+\Delta z)$$

$$\therefore \quad \Delta y=f(z+\Delta z)-f(z)$$

and $\Delta z=\phi(x+\Delta x)-\phi(x)\to 0$ as $\Delta x\to 0$

$$\therefore \quad \frac{dy}{dx}=\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\lim_{\Delta x\to 0}\left\{\frac{\Delta y}{\Delta z}\,\frac{\Delta z}{\Delta x}\right\}$$

[Taking $\Delta z\neq 0$]

$$=\lim_{\Delta z\to 0}\frac{f(z+\Delta z)-f(z)}{\Delta z}\lim_{\Delta\to 0}\frac{\Delta z}{\Delta x}$$

[$\because$ when $\Delta x\to 0$ then $\Delta z\to 0$]

$$=\frac{d}{dz}\{f(z)\}\frac{dz}{dx}=\frac{dy}{dz}.\frac{dz}{dx}.$$

Hence the rate of change of y with respect to x is the product of the rate of change of y with respect to z and the rate of change of z with respect to x.

Similarly if, $y=f(u)$, $u=h(v)$ and $v=\phi(x)$ be three functions, i.e., if $y=f[h\{(x)\}]$, then

$$\frac{dy}{dx}=\frac{dy}{du}.\frac{du}{dv}.\frac{dv}{dx}.$$

In general, if $y=f_1(u_1)$, $u_1=f_2(u_2),\ldots\ldots u_{n-1}=f_{n-1}(u_n)$ and $u_n=f_n(x)$ then

$$\frac{dy}{dx}=\frac{dy}{du_1}.\frac{du_1}{du_2}\ldots\ldots\frac{du_{n-1}}{du_n}.\frac{du_n}{dx}.$$

Note : (i) Here the functions have been assumed to be differentiable. (ii) The above rule is called the chain rule.

Ex. 1. Find $\frac{dy}{dx}$ when

(i) $y=(1+x^2)^3$ (ii) $y=\sqrt{1-x^2}$

(iii) $y=(ax^2+bx+c)^{\frac{3}{2}}$ (iv) $y=\sin^2 x$

(v) $y=\sin(x^2)$ (vi) $y=\log(\sin x)$

(i) $y=(1+x^2)^3$ Let $u=1+x^2$ $\therefore$ $y=u^3$.

$$\therefore\quad \frac{dy}{du}=3u^2.\qquad \frac{du}{dx}=0+2x=2x$$

$$\therefore\quad \frac{dy}{dx}=\frac{dy}{du}\frac{du}{dx}=3u^2.2x=6x(1+x^2)^2$$

[putting the value of u]

(ii) $y=\sqrt{1-x^2}=\sqrt{u}$ where $u=1-x^2$

$$\therefore\quad \frac{dy}{du}=\frac{d}{du}\left(u^{\frac{1}{2}}\right)=\tfrac{1}{2}u^{\frac{1}{2}-1}=\tfrac{1}{2}.\frac{1}{\sqrt{u}},\text{ and }\frac{du}{dx}=0-2x=-2x$$

$$\therefore\quad \frac{dy}{dx}=\frac{dy}{du}\frac{du}{dx}=\tfrac{1}{2}.\frac{1}{\sqrt{u}}.(-2x)=-\frac{x}{\sqrt{1-x^2}}$$

(iii) $y=(ax^2+bx+c)^{\frac{3}{2}}=u^{\frac{3}{2}}$, where $u=ax^2+bx+c$

$\therefore \quad \dfrac{dy}{du}=\frac{3}{2}x^{\frac{3}{2}-1}=\frac{3}{2}\sqrt{u}$ and $\dfrac{du}{dx}=a.2x+b.1+0=2ax+b$

$\therefore \quad \dfrac{dy}{dx}=\dfrac{dy}{du}\cdot\dfrac{du}{dx}=\frac{3}{2}\sqrt{u}(2ax+b)=\frac{3}{2}(2ax+b)\sqrt{ax^2+bx+c}$

(iv) $y=\sin^2 x=(\sin x)^2=u^2$, where $u=\sin x$

$\therefore \quad \dfrac{dy}{du}=2u$ and $\dfrac{du}{dx}=\cos x$

$\therefore \quad \dfrac{dy}{dx}=\dfrac{dy}{du}\times\dfrac{du}{dx}=2u.\cos x=2\sin x.\cos x=\sin 2x.$

(This differentiation was done in the last article with the help of the formula for the derivative of the product of two functions.)

(v) $y=\sin x^2=\sin u$, where $u=x^2$

$\therefore \quad \dfrac{dy}{du}=\cos u$ and $\dfrac{du}{dx}=2x$

$\therefore \quad \dfrac{dy}{dx}=\dfrac{dy}{du}\times\dfrac{du}{dx}=\cos u.2x=2x\cos x^2.$

(vi) $y=\log\sin x=\log u$, where $u=\sin x$.

$\therefore \quad \dfrac{dy}{du}=\dfrac{1}{u}$, $\dfrac{du}{dx}=\cos x$

$\therefore \quad \dfrac{dy}{dx}=\dfrac{dy}{du}\cdot\dfrac{du}{dx}=\dfrac{1}{u}.\cos x=\dfrac{\cos x}{\sin x}=\cot x.$

Ex. 2. Differentiate the following functions with respect to x :

(i) $\sin^2 3x$ (ii) $e^{\tan(ax+b)}$ (iii) $\sqrt{\tan(e^x)}$

(iv) $\sin\left[e^{\tan^2 2x}\right]$.

(i) Let $y=\sin^2 3x=(\sin 3x)^2=u^2$, where
$u=\sin 3x=\sin v$, where $v=3x$

$\therefore$ differentiating $y=u^2$ with respect to u we have

$\dfrac{dy}{du}=2u$. Similarly as $u=\sin v$ then $\dfrac{du}{dv}=\cos v$

and from $v=3x\ \dfrac{dv}{dx}=3.$

$\therefore \quad \dfrac{dy}{dx}=\dfrac{dy}{du}\cdot\dfrac{du}{dv}\cdot\dfrac{dv}{dx}=2u\cos v.3=6\sin 3x.\cos 3x.$

(ii) $y=e^{\tan(ax+b)}=e^{u}$, where $u=\tan(ax+b)=\tan v$ where $v=ax+b$.

$\therefore\quad \frac{dy}{du}=e^{u},\ \frac{du}{dv}=\sec^2 v,\ \frac{dv}{dx}=a$

$$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dv}\cdot\frac{dv}{dx}=e^{u}.\sec^2 v.a$$

$$=a\sec^2(ax+b)\,e^{\tan(ax+b)}$$

(iii) $y=\sqrt{\tan(e^{x})}$. Let $u=\tan(e^{x})$ and $v=e^{x}$.

$\therefore\quad y=\sqrt{u},\ u=\tan v$ and $v=e^{x}$.

$\therefore\quad \frac{dy}{dx}=\frac{1}{2}\cdot\frac{1}{\sqrt{u}},\ \frac{du}{dv}=\sec^2 v,\ \frac{dv}{dx}=e^{x}$

$\therefore\quad \frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dv}\,\frac{dv}{dx}=\frac{1}{2}\cdot\frac{1}{\sqrt{u}}\sec^2 v.e^{x}=\frac{e^{x}.\sec^{3}(e^{x})}{2\sqrt{\tan(e^{x})}}$

(iv) $y=\sin\left(e^{\tan^2 2x}\right)=\sin u$, where $u=e^{\tan^2 2x}=e^{v}$.

where $v=\tan^2 2x=(\tan 2x)^2=w^2$.

where $w=\tan 2x=\tan z$ where $z=2x$.

$\therefore\quad \frac{dy}{du}=\cos u,\ \frac{du}{dv}=e^{v},\ \frac{dv}{dw}=2w,\ \frac{dw}{dz}=\sec^2 z,\ \frac{dz}{dx}=2.$

$\therefore\quad \frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dv}\cdot\frac{dv}{dw}\,\frac{dw}{dz}\cdot\frac{dz}{dx}=\cos u.e^{v}.\ 2w.\ \sec^2 z.2$

$$=4\tan 2x\sec^2 2x.\ e^{\tan^2 2x}.\ \cos\left(e^{\tan^2 2x}\right)$$

Ex. 3. If $y=f(cx)$, determine $\frac{dy}{dx}$. Use it to find the derivative of

(i) $y=\sin 2x$ (ii) $y=e^{kx}$ (iii) $y=a^{x}$.

$y=f(cx)=f(u)$, where $u=cx$.

$\therefore\quad \frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}=f'(u).c=cf'(cx).$

(i) $\frac{d}{dx}(\sin 2x)=2\frac{d}{du}(\sin u)$, (where $u=2x$)

$$=2\cos u=2\cos 2x.$$

(ii) $\frac{d}{dx}\left(e^{kx}\right)=k\frac{d}{du}e^{u}$, (where $u=kx$)$=ke^{u}=ke^{kx}$.

(iii) $\frac{d}{dx}\left(a^{x}\right)=\frac{d}{dx}\left(e^{x\log a}\right)$

$=\frac{d}{dx}\left(e^{kx}\right)$, (where $k=\log a$)$=ke^{kx}=a^{x}.\log a$.

Exercise 4C

1. Differentiate the following functions with respect to x :

(i) $(x^2+1)^5$ (ii) $\dfrac{1}{(x^2-1)^2}$ (iii) $(x^2+a^2)^{10}$

(iv) $(2x^2+4x+1)^{\frac{3}{2}}$ (v) $(ax^2+bx+c)^n$ (vi) $\sqrt{x^2-3x+7}$

(vii) $\sin 2x$ (viii) $\cos 3x$ (ix) $\cot 5x$

(x) $\sec nx$ (xi) $\sqrt{\sin x}$ (xii) $\sin (x^3)$.

(xiii) $\log \tan x$ (xiv) $\log \operatorname{cosec} x$ (xv) $\log \cos x$ (xvi) $\cos (\log x)$

(xvii) $\log (\sec x-\tan x)$ (xviii) $\log \dfrac{e^x}{e^x+1}$ (xix) $\log (\log x)$.

2. Differentiate the following functions with respect to x :

(i) $\dfrac{1}{(x^2+a^2)^{\frac{3}{2}}}$ (ii) $\sqrt{\dfrac{2x}{1-x^2}}$ (iii) $\sqrt{\dfrac{x-a}{x+a}}$

(iv) $\sqrt{\sin nx}$ (v) $\cos (\sin x^3)$ (vi) $\sin^2 2x$

(vii) $\tan \sqrt{2x+1}$ (viii) $\sin (x^2+x+1)^{\frac{1}{2}}$

(ix) $\tan \left\{e^{\sin^2 \frac{1}{2}x}\right\}$ (x) $\log \tan \left(\dfrac{\pi}{4}+\dfrac{x}{2}\right)$ (xi) $10^x \log x$

(xii) $\log (\sqrt{x-a}+\sqrt{x-b})$

§ 4·6. Derivatives of Inverse function

Let $y=f(x)$ be a differentiable function. Suppose, solving the equation $y=f(x)$ we obtain $x=\phi(y)$.

$\phi(y)$ is called the inverse function of $f(x)$.

It is possible to differentiate $x=\phi(y)$ with respect to y and determine $\dfrac{dx}{dy}$.

We shall now find the relation between $\dfrac{dy}{dx}$ and $\dfrac{dx}{dy}$.

As $y=f(x)$ and $x=\phi(y)$, so $x=\phi(y)=\phi\{f(x)\}$

Now, let in $y=f(x)$, Δy be the increment of y for the increment Δx of x.

i.e., $y+\Delta y=f(x+\Delta x)$ $\qquad \dfrac{dy}{dx}=\lim\limits_{\Delta x\to 0} \dfrac{\Delta y}{\Delta x}$

Let in $x=\phi(y)$, the value of y be increased by Δy. Now let us determine the corresponding increment of x. The increment of x is,

$$\begin{aligned}\phi(y+\Delta y)-\phi(y)&=\phi\{f(x+\Delta x)\}-\phi\{f(x)\}\\&=(x+\Delta x)-x \qquad [\because \phi\{f(x)\}=x]\\&=x.\end{aligned}$$

$$\therefore \quad \frac{dx}{dy}=\lim_{\Delta y\to 0}\frac{\Delta x}{\Delta y}=\frac{1}{\lim\limits_{\Delta x\to 0}\frac{\Delta y}{\Delta x}}$$

$$=\frac{1}{\lim\limits_{\Delta x\to 0}\frac{\Delta y}{\Delta x}}\ [\text{ For, } \Delta x\to 0 \text{ as } \Delta y\to 0\ ]$$

$$=\frac{1}{\frac{dy}{dx}}\left[\text{ taking } \frac{dy}{dx}\neq 0\ \right]$$

$$\therefore \quad \frac{dy}{dx}\times\frac{dx}{dy}=1,$$

Hence the rate of change of y with respect to x is the reciprocal of the rate of change of x with respect to y.

[Note : the above formula can also be obtained as follows :

In the formula $\frac{dy}{dx}=\frac{dy}{dz}\cdot\frac{dz}{dx}$, put $y=x$.

$$\therefore \quad \frac{dx}{dx}=\frac{dx}{dz}\cdot\frac{dz}{dx}\quad \text{or,}\quad 1=\frac{dy}{dz}\times\frac{dz}{dy}.$$

Now, let $z=x$, $\therefore$ $1=\frac{dy}{dx}\times\frac{dx}{dy}$]

With the help of the above formula, we shall now determine the derivatives of a few more functions.

Ex. Show that if

(i) $y=\log x$, then $\frac{dy}{dx}=\frac{1}{x}$

(ii) $y=\sin^{-1} x$, then $\frac{dy}{dx}=\frac{1}{\sqrt{1-x^2}}$.

(iii) $y=\tan^{-1} x$, then $\frac{dy}{dx}=\frac{1}{1+x^2}$

(iv) $y=\sec^{-1} x$, then $\frac{dy}{dx}=\frac{1}{x\sqrt{x^2-1}}$

(i) $\because$ $y=\log_e x$, $\therefore$ $x=e^y$

Now, $\frac{dx}{dy}=\frac{d}{dy}(e^y)=e^y$ $\quad\therefore$ $\frac{dy}{dx}=\frac{1}{\frac{dx}{dy}}=\frac{1}{e^y}=\frac{1}{x}$.

(ii) $\because$ $y=\sin^{-1} x$, $\therefore$ $x=\sin y$,

$$\therefore \quad \frac{dx}{dy}=\cos y=\sqrt{1-\sin^2 y}=\sqrt{1-x^2}$$

[The positive sign of the square root has been taken as $\frac{-\pi}{2} \leqslant \sin^{-1} x \leqslant \frac{\pi}{2}$, $\therefore\ -\frac{\pi}{2} \leqslant y \leqslant \frac{\pi}{2}$ $\therefore\ \cos y \geqslant 0$]

$$\therefore\quad \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{\sqrt{1-x^2}}.$$

(iii) $\because\ y=\tan^{-1} x,\ x=\tan y$

$$\therefore\quad \frac{dx}{dy} = \sec^2 y = 1+\tan^2 y = 1+x^2, \quad \therefore\quad \frac{dy}{dx} = \frac{1}{1+x^2}.$$

(iv) $\because\ y=\sec^{-1} x,\ \therefore\ x=\sec y,$

$$\therefore\quad \frac{dx}{dy} = \sec y.\tan y = x\sqrt{\sec^2 y - 1} = x\sqrt{x^2-1}$$

$$\therefore\quad \frac{dy}{dx} = 1 \div \frac{dx}{dy} = \frac{1}{x\sqrt{x^2-1}}.\ \text{when } |x| > 1.$$

Cor. $\because\ \sin^{-1} x + \cos^{-1} x = \frac{\pi}{2},\ \tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}$

$$\sec^{-1} x + \operatorname{cosec}^{-1} x = \frac{\pi}{2};$$

$$\therefore\quad \frac{d}{dx}(\cos^{-1} x) = -\frac{d}{dx}(\sin^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\cot^{-1} x) = -\frac{d}{dx}(\tan^{-1} x) = -\frac{1}{1+x^2}$$

$$\frac{d}{dx}(\operatorname{cosec}^{-1} x) = -\frac{d}{dx}(\sec^{-1} x) = -\frac{1}{x\sqrt{x^2-1}}.$$

These derivatives can be deduced directly without taking the help of the derivatives of $\sin^{-1} x$, $\sec^{-1} x$ and $\tan^{-1} x$. This is left as an exercise in exercises 4(I).

Ex. 2. (i) For the function $y=\sqrt{x}$, find $\frac{dy}{dx}$ at $x=4$ and the inverse function $x=y^2$ find $\frac{dx}{dy}$ at $y=2$ and hence prove that

$$\frac{dy}{dx} \times \frac{dx}{dy} = 1.$$

(ii) for the function $y=\frac{2x}{1+x}$ prove that $\frac{dx}{dy} \times \frac{1}{\frac{dy}{dx}}$.

(i) $\because \quad y=\sqrt{x}=x^{\frac{1}{2}}, \quad \therefore \quad \frac{dy}{dx}=\frac{1}{2}x^{\frac{1}{2}-1}=\frac{1}{2}\cdot\frac{1}{\sqrt{x}}$

$$\therefore \quad \left(\frac{dy}{dx}\right)_{x=4}=\frac{1}{2}\cdot\frac{1}{\sqrt{4}}=\frac{1}{2}\cdot\frac{1}{2}=\frac{1}{4}$$

Again, $\because \ x=y^2, \frac{dx}{dy}=2y, \quad \therefore \quad \left(\frac{dx}{dy}\right)_{x=2}=2.2=4$

$$\therefore \quad \left(\frac{dy}{dx}\right)_{x=4}\times\left(\frac{dx}{dy}\right)_{x=2}=\frac{1}{4}\times 4=1.$$

(ii) $\because \quad y=\frac{2x}{1+x} \quad \therefore \quad \frac{dy}{dx}=\frac{(1+x)\ \frac{d}{dx}(2x)-2x\frac{d}{dx}(1+x)}{(1+x)^2}$

$$=\frac{(1+x).2-2x.1}{(1+x)^2}=\frac{2}{(1+x)^2}.$$

Now, from $y=\frac{2x}{1+x}$, $y(1+x)=2x$, or, $x=\frac{y}{2-y}$

$$\frac{dx}{dy}=\frac{1.(2-y)-y(-1)}{(2-y)^2}=\frac{2}{(2-y)^2}$$

$$=\frac{2}{\left(2-\frac{2x}{1+x}\right)^2} \text{ [Putting the value of } y]$$

$$=\frac{2(1+x)^2}{(2+2x-2x)^2}=\frac{(1+x)^2}{2}=\frac{1}{\frac{2}{(1+x)^2}}=\frac{1}{\frac{dy}{dx}} \qquad \therefore \quad \frac{dx}{dy}=\frac{1}{\frac{dy}{dx}}.$$

Ex. 3. Find the derivative of the following functions :

(i) $\sin^{-1}\frac{x}{a}$, (ii) $\tan^{-1}(\cot x)$, (iii) $\sec^{-1}(\log x)$

(i) $\frac{d}{dx}\left(\sin^{-1}\frac{x}{a}\right)=\frac{d}{du}(\sin^{-1}u).\ \frac{du}{dx}$, where $u=\frac{x}{a}$

$$=\frac{1}{\sqrt{1-u^2}}.\frac{1}{a}=\frac{1}{\sqrt{a^2-x^2}}$$

(ii) $\frac{d}{dx}\left\{\tan^{-1}(\cot x)\right\}=\frac{d}{dx}(\tan^{-1}u)$, where $u=\cot x$

$$=\frac{1}{1+u^2}(-\operatorname{cosec}^2 x)=\frac{-\operatorname{cosec}^2 x}{1+\cot^2 x}=-1.$$

(iii) $\frac{d}{dx}\left\{\sec^{-1}(\log x)\right\}=\frac{1}{\log x\sqrt{\log x^2-1}}\cdot\frac{1}{x}$

Exercise 4D

1. Prove that

(i) If $y=\cos^{-1} x$ then $\dfrac{dy}{dx}=-\dfrac{1}{\sqrt{1-x^2}}$, $0 \leqslant \cos^{-1} x \leqslant \pi$

(ii) If $y=\cot^{-1} x$, then $\dfrac{dy}{dx}=-\dfrac{1}{1+x^2}$.

(iii) If $y=\operatorname{cosec}^{-1} x$, then $\dfrac{dy}{dx}=-\dfrac{1}{x\sqrt{x^2-1}}$.

2. Find $\dfrac{dy}{dx}$ at the point $x=2$ for the function $y=x^2+1$ and also find $\dfrac{dx}{dy}$ for the function $x=\sqrt{y-1}$ at the point $y=5$ and hence prove that $\dfrac{dy}{dx}\times\dfrac{dx}{dy}=1$.

3. Justify the formula $\dfrac{dy}{dx}=1\div\dfrac{dx}{dy}$ for the following functions :

(i) $y=3x+2$ (ii) $y=\sin(2x+1)$ (iii) $y=\dfrac{1-x^2}{1+x^2}$ (iv) $y=e^{x^2}$

4. Prove that :

(i) $\dfrac{d}{dx}\left(\tan^{-1}\dfrac{x}{a}\right)=\dfrac{a}{a^2+x^2}$ (ii) $\dfrac{d}{dx}\left(\sec^{-1}\dfrac{x}{a}\right)=\dfrac{a}{x\sqrt{x^2-a^2}}$

(iii) $\dfrac{d}{dx}(\sin^{-2} x^2)=\dfrac{2x}{\sqrt{1-x^4}}$, (iv) $\dfrac{d}{dx}\tan^{-1}(\sec x)=\dfrac{\sin x}{1+\cos^2 x}$

(v) $\dfrac{d}{dx}\left\{\cos^{-1}\dfrac{1-x^2}{1+x^2}\right\}=\dfrac{2}{1+x^2}$

(vi) $\dfrac{d}{dx}\left\{\sin^{-1}\dfrac{a+b\cos x}{a-b\cos x}\right\}=\dfrac{-ab\sin x}{(a-b\cos x)\sqrt{-ab\cos x}}$.

§ 4 7. Derivatives of implicit functions.

Sometimes the relation between two variables is expressed in the form $f(x, y)=0$. If $f(x \,.\, y)=0$ can be solved in the explicit form $y=f(x)$ or, $x=f(y)$ then from these relations $\dfrac{dy}{dx}$ can be determined. But it is not always possible to solve $f(x \,.\, y)=0$ in the forms $y=f(x)$ or, $x=f(y)$. In such cases to determine $\dfrac{dy}{dx}$ $f(x \,.\, y)$ should be differ-

entiated term by term with respect to x. Terms containing y should be multiplied by $\frac{dy}{dx}$ after differentiation with respect to y. So, one shall get an equation involving $\frac{dy}{dx}$ and $\frac{dy}{dx}$ should be solved from this equation. The method is illustrated in the following examples.

Ex. 1. If $x^2+y^2=a^2$ then, find $\frac{dy}{dx}$.

$x^2+y^2=a^2 \quad \therefore \quad \frac{d}{dy}(x^2+y^2)=\frac{d}{dx}(a^2),$

or, $\frac{d}{dx}(x^2)+\frac{d}{dx}(y^2)=0 \quad [\because \quad a^2=\text{constant}]$

or, $2x+\frac{d}{dy}(y^2).\frac{dy}{dx}=0,$

or, $2x+2y\frac{dy}{dx}=0. \quad \therefore \quad \frac{dy}{dx}=-\frac{2x}{2y}=-\frac{x}{y}.$

Here the equation $x^2+y^2=a^2$ can be solved for y in the form $y=\sqrt{a^2-x^2}$.

From this relation we get $\frac{dy}{dx}=\frac{d}{dx}(\sqrt{a^2-x^2})=\frac{d}{dx}\sqrt{u}$

(where $u=a^2-x^2$)$=\frac{d}{du}\sqrt{u}.\frac{du}{dx}=\frac{1}{2}u^{-\frac{1}{2}}(-2x)=\frac{-x}{\sqrt{a^3-x^2}}=-\frac{x}{y}.$

Ex. 2. If $x^3+y^3=3xy$, then find $\frac{dy}{dx}$.

$\frac{d}{dx}(x^3+y^3)=\frac{d}{dx}(3xy)$

or, $\frac{d}{dx}(x^3)+\frac{d}{dx}(y^3)=3\left(x.\frac{dy}{dx}+1.y\right)$

or, $3x^2+3y^2\frac{dy}{dx}=3x\frac{dy}{dx}+3y$, or, $(y^2-x)\frac{dy}{dx}=y-x^2$

$\frac{dy}{dx}=\frac{y-x^2}{y^2-x}.$

Ex. 3. If $x^m y^n=(x+y)^{m+n}$, then show that $\frac{dy}{dx}=\frac{y}{x}$ [C. U.]

Differentiating both sides with respect to x,

$\frac{d}{dx}(x^m y^n)=\frac{d}{dx}(x+y)^{m+n}$

or, $x^m.\frac{d}{dx}(y^n)+y^n.\frac{d}{dx}(x^m)=(m+n)(x+y)^{m+n-1}\frac{d}{dx}(x+y)$,

or, $x^m.ny^{n-1}\frac{dy}{dx}+y^n mx^{m-1}=(m+n)(x+y)^{m+n-1}\left(1+\frac{dy}{dx}\right)$

or, $\{(m+n)(x+y)^{m+n-1}-nx^my^{n-1}\}\frac{dy}{dx}$

$=mx^{m-1}y^n-(m+n)(x+y)^{m+n-1}$

$\therefore\quad \frac{dy}{dx}=\frac{mx^{m-1}y^n-(m+n)(x+y)^{m+n-1}}{(m+n)(x+y)^{m+n-1}-nx^my^{n-1}}$

$=\frac{mx^{m-1}y^n(x+y)-(m+n)x^my^n}{(m+n)x^my^n-nx^my^{n-1}(x+y)}$

[Putting $(x+y)^{m+n}=x^my^n$]

$=\frac{x^{m-1}y^{n-1}\{my(x+y)-(m+n)xy\}}{x^{m-1}y^{n-1}\{(m+n)xy-nx(x+y)\}}=\frac{(my-nx)y}{(my-nx)x}=\frac{y}{x}$

Exercise 4E

1. Find $\frac{dy}{dx}$ from the following relations

(i) $\frac{x^2}{4}+\frac{y^2}{9}=1$ (ii) $x^3y^4=(x+y)^7$ (iii) $y=(x+y)^2$

(iv) $\frac{x^m}{a^m}+\frac{y^m}{b^m}=1$ (v) $ax^2+2hxy+by^2=1$ [a, b, h are constants]

(vi) $y=\sin(x+y)$ (vii) $x+y=\sin(xy)$

(viii) $xy=\sin(x+y)$ (ix) $x+y=\tan(xy)$

(x) $\sin 3x=\cos 4y$ (xi) $\log xy=x+y$

(xii) $e^{x+y}=xy$ (xiii) $ax^2+2hxy+by^2+2gx+2fy+c=0$

(xiv) $e^{xy}-2xy=4$.

§ 4·8. Differentiation of parametric functions and logarithmic differentiation.

Sometimes the relation between x and y is expressed in terms of a third variable e.g., $x=a\cos\theta$ and $y=b\sin\theta$. Here both x and y are functions of θ. Again eliminating θ we can get

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\cos^2\theta+\sin^2\theta=1.$$

$\therefore$ There is relation between x and y. If x and y be both functions of a third variable t (say), then we get parametric expressions of x and y. t is called the parameter.

Let $x=f(t)$, $y=g(t)$

$\therefore \quad \frac{dx}{dt}=f'(t)$ and $\frac{dy}{dt}=g'(t)$

$$\therefore \quad \frac{dy}{dx}=\frac{dy}{dt}\frac{dt}{dx}=\frac{dy}{dt}.\frac{1}{\frac{dx}{dt}}=\frac{g'(t)}{f'(t)}=\frac{dy}{dt}\div\frac{dx}{dt}$$

Ex. 1. Differentiate $x=a\cos\theta$, $y=b\sin\theta$ where the parameter is θ.

$$\frac{dx}{d\theta}=-a\sin\theta,\ \frac{dy}{d\theta}=b\cos\theta$$

$$\therefore \quad \frac{dy}{dx}=\frac{dy}{d\theta}\div\frac{dx}{d\theta}=-\frac{b}{a}\cot\theta.$$

Ex. 2. $y=at^2$, $x=2at$, parameter, t

$$\therefore \quad \frac{dy}{dt}=2at,\ \frac{dx}{dt}=2a.$$

$$\therefore \quad \frac{dy}{dx}=\frac{dy}{dt}\div\frac{dx}{dt}=\frac{2at}{2a}=t.$$

If $y=f(x)$ be a given function of x, sometimes it is found convenient to determine the derivative of $f(x)$ with respect to x by taking logarithm of both sides of $y=f(x)$ and then differentiating both sides of $\log y=\log f(x)$. This method of differentiation is referred to as logarithmic differentiation. The method is illustrated in the following examples.

Ex. 3. If $y=x^x$, find $\frac{dy}{dx}$.

Taking logarithm of both sides of $y=x^x$.
we get $\log y=\log x^x=x\log x$

$$\therefore \quad \frac{d}{dx}(\log y)=\frac{d}{dx}(x\log x)$$

$$\text{or,} \quad \frac{1}{y}\frac{dy}{dx}=1.\log x+x.\frac{1}{x}=1+\log x.$$

$$\therefore \quad \frac{dy}{dx}=y(1+\log x)=x^x(1+\log x).$$

Ex. 4. Find $\frac{dy}{dx}$, where $x^m y^n=(x+y)^{m+n}$

Taking logarithm of both sides,
$m\log x+n\log y=(m+n)\log(x+y)$.

Differentiating both sides with respect to x, we obtain,

$$m.\frac{1}{x}+n.\frac{1}{y}\frac{dy}{dx}=(m+n).\frac{1}{x+y}\left(1+\frac{dy}{dx}\right)$$

or, $\left(\frac{n}{y}-\frac{m+n}{x+y}\right)\frac{dy}{dx}=\frac{m+n}{x+y}-\frac{m}{x}$

or, $\frac{nx-my}{y(x+y)}.\frac{dy}{dx}=\frac{nx-my}{x(x+y)}$ $\therefore$ $\frac{dy}{dx}=\frac{y}{x}$.

[Note. In the previous article the same sum was done by a different method.]

Ex. 5. Differentiate $(\sin x)^{\cos x}$ with respect to x

Let $y=(\sin x)^{\cos x}$

$\therefore$ $\log y=\log (\sin x)^{\cos x}=\cos x \log (\sin x)$

Differentiating both sides with respect to x, we get,

$$\frac{1}{y}\frac{dy}{dx}=-\sin x \log (\sin x)+\cos x\frac{1}{\sin x}\cos x$$

$$\therefore \quad \frac{dy}{dx}=y(-\sin x \log \sin x+\cos x \cot x)$$

$$=(\sin x)^{\cos x}(-\sin x \log \sin x+\cos x \cot x).$$

Ex. 6. If $x^y=y^x$ show that, $\frac{dy}{dx}=\frac{y(x \log y-y)}{x(y \log x-x)}$ [C. U. '45]

Taking logarithm of both sides of $x^y=y^x$,

$y \log x=x \log y.$

Differentiating both sides with respect to x,

$$y.\frac{1}{x}+\frac{dy}{dx}.\log x=1.\log y+x\frac{1}{y}\frac{dy}{dx}$$

or, $\left(\log x-\frac{x}{y}\right)\frac{dy}{dx}=\log y-\frac{y}{x}$

$$\therefore \quad \frac{dy}{dx}=\frac{y(x \log y-y)}{x(y \log x-x)}.$$

Ex. 7. Find $\frac{dy}{dx}$ when $y=x^3.\sqrt{\frac{x^2+4}{x^2+3}}$. [C. U. '41]

$$\log y=3 \log x+\tfrac{1}{2} \log (x^2+4)-\tfrac{1}{2} \log (x^2+3)$$

$$\therefore \quad \frac{1}{y}\frac{dy}{dx}=\frac{3}{x}+\frac{1}{2}\frac{2x}{x^2+4}-\frac{1}{2}.\frac{1}{x^2+3}2x$$

$$\therefore \quad \frac{dy}{dx}=y\left[\frac{3}{x}+\frac{x}{x^2+4}-\frac{x}{x^2+3}\right]$$

$$=x^3.\sqrt{\frac{x+4}{x^2+3}}\left[\frac{3}{x}+\frac{x}{x^2+4}-\frac{x}{x^2+3}\right]$$

Exercise 4F

1. Find $\frac{dy}{dx}$ in the following cases :

(i) $x=a\cos\theta, y=a\sin\theta$. (ii) $x=a\sec\theta, y=b\tan\theta$

(iii) $x=\frac{3at}{1+t^2}, y=\frac{3at^2}{1+t^2}$ (iv) $x=at, y=\frac{a}{t}$

(v) $x=a\cos^3\theta, y=b\sin^3\theta$

(vi) $x=a(t+\sin t), y=a(1-\cos t)$

2. Find $\frac{dy}{dx}$ where

(i) $y=x^{2x}$ (ii) $y=x^{\tan x}$ (iii) $y=(\cos x)^{\sin x}$

(iv) $y=x^{\log x}$ (v) $y=(x+y)^{x+y}$

(vi) $y=\sin x.e^x x^x$ (vii) $y=\frac{x(x^2+4)^{\frac{1}{3}}}{(x^3+5)^{\frac{1}{4}}}$

(viii) $x.y^x=a$ (ix) $y=(1+x)(1+2x)(1+3x)(1+4x)$

(x) $y=\sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)}}$

3. If $y=u^v$, where u and v are functions of x, then show that

$$\frac{dy}{dx}=u^v.\left(\frac{dv}{dx}\log u+\frac{v}{u}\frac{du}{dx}\right).$$

From the above formula find $\frac{dy}{dx}$ when

(i) $y=(e^x)^{e^x}$ (ii) $y=(\log x)^x$ (iii) $y=(x^2+2)^{(x^2+3x+1)}$

§ 4·9. Recapitulation

In the preceding articles we have defined the derivative of a function and also shown methods of differentiation of different types of functions. We give below a list of derivatives of functions. Remember tha thet derivative with respect to x of the function

$y=f(x)$ is denoted by the symbol $\frac{dy}{dx}$ or, $f'(x)$ and its definition is $f'(x)=\frac{dy}{dx}=\underset{h\to 0}{\text{Lt}}\ \frac{f(x+h)-f(x)}{h}$.

If $y=x^n$, then $\frac{dy}{dx}=nx^{n-1}$ for all n,

$\frac{d}{dx}(c)=0$, where c is a constant

$y=e^x$, then $\frac{dy}{dx}=e^x$.

If $y=a^x$ then $\frac{dy}{dx}=a^x \log_e^a$

If $y=\log x$, then $\frac{dy}{dx}=\frac{1}{x}$.

If $y=\sin x$, then $\frac{dy}{dx}=\cos x$.

If $y=\cos x$, then $\frac{dy}{dx}=-\sin x$.

If $y=\tan x$, then $\frac{dy}{dx}=\sec^2 x$.

If $y=\cot x$, then $\frac{dy}{dx}=-\text{cosec}^2 x$

If $y=\sec x$, then $\frac{dy}{dx}=\sec x \tan x$.

If $y=\text{cosec}\, x$, then $\frac{dy}{dx}=-\text{cosec}\, x \cot x$

If $y=\sin^{-1} x$, then $\frac{dy}{dx}=\frac{1}{\sqrt{1-x^2}}$

If $y=\cos^{-1} x$, then $\frac{dy}{dx}=-\frac{1}{\sqrt{1-x^2}}$

If $y=\tan^{-1} x$, then $\frac{dy}{dx}=\frac{1}{1+x^2}$

If $y=\cot^{-1} x$, then $\frac{dy}{dx}=-\frac{1}{1+x^2}$

If $y=\sec^{-1} x$, then $\frac{dy}{dx}=\frac{1}{x\sqrt{x^2-1}}$

If $y=\text{cosec}^{-1} x$, then $\frac{dy}{dx}=-\frac{1}{x\sqrt{x^2-1}}$

Remember the following general results together with the above formulas.

If u and v be two differentiable functions of x, then

(i) $\frac{d}{dx}(c.\,u)=c.\,\frac{d}{dx}(u)$, where c is a constant.

(ii) $\frac{d}{dx}(u+v)=\frac{du}{dx}+\frac{dv}{dx}$.

(iii) $\frac{d}{dx}(u-v)=\frac{du}{dx}-\frac{dv}{dx}$.

(iv) $\frac{d}{dx}(uv)=u\frac{dv}{dx}+v\frac{du}{dx}$.

(v) $\frac{d}{dx}\left(\frac{u}{v}\right)=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$

(vi) If $y=f\{g(x)\}$, then $\frac{dy}{dx}=f'\{g(x)\}g'(x)$

or, $\frac{dy}{dx}=\frac{dy}{du}.\frac{du}{dx}$ where $u=g(x)$.

With the help of the above formulas and rules one can determine the derivative of any differentiable function. In the previous articles we have illustrated the methods of determination of derivatives of functions with the help of these formulas and rules.

§ 5·10. Differential.

If $y=f(x)$ be a differentiable function, then we know that,

$\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=f'(x)$. Let $\frac{\Delta y}{\Delta x}-f'(x)=\alpha$.

Now, $\lim_{\Delta x\to 0}\alpha=\lim_{\Delta x\to 0}\left[\frac{\Delta y}{\Delta x}-f'(x)\right]$

$=\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}-f'(x)$ [as $f'(x)$ is independent of Δx]

$=f'(x)-f'(x)=0$.

Hence we can write,

$\Delta y=f'(x).\Delta x+\alpha.\Delta x$ where $\alpha\to 0$ as $\Delta x\to 0$.

Hence the increment Δy of the function $y=f(x)$ has two parts. The part $f'(x).\Delta x$ is called the differential of y and is denoted by dy. $\therefore$ $dy=f'(x)\,\Delta x$.

Now, let $y=f(x)=x$

$\therefore$ $f'(x)=1$ and $dx=1.\Delta x=\Delta x$.

Hence the differential dx of the independent variable x is equal to its increment Δx.

Hence for any function $y=f(x)$, the differential $dy=f'(x)dx$.

Hence differential of a function can be defined in the following way :

(i) The differential dx of the independent variable x is equal to its small increment Δx.

(ii) For the dependent variable $y=f(x)$, the differential dy is the product of the derivative of $f(x)$ and the differential dx of the independent variable.

[Note 1. dy is proportional to dx.]

For, $dy=k\ dx$ where $k=f'(x)$, a quantity independent of dx.

Also $\dfrac{\Delta y-dx}{\Delta x}\to 0$ when $\Delta x\to 0$.

Conversely, if a quantity z be such that

(i) z is proportional to Δx and

(ii) $\dfrac{\Delta y-z}{\Delta x}\to 0$ as $\Delta x\to 0$, then the quantity z is nothing but the differential dy of y.

2. The derivative $f'(x)$ of the function $y=f(x)$ is the ratio of the two differentials dy and dx *i.e.*, $f'(x)=dy\div dx$.

$\therefore$ $\dfrac{dy}{dx}$ has got two meanings.

In the first place $\dfrac{dy}{dx}$ is $\lim\limits_{\Delta x\to 0}\dfrac{\Delta y}{\Delta x}$ (when the limit exists) and in the second place it is the ratio of the two differentials dy and dx.

As in both cases we get the same value of $\dfrac{dy}{dx}$ it can be used either way.

3. If u and v be two differentiable functions of x, then it can be proved that

$$d(u+v)=du+dv,\ d(u-v)=du-dv$$

$$d(uv)=udv+v.du,\ d\left(\frac{u}{v}\right)=\frac{vdu-udv}{v^2}.$$

4. If $y=f(x)$, then $y+\Delta y=f(x+\Delta x)$

Again, $\Delta y=f'(x).\Delta x+\alpha.\Delta x$ where

$\alpha\to 0$ as $\Delta x\to 0$

$\therefore\quad f(x+\Delta x)=f(x)+f'(x).\Delta x+\alpha\Delta x.$

Now if Δx be taken sufficiently small, then α will also be very small for as $\Delta x\to 0$, then $\alpha\to 0$ and hence $\alpha\Delta x$ will be smaller.

Hence neglecting the small quantity $\alpha\Delta x$, we can take $f(x+\Delta x)$ to be nearly equal to $f(x)+f(x)\Delta x$.

i.e., $f(x+\Delta x)=f(x)+f'(x).\Delta x$ when Δx is very small.

With the help of this formula one can easily determine the approximate value of a function at a point if its value at a neighbouring point is known.

Ex. 1. Determine the increment Δy and the differential dy of the function $y=x^2$ for the increment Δx of x.

Here $y=f(x)=x^2 \quad\therefore\quad f'(x)=2x.$

Now for the increment Δx of x, the increment of y is

$$\Delta y=f(x+\Delta x)-f(x)=(x+\Delta x)^2-x^2=2x.\Delta x+(\Delta x)^2$$

The differential $dy=f'(x)dx=2x.\Delta x\ [\because\quad \Delta x=dx\ ]$

Ex. 2. Find the differentials of the following functions :

(i) $y=\sin x$ (ii) $y=\sqrt{1+x}$ (iii) $y=\sin\sqrt{x}$

(iv) $y=\sqrt{1+\log x}$.

(i) $dy=f'(x)dx=\cos x.dx.$

(ii) $f(x)=\sqrt{1+x},\quad\therefore\quad f'(x)=\dfrac{1}{2\sqrt{1+x}}\quad\therefore\quad dy=\dfrac{1}{2\sqrt{1+x}}dx.$

(iii) $f(x)=\sin\sqrt{x}\quad\therefore\quad f'(x)=\cos\sqrt{x}.\dfrac{1}{2\sqrt{x}}$

$$\therefore\quad dy=f'(x)dx=\cos\sqrt{x}\cdot\frac{1}{2\sqrt{x}}dx.$$

Note that if $\sqrt{x}=u$, then $du=\dfrac{1}{2\sqrt{x}}dx$, and $dy=\cos u.du$.

(iv) $dy=\dfrac{1}{2\sqrt{1+\log x}}\cdot\dfrac{1}{x}dx$

Ex. 3. Determine the increments and differentials of the following functions :

(i) $f(x)=x^2+2x-1$, when the value of x changes from 2 to 2·1.

(ii) $f(x)=\dfrac{2}{x-1}$ when the value of x changes from 3 to 3·001.

(i) Here, $\triangle x=2{\cdot}1-2={\cdot}1$, and $f(x)=x^2+2x-1$,

$\therefore\ f(2)=2^2+2.2-1=7,\ f(2{\cdot}1)=(2{\cdot}1)^2+2(2{\cdot}1)-1=7{\cdot}61$

$\therefore\ \triangle y=f(2{\cdot}1)-f(2)=7{\cdot}61-7={\cdot}61$

Again, $f'(x)=2x+2$

$\therefore\ dy=f'(2).\triangle x=(2.2+2)\times{\cdot}1={\cdot}6$

(ii) Here, $\triangle x=3{\cdot}001-3={\cdot}001$, $f(x)=\dfrac{2}{x-1}$,

$$f'(x)=-\frac{2}{(x-1)^2},$$

$$\therefore\ \triangle y=f(3{\cdot}001)-f(3)=\frac{2}{2{\cdot}001}-\frac{2}{2}=-\frac{{\cdot}001}{2{\cdot}001}=-\frac{1}{2001}$$

$$dy=-\frac{2}{(3-1)^2}\times{\cdot}001=-\frac{1}{2000},$$

Ex. 4. If $\log_{10}200=2{\cdot}30103$ then find $\log_{10}200{\cdot}2$ (nearly)
(Given that $\log_{10}{}^{e}={\cdot}43429$)

Let, $y=\log_{10}{}^{x}$, $x=200$ and $\triangle x={\cdot}2$

$$\therefore\ y=f(x)=\frac{\log_e{}^{x}}{\log_e{}^{10}}=\log_e{}^{x}.\log^e{}_{10}$$

$$\therefore\ f'(x)=\frac{1}{x}\times\log_{10}{}^{e}={\cdot}43429\times\frac{1}{x}$$

Now as, $f(x+\triangle x)=f(x)+f'(x)\triangle x$,

so $f(200+{\cdot}2)=f(200)+f'(200)\times{\cdot}2$

or, $\log_{10}200{\cdot}2=\log 200+{\cdot}43429\times\dfrac{1}{200}\times{\cdot}2$

$=2{\cdot}30103+{\cdot}00043429=2{\cdot}30146.$

Exercise 4G

1. Find the differentials of the following functions :

(i) $y=x^3-2x+5$ (ii) $y=ax^3+bx^2+cx+d$

(iii) $y=\sqrt{x}+\dfrac{1}{\sqrt{x}}$ (iv) $y=\dfrac{x}{c}-\dfrac{c}{x}$ (v) $y=\sqrt{1+x^2}$

(vi) $y=x.\log x$ (vii) $y=\dfrac{x\log x}{1-x}+\log(1-x)$.

(viii) $y=\dfrac{\sin x+\cos x}{\sin x-\cos x}$ (ix) $y=\dfrac{1}{3}\tan^3 x-\tan x+x$

(x) $y=e^x(\sin x+\cos x)$.

2. Determine the increment and differentials of the following functions :

(i) $f(x)=x^2-x$, when the value of x changes from 1 to 1·01

(ii) $f(x)=\sin x$, when $x=\frac{\pi}{3}$ and $\triangle x=\frac{\pi}{18}$.

(iii) $f(x)=x^3+2x$; when $x=-1$, $\triangle x=\cdot 02$.

(iv) $f(x)=\sqrt{1+x^2}$ when the value of x changes from 3 to 3·2.

3. Given $\sin 60°=\cdot 866025$ and $\cos 60°=\frac{1}{2}$.

Find approximate values of $\sin 60°18'$ and $\cos 60°30'$.

4. Find the approximate value of $\tan 45°4'30''$.

5. Given (i) $\log_{10}300=2\cdot 47712$. Find $\log_{10} 300\cdot 3$

(ii) $\log_{10}540=3\cdot 73239$. Find $\log_{10}540\cdot 7$.

[Given $\log_{10} e=\cdot 43429$]

§ 4·11. Geometrical significances of derivative and differential.

A straight line is a tangent to the curve $y=f(x)$, if it intersects the curve in coincident points. Hence the tangent to a curve at a point P is the limiting position $\overline{PT}$ of the chord $\overline{PQ}$, as the point $\overline{Q}$ approaches the point P along the curve.

Let the co-ordinates of the point P of the curve $y=f(x)$ be (x, y) and those of a point Q (also on the curve) close to P be $(x+\triangle x, y+\triangle y)$. Let the chord $\overline{PQ}$ makes an angle α with the positive direction of the x-axis. Now as the point Q approaches the point P along the curve, the limiting position of the chord $\overline{PQ}$ is the tangent $\overline{PT}$ at P. Let $\overline{PT}$ be inclined to the positive direction of the x-axis at an angle ψ. $\overline{PM}$ and $\overline{QN}$ are perpendiculars from P and Q respectively on the x-axis, $\overline{PR}$ is drawn perpendicular from P on QN.

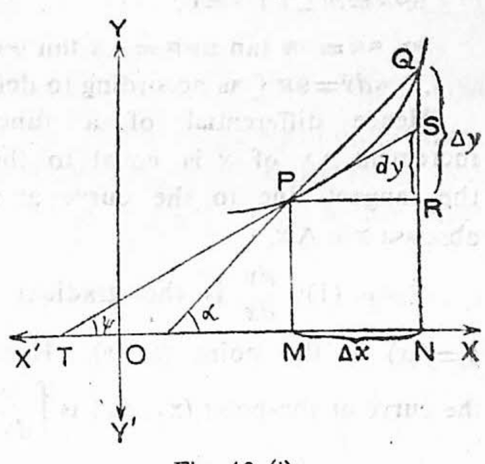


Fig. 40 (i)

Now, $OM=x$, $ON=x+\Delta x$, $PM=y=RN$, $QN=y+\Delta y$.

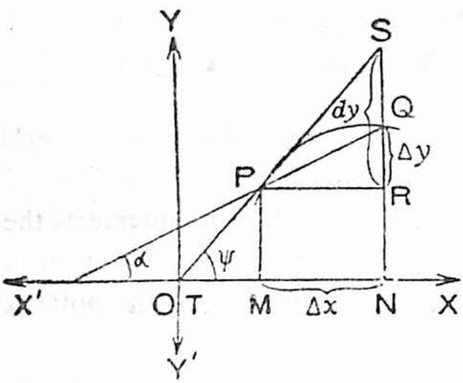
$\therefore \quad MN=ON-OM=\Delta x, \quad QR=QN-RN=\Delta y,$

and $\tan \alpha=\dfrac{QR}{PR}=\dfrac{\Delta y}{\Delta x}$. Now, as $Q\to P$, then $\Delta x\to 0$, the chord $\overline{PQ}\to\overline{PT}$ and the angle $\alpha\to\psi$.

$$\therefore \quad \frac{dy}{dx}=\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\lim_{Q\to P}\tan\alpha=\tan\psi.$$

or, $f'(x)=\tan\psi$.

Hence the derivative of a function $y=f(x)$ at a point (x, y) is the trigonometric tangent of the angle ψ that the tangent to the curve at the point makes with the positive direction of the x-axis.



(ii)

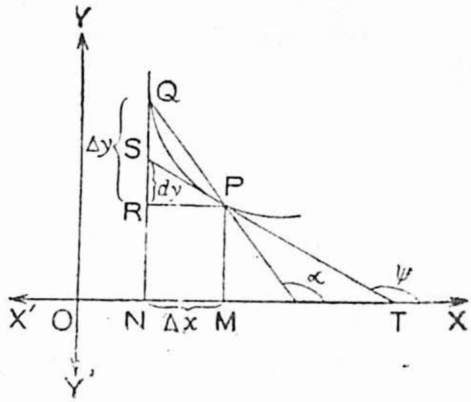


Fig. 40 (iii)

Let the tangent $\overline{PT}$ at the point P, intersect $\overline{QN}$ at S. Now $m\angle S\hat{P}R=m\angle PTM=\psi$.

So, $SR=PR\tan SPR=\Delta x\tan\psi=f'(x)\Delta x$.

$\therefore \quad dy=SR$ [as according to definition, $f'(x)\Delta x=dy$].

Hence differential of a function $f(x)$ corresponding to an increment Δx of x is equal to the increment in the ordinate of the tangent line to the curve at the point x up to the point with abscissa $x+\Delta x$.

Note. (1) $\dfrac{dy}{dx}$ is the gradient of the tangent to the curve $y=f(x)$ at the point (x, y). Hence the gradient of the tangent to the curve at the point (x_1, y_1) is $\left[\dfrac{dy}{dx}\right]_{(x_1, y_1)}$, where $\left[\dfrac{dy}{dx}\right]_{(x_1, y_1)}$ is the value of $\dfrac{dy}{dx}$ at the point (x_1, y_1).

$\therefore$ The equation of the tangent at the point (x_1, y_1) is $y-y_1=\left(\frac{dy}{dx}\right)_{(x_1,\, y_1)}(x-x_1)$.

(2) The normal to a curve at the point (x_1, y_1) is the perpendicular to the tangent to the curve at the point (x_1, y_1).

Hence the equation of the normal to the curve at the point (x_1, y_1) is $y-y_1=\left(-\frac{dx}{dy}\right)_{(x_1,\, y_1)}(x-x_1)$.

(3) In the figure $\overline{QR}$ is the increment $\triangle y$ of y and $\overline{SR}$ is the differential dy of y. In fig. 40 (i) $\triangle y>dy$ and in fig. 40 (ii) $\triangle y<dy$.

Hence the value of dy for the increment $\triangle x$ of x can be greater, less than or equal to $\triangle y$.

(4) Note that in fig. 40 (i) and 40 (ii) the value of ψ increases with the increase of the value in x and the angle ψ is an acute angle ;

$\therefore$ $\frac{dy}{dx}=\tan\psi>0$. But in fig. 40 (iii) the value of ψ decreases as the value of x increases and the angle ψ is an obtuse angle. So $\tan\psi<0$.

Hence we find that if $\frac{dy}{dx}>0$, then the value of $y=f(x)$ increases as the value of x increases and if $\frac{dy}{dx}<0$, then the value of y decreases as the value of x increases.

§ 4·12. Second order derivative.

Let $y=f(x)$ be a differentiable function. The differential coefficient $f'(x)$ is a function of x. So one can again differentiate $f'(x)$ with respect to x. The derivative of $f'(x)$ is called the second order derivative of $f(x)$ and is expressed as $f''(x)$. $f'(x)$ is also called the first derivative of $f(x)$.

Now, you know that the first derivative $f'(x)$ of $y=f''(x)$ is denoted as $\frac{dy}{dx}$. The second derivative $f''(x)$ is denoted as

$$\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d^2y}{dx^2}.$$

Hence $\frac{dy}{dx}$ is the first derivative of $y=f(x)$.

The second derivative of $y=f(x)$ is $f''(x)=\dfrac{d^2y}{dx^2}$

For example, let $y=x^4$,

$\therefore$ The first derivative of $y=\dfrac{dy}{dx}=4x^3$.

The second derivative of $y=\dfrac{d^2y}{dx^2}=\dfrac{d}{dx}\left(\dfrac{dy}{dx}\right)=\dfrac{d}{dx}(4x^3)=12x^2$.

Note : (i) The third derivative or the derivative of the third order of $y=f(x)$ is the derivative of the second derivative

$$\frac{d^2y}{dx^2} \text{ or } f''(x) \text{ of } y=f(x).$$

$\therefore$ The third derivative is denoted as

$$f'''(x)=\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right)=\frac{d^3y}{dx^3}.$$

Similarly the fourth derivative of $y=f(x)$ is denoted as

$$f^{iv}(x) \text{ and } f^{iv}(x)=\frac{d}{dx}\left(\frac{d^3y}{dx^3}\right)=\frac{d^4y}{dx^4}.$$

In general the nth derivative of $y=f(x)$ is denoted of $f^n(x)$ and $f(x)=\dfrac{d}{dx}\left(\dfrac{d^{n-1}y}{dx^{n-1}}\right)=\dfrac{d^ny}{dx^n}$.

(ii) The derivative of $y=f(x)$ is denoted by various symbols such as $\dfrac{dy}{dx}$, $f'(x)$, y_1, y', $D\{f(x)\}$, $D(y)$, Dy etc. Similarly, the second derivative of $y=f(x)$ is also denoted by various symbols such as $\dfrac{d^2y}{dx^2}$, $f''(x)$, y_2, y'', $D^2\{f(x)\}$, $D^2(y)$, D^2y etc. The various symbols for the nth derivative are, $f^n(x)$, $\dfrac{d^ny}{dx^n}$, y_n, $y^{n)}$, $D^n\{(fx)\}$, $D^n\{y\}$, D^ny etc.

(iii) The differential of $y=f(x)$ for the increment dx of x is $dy=f'(x)dx$, where $f'(x)$ is a function of x and dx is independent of x. $\therefore$ dy can be taken as a function of x and its differential with respect to x can be determined. This differential of the differential of y is said to be the differential of the second order or the second differential of $y=f(x)$ and is denoted as d^2y.

$$\begin{aligned}\therefore\quad d^2y&=d(dy)=d\{f'(x)dx\}=\{f'(x)dx\}'dx\\&=dx\{f'(x)\}'dx \ [\text{ as } dx \text{ is independent of } x\ ]\\&=dxf''(x)dx=f''(x)(dx)^2.\end{aligned}$$

Similarly the third order differential of $y=f(x)$ is

$$d^3y=d(d^2y)=(d^2y)'dx=\{f''(x)(dx)^2\}dx=f'''(x)(dx)^3.$$

In general the nth differential or the differential of the nth order is $d^n(y)=d\{d^{n-1}y\}=f^n(x)(dx)^n$.

Ex. 1. Find the second derivative when

(i) $y=x^3-3x^2+4x-1$ (ii) $y=e^{x^2}$ (iii) $y=x.\sin x$.

(i) $\frac{dy}{dx}=3x^2-6x+4$; $\frac{d^2y}{dx^2}=\frac{d}{dx}(3x^2-6x+4)$

$=6x-6=6(x-1)$.

(ii) $\frac{dy}{dx}=e^{x^2}, 2x=2x.e^{x^2}, \frac{d^2y}{dx^2}=\frac{d}{dx}\left(2x.e^{x^2}\right)$

$=2.e^{x^2}+2x.e^{x^2}.2x=2(1+2x^2)e^{x^2}$.

(iii) $\frac{dy}{dx}=1.\sin x+x\cos x=\sin x+x\cos x$

$\frac{d^2y}{dx^2}=\cos x+1.\cos x+x(-\sin x)=2\cos x-x\sin x$.

Ex. 2. If $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ then find $\frac{d^2y}{dx^2}$.

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

Differentiating both sides with respect to x we get,

$$\frac{2x}{a^2}+\frac{2y}{b^2}.\frac{dy}{dx}=0, \text{ or, } \frac{dy}{dx}=-\frac{x}{a^2}.\frac{b^2}{y}=-\frac{b^2}{a^2}.\frac{x}{y}.$$

$$\therefore \frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dx}\left(\frac{b^2}{a^2}.\frac{x}{y}\right)=-\frac{b^2}{a^2}.\frac{d}{dx}\left(\frac{x}{y}\right)$$

$$=-\frac{b^2}{a^2}.\frac{y.\frac{d}{dx}(x)-x.\frac{d}{dx}(y)}{y^2}=-\frac{b^2}{a^2}.\frac{y\frac{d}{dx}(x)-x\frac{dy}{dx}}{y^2}$$

$$=-\frac{b^2}{a^2}.\frac{y.1-x.\left(-\frac{b^2}{a^2}.\frac{x}{y}\right)}{y^2}=-\frac{b^2}{a^2}.\frac{a^2y^2+x^2b^2}{a^2y^3}$$

$$=-\frac{b^2}{a^2}.\frac{a^2b^2}{a^2y^3}\left[\because \frac{x^2}{a^2}+\frac{y^2}{b^2}=1, \therefore b^2x^2+a^2y^2=a^2b^2\right]$$

$$=-\frac{b^4}{a^2y^3}.$$

Ex. 3. If $x=\phi(t)$, $y=\psi(t)$, then show that

$$\frac{d^2y}{dx^2}=\frac{\phi'(t)\psi''(t)-\psi'(t).\phi''(t)}{\{\phi'(t)\}^3}$$ and hence find $\frac{d^2y}{dx^2}$ when

(i) $x=a\cos t$, $y=b\sin t$

(ii) $x=a\cos 2t$, $y=b\sin^2 t$.

Now, $$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{\psi'(t)}{\phi'(t)}$$

$$\therefore \quad \frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dt}\left(\frac{dy}{dx}\right)\frac{dt}{dx}=\frac{d}{dt}\left\{\frac{\psi'(t)}{\phi'(t)}\right\}\cdot\frac{1}{\frac{dx}{dt}}$$

$$=\frac{\psi''(t).\phi'(t)-\psi'(t).\phi''(t)}{\{\phi'(t)\}^2}\cdot\frac{1}{\phi'(t)}=\frac{\phi'(t)\psi''(t)-\psi'(t).\phi''(t)}{\{\phi'(t)\}^3}$$

(i) $x=a\cos t$, $\therefore \quad x=\frac{dx}{dt}=-a\sin t$, $x''=-a\cos t$

$y=b\sin t$, $\therefore \quad y'=\frac{dy}{dt}=b\cos t$, $y''=-b\sin t$.

$$\therefore \quad \frac{y}{dx^2}=\frac{x'y''-y'x''}{(x')^3}$$

$$=\frac{(-a\sin t).(-b\sin t)-(b\cos t).(-a\cos t)}{(-a\sin t)^3}$$

$$=\frac{ab(\cos^2 t+\sin^2 t)}{-a^3\sin^3 t}=-\frac{b}{a^2}\,\frac{1}{\sin^3 t}.$$

(ii) $x=a\cos 2t$, $x'=-2a\sin 2t$, $x''=-4a\cos 2t$;
$y=b\sin^2 t$, $y'=2b\sin t.\cos t=b\sin 2t$, $y''=2b\cos 2t$;

$$\therefore \quad \frac{d^2y}{dx^2}=\frac{x'y''-x''y'}{(x')^3}$$

$$=\frac{-2a\sin 2t.2b\cos 2t-(-4a\cos 2t).b\sin 2t}{(-2a\sin 2t)^3}=0$$

Ex. 4. If $y=ae^x+be^{2x}$, show that $\frac{d^2y}{dx^2}-3\frac{dy}{dx}+2y=0$.

$$y=ae^x+be^{2x},\ \frac{dy}{dx}=ae^x+2be^{2x},\ \frac{d^2y}{dx^2}=ae^x+4be^{2x}$$

$$\therefore \quad \frac{d^2y}{dx^2}-3\frac{dy}{dx}+2y=ae^x+4be^{2x}-3(ae^x+2be^{2x})$$
$$+2(ae^x+be^{2x})=0.$$

Ex. 5. If $y=\sin(m \sin^{-1} x)$, then show that

$$(1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}+m^2y=0.$$

$\because \quad y=\sin(m \sin^{-1}x), \quad \therefore \quad \frac{dy}{dx}=\cos(m \sin^{-1}x).\frac{m}{\sqrt{1-x^2}}$

or, $\quad \sqrt{1-x^2}.\frac{dy}{dx}=m\cos(m \sin^{-1}x).$

Differentiating both sides again with respect to x we get,

$$\sqrt{1-x^2}.\frac{d^2y}{dx^2}+\frac{1}{2\sqrt{1-x^2}}(-2x)\frac{dy}{dx}$$

$$=-m\sin(m \sin^{-1}x).\frac{m}{\sqrt{1-x^2}}$$

or, $\quad (1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}+m^2y=0 \quad \left[\because \quad y=\sin(m \sin^{-1}x)\right]$

Ex. 6. Find the equations of the tangent and normal to the curve $y=x^3-3x^2-x+4$ at the point (3, 1),

Here $\frac{dy}{dx}=3x^2-6x-1, \quad \therefore \quad \left(\frac{dy}{dx}\right)_{(3,\ 1)}=3.3^2-6.3-1=8.$

Hence the equation of the tangent to the curve at the point (3, 1) is

$$y-1=\left(\frac{dy}{dx}\right)_{(3,\ 1)}(x-3)$$

or, $\quad y-1=8(x-3), \qquad$ or, $\quad 8x-y-23=0.$

The equation of the normal at the point (3, 1) is

$$y-1=-\left(\frac{dx}{dy}\right)_{(3,\ 1)}(x-3), \quad \text{or,} \quad y-1=-\frac{1}{8}(x-3)$$

or, $\quad x+8y-11=0.$

Ex. 7. Show that the function

(i) $f(x)=x^3-x^2-2x$ is decreasing at the point $x=1$ and increasing at the point $x=2$.

(ii) $f(x)=x^3-6x^2+12x-1$ is increasing everywhere,

(i) $\because \quad f(x)=x^3-x^2-2x,$

$\therefore \quad f'(x)=3x^2-2x-2$

Now, $\quad f'(1)=3.1^2-2.1-2=-1<0.$

$\therefore \quad f(x)$ is decreasing at the point $x=1$.

$f'(2)=3.2^2-2.2-2=6>0$ and so the function is increasing at the point $x=2$.

(ii) $\because\ f'(x)=3x^2-12x+12=3(x^2-4x+4)$
$=3(x-2)^2\geqslant 0$ everywhere.

Hence the function is increasing everywhere.

Ex. 8. Determine the interval in which the function $f'(x)=2x^3-15x^2+36x+2$ is increasing and is decreasing.

$f(x)=2x^3-15x^2+36x+2$

$\therefore\ f'(x)=6x^2-30x+36=6(x-2)(x-3)$

Now, in $2<x<3$, $x-2$ is positive and $x-3$ is negative, so that $6(x-2)(x-3)$ is negative. Therefore, in the interval $2<x<3$, $f(x)$ is decreasing, and is increasing elsewhere.

Exercise 4H

1. Find the second order derivatives of the following functions

(i) $y=x^2-3x+4$ (ii) $y=ax^2+bx+c$ (iii) $y=\frac{1}{x}$

(iv) $y=x\sin x+\cos x$ (v) $y=ax^n+\frac{b}{x^n}$

(vi) $y=\log\sin x$ (vii) $y=x\log x-x$ (viii) $y=\frac{x-1}{x+1}$.

2. Find $\frac{d^2y}{dx^2}$:

(i) $y^2=4ax$ (ii) $x^2+y^2=a^2$ (iii) $xy=c^2$

(iv) $x^my^n=(x+y)^{m+n}$.

3. Find $\frac{d^2y}{dx^2}$:

(i) $x=a\cos\theta$, $y=a\sin\theta$, (parameter$=\theta$)

(ii) $x=a\sec\theta$, $y=b\tan\theta$, (iii) $x=at$ $y=\frac{a}{t}$, (Parameter t)

(iv) $x=a\cos^3\theta$, $y=b\sin^3\theta$,

(v) $x=a(t+\sin t)$, $y=a(1-\cos t)$

(vi) $x=\frac{3at}{1+t^2}$, $y=\frac{3at^2}{1+t^2}$.

4. Show that if

(i) $y=c_1 e^{2x}+c_2 e^{-x}$, then $\frac{d^2y}{dx^2}-\frac{dy}{dx}-2y=0$

(ii) $y=c_1 \sin nx+c_2 \cos nx$, then $\frac{d^2y}{dx^2}+n^2y=0$

(iii) $y=a \cos(\log x)+b \sin(\log x)$, then $x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+y=0$

(iv) $y(1-x)=x^2$, then $(1-x)\frac{d^2y}{dx^2}-2\frac{dy}{dx}=2$

(v) $y=\sin^{-1}x$, then, $(1-x^2)y_2-xy_1=0$

(vi) $y=c_1 \cos 3x+c_2 \sin 3x+\left(\frac{1}{18}x^2-\frac{1}{27}x+\frac{5}{81}\right)e^{2x}$, then,

$$\frac{d^2y}{dx^2}+9y=(x^2+1)e^{3x}.$$

(vii) $y=c_1 e^{mx}+c_2 e^{-mx}$, then $\frac{d^2y}{dx^2}-m^2y=0$

(viii) $y=\log(x+\sqrt{a^2+x^2})$, then $(a^2+x^2)\frac{d^2y}{dx^2}+x\frac{dy}{dx}=0$

(ix) $y=(x+\sqrt{1+x^2})^m$, then $(1+x^2)\frac{d^2y}{dx^2}+x\frac{dy}{dx}-m^2y=0$

(x) $y=(\sin^{-1}x)^2$, then $(1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}-2y=0$

(xi) $y=e^{\tan^{-1}x}$, then $(1+x^2)\frac{d^2y}{dx^2}+(2x-1)\frac{dy}{dx}=0$

5. Find the equations of the tangent and normal to the curve $y=x^4-3x^2+4$ at the point (1, 2).

6. Find the equations of the tangent and normal to the parabola $y^2=4ax$ at the point $(at^2, 2at)$.

7. Show that the equation of the tangent to the curve $\left(\frac{x}{a}\right)^n+\left(\frac{y}{b}\right)^n=2$ at the point (a, b) is $\frac{x}{a}+\frac{y}{b}=2$. Find the equation of the normal to the curve at this point.

8. Show that the function $f(x)=x^2-3x+1$ is decreasing at the point $x=1$ but increasing at the point $x=2$.

9. Show that (i) the function $3x(x^2+x+1)$ is increasing for all values of x and (ii) the function $-3x-x^3$ is decreasing for all values of x.

10. Show that the function $y=\sin z+\cos x$ is increasing in the interval $0<x<\frac{\pi}{2}$ but decreasing in the interval $\frac{\pi}{2}<x<\pi$.

11. (i) Find the interval of x in which the following functions are decreasing :

(*a*) $x^3-3x^2-24x+29$, (*b*) $2x^3-9x^2+12x+7$

(ii) Find the interval of x in which the function $6x^2-9x-x^3$ is increasing.

§ 4·13. Applications of derivatives.

In the previous articles we have given two types of interpretation of derivatives of functions. They are (i) $\frac{dy}{dx}$ is the rate of change of y with respect to x.

(ii) $\frac{dy}{dx}$ is the gradient of the tangent to the curve $y=f(x)$ at the the point (x, y).

From the first view point one can define various concepts such as velocity, acceleration, etc. The second view point helps in the discussion of different geometrical properties. We discuss below some applications of the concept of derivatives.

(A) Subtangent and subnormal.

If the tangent and normal to a curve at the point P intersect the x-axis at the points T and N respectively, then PT and PN are respectively the lengths of the tangent and normal at the the point P. The

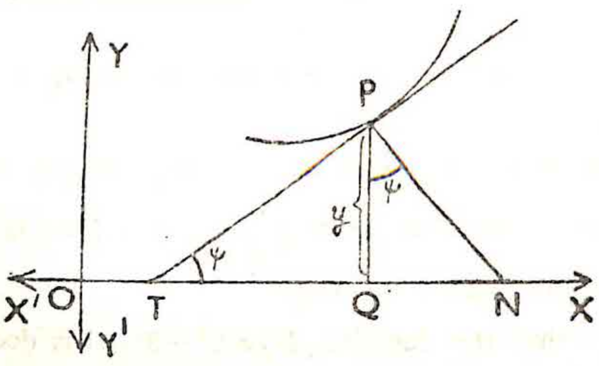


Fig. 41

orthogonal projections of these lengths on the the x-axis are respectively called the subtangent and the subnormal to the curve at the point P. In the figure the ordinate of P intersects the x-axis at the

point Q. $\overline{TQ}$ is the subtangent and $\overline{QN}$ the subnormal to the curve at the point P.

If the co-ordinates of the point P be (x, y), then $PQ=y$ and $\tan PTQ=\tan\psi=\dfrac{dy}{dx}$.

Hence the length of the tangent$=PT=\dfrac{PT}{PQ}.PQ$.

$$=y\operatorname{cosec}\psi=y\sqrt{1+\cot^2\psi}$$

$$=y\sqrt{1+\left(\frac{dx}{dy}\right)^2}=\frac{y\sqrt{1+y_1^{\,2}}}{y_1}\text{ where } y_1=\frac{dy}{dx}.$$

The length of the normal$=PN=\dfrac{PN}{PQ}.PQ$

$$=y\sec\psi=y\sqrt{1+\tan^2\psi}$$

$$=y\sqrt{1+\left(\frac{dy}{dx}\right)^2}=y\sqrt{1+y_1^{\,2}}.$$

The subtangent$=TQ=\dfrac{TQ}{PQ}.PQ=y\cot\psi=y.\dfrac{dx}{dy}=\dfrac{y}{y_1}$.

The subnormal$=NQ=\dfrac{NQ}{PQ}PQ=y\tan\psi=y\dfrac{dy}{dx}=yy_1$.

Example. 1. Find the lengths of the tangent, normal, subtangent and subnormal to the curve $y=x+x^3$ at the point (1, 2).

$\because\ y=x+x^3,\quad\therefore\ y_1=1+3x^2$.

Hence at the point (1, 2), $y_1=1+3\ 1^2=4$.

$\therefore$ the length of the tangent$=\dfrac{y\sqrt{1+y_1^{\,2}}}{y_1}=\dfrac{2\sqrt{1+16}}{4}=\dfrac{\sqrt{17}}{2}$,

the length of the normal$=y\sqrt{1+y_1^{\,2}}=2\sqrt{1+16}=2\sqrt{17}$;

the length of the subtangent$=\dfrac{y}{y_1}=\dfrac{2}{4}=\dfrac{1}{2}$ and

the length of the subnormal$=yy_1=2.4=8$.

Ex. 2. Find the lengths of the tangent, normal, subtangent and subnormal to the curve $y^2=4ax$ at any point (x, y) of the curve.

$$y^2=4ax,\quad\therefore\quad 2y.y_1=4a,\quad\therefore\quad y_1=\frac{2a}{y}.$$

Hence the length of the tangent$=\dfrac{y\sqrt{1+y_1^2}}{y_1}$

$$=\frac{y\sqrt{1+\dfrac{4a^2}{y^2}}}{\dfrac{2a}{y}}=\frac{y^2}{2a}\sqrt{1+\frac{4a^2}{4ax}}$$

$$=\frac{4ax}{2a}\sqrt{1+\frac{a}{x}}=\frac{2x\sqrt{x+a}}{\sqrt{x}}=2\sqrt{x}\cdot\sqrt{x+a}$$

$$=2\sqrt{x(x+a)}.$$

The length of the normal$=y\sqrt{1+y_1^2}$

$$=y\sqrt{1+\frac{4a^2}{4ax}}=2\sqrt{ax}\sqrt{1+\frac{a}{x}}$$

$$=2\sqrt{a}\sqrt{x}\frac{\sqrt{x+a}}{\sqrt{x}}=2\sqrt{a(x+a)}.$$

The length of the subtangent$=\dfrac{y}{y_1}=\dfrac{y}{\dfrac{2a}{y}}=\dfrac{y^2}{2a}=\dfrac{4ax}{2a}=2x.$

The length of the subnormal$=yy_1=y.\dfrac{2a}{y}=2a.$

(B) Maxima and Minima.

In the domain of definition of a function $f(x)$, the function is said to have a maximum value at a point $x=a$ if there exists an interval including the point, so that the maximum value of the function in the interval is attained at the point $x=a$. Hence the function $f(x)$ has a maximum at the point $x=a$, if one can find an interval $a-h\leqslant x\leqslant a+h$ including the point a, so that $f(a)$ is the maximum value of the function in that interval, *i.e.*, for all x in $a-h\leqslant x\leqslant a+h$, $f(x)\leqslant f(a)$.

If the function $f(x)$ possesses a maximum value at the point $x=a$ and if $f(x)$ be differentiable at the point $x=a$, then $f'(a)=0$; for if $f(a)>0$, then $f(x)$ is increasing at $x=a$ and for every x near the point $x=a$, $f(x)>f(a)$, i e, we shall not get any interval including the point $x=a$, such that the maximum value of $f(x)$ in the interval is $f(a)$. Similarly if $f'(a)<0$, then $f(x)$ is decreasing at $x=a$ and so for every $x<a$ and near a, $f(x)>f(a)$ and so we shall not get any interval within which the maximum value of $f(x)$ is $f(a)$. Hence $f'(a)=0$.

Again, if $f'(a)=0$, then the tangent to the curve at the point $x=a$ is parallel to the x-axis.

In the figure notice that the function $f(x)$ has maximum value at the point A $(x=a)$ and the values of $f(x)$ in the neighbouring points B and C are less than $f(a)$.

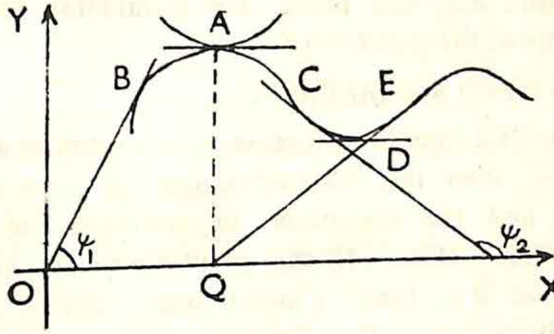


Fig. 42

The slope ψ_1 of the tangent to the curve at the point is less than 90° and that at the point C is $\psi_2>90°$. So, $\tan\psi_1>0$ and $\tan\psi_2<0$. Hence if $f(x)$ possessess a maximum, then at the neighbouring points of A which are on its left, $\tan\psi>0$ and at those points which are to the right of A, $\tan\psi<0$. As $\tan\psi=\frac{dy}{dx}$, so one may say that the value of $\frac{dy}{dx}$ is positive at points to the left of the point of maximum and $\frac{dy}{dx}$ is negative at points to the right of the point of maximum. Hence the value of $\frac{dy}{dx}$ is decreasing at the point $x=a$. Hence the derivative of $\frac{dy}{dx}$ i.e. $\frac{d}{dx}\left(\frac{d}{dx}\right)=\frac{d^2y}{dx^2}<0$ at $x=a$. The above discussion can be summarised as follows.

A function $y=f(x)$ will have a maximum at the point $x=a$ if (i) $f'(a)=0$ and (ii) $f''(a)<0$.

Minimum : A function $f(x)$ is said to have a minimum at the point $x=a$ if one can find an interval $a-h\leqslant x\leqslant a+h$ including the point $x=a$, such that $f(x)$ has the minimum value within the interval i e, at every point of the interval $f(x)\geqslant f(a)$.

In figure 42 the function $f(x)$ has a minimum at the point D $\tan\psi<0$ at the point C on the left of the point D and $\tan\psi>0$ at the point E to the right of D. Hence as a point moves from the left of the point D to its right along the curve, the value of $\frac{dy}{dx}$, i.e., $f'(x)$ changes from negative to positive. Hence as before one can prove that a function $f(x)$ will have a minimum value at the point $x=a$ if

(i) $f'(a)=0$ and (ii) $f''(a)>0$.

Note. (i) If a function possesses a maximum or a minimum at the point $x=a$, then the function is said to have an *extremum* at the point and the maximum or minimum values are said to be its extreme values. If the point $x=a$ is an extreme point of the function $f(x)$, then $f'(a)=0$ and $f''(a)\neq 0$. If $f''(a)<0$, then at the point $x=a$, the function possesses a maximum and if $f''(a)>0$, then the function will have a minimum value at the point.

(ii) If $f'(a)=0, f''(a)=0$, then the function may not have an extreme value at the point $x=a$. We state below the conditions of existence of extreme value of a function $f(x)$ at the point $x=a$. If $f'(a)=f''(a)=\cdots=f^{n-1}(a)=0, f^{n}(a)\neq 0$, then the function will possess an extreme value at $x=a$, if n is even and the extreme value is maximum or minimum according as $f^{n}(a)<0$ or $f^{n}(a)>0$. If n be odd, the function will be neither maximum nor minimum.

(iii) If a function $f(x)$ possesses a maximum value at the point $x=a$, then $f(a)$ may not be the greatest value of $f(x)$. That a function possesses a maximum value at the point $x=a$ means that, the value of $f(x)$ is maximum at the point $x=a$ within an interval including the point $x=a$. The value of the function outside the interval may be greater than $f(a)$. Similarly one can say that a function possesses a minimum at a point $x=a$ does not mean that the least value of $f(x)$ is $f(a)$.

In fact a function may possess a number of extreme values but the function will have not more than one greatest or one least value.

Ex. 1. Find the maximum and minimum values of the function $y=6-x^2-x^3-\frac{1}{4}x^4$.

$\because\ y=6-x^2-x^3-\frac{1}{4}x^4,\ \therefore\ \frac{dy}{dx}=-2x-3x^2-x^3$

and $\frac{d^2y}{dx^2}=-2-6x-3x^2$.

For extreme values, $\frac{dy}{dx}=0$, i.e , $-2x-3x^2-x^3=0$

or, $-x(x+1)(x+2)=0$

$\therefore$ the function may possess extreme values at the points

$$x=0,\ -1,\ -2.$$

Now $\left(\frac{d^2y}{dx^2}\right)_{x=0}=-2$ which is negative and hence the function $f(x)$ possesses a maximum at $x=0$.

$\left(\frac{d^2y}{dx^2}\right)_{x=-1}=-2+6-3=1>0$. Hence the function possesses a minimum at the point $x=-1$.

Again at $x=-2$, $\left(\frac{d^2y}{dx^2}\right)_{x=-2}=-2<0$.

Hence the function possesses a maximum at the points $x=0,\ -2$.

The maximum values are 6 and 6 and the minimum value is $f(-1)=5\frac{3}{4}$.

Ex 2. Show that the maximum and minimum values of $3\sin x+4\cos x$ are respectively $+5$ and -5.

Let $y=3\sin x+4\cos x$

$\therefore\ \frac{dy}{dx}=3\cos x-4\sin x$

Now for extreme values $\frac{dy}{dx}=0$.

$\therefore\ 3\cos x-4\sin x=0$.

$\therefore\ \frac{\sin x}{3}=\frac{\cos x}{4}=\pm\frac{1}{\sqrt{3^2+4^2}}=\pm\frac{1}{5}$.

$\therefore$ The function will possess extreme values if $\sin x=\pm\frac{3}{5}$ and $\cos x=\pm\frac{4}{5}$.

When $\sin x = \frac{3}{5}$, $\cos x = \frac{4}{5}$, then $\dfrac{d^2y}{dx^2} = -3 \sin x - 4 \cos x$

$$= -\tfrac{9}{5} - \tfrac{16}{5} = -5 < 0.$$

$\therefore$ For the value of x for which $\cos x = \frac{4}{5}$, $\sin x = \frac{3}{5}$, $f(x)$ possesses an extreme value which is maximum and this maximum value is $3.\frac{3}{5} + 4.\frac{4}{5} = \frac{9}{5} + \frac{16}{5} = \frac{25}{5} = 5$.

When $\sin x = -\frac{3}{5}$, $\cos x = -\frac{4}{5}$, then the value of

$$\frac{d^2y}{dx^2} = \tfrac{9}{5} + \tfrac{16}{5} = +5 > 0.$$

$\therefore$ For the value of x for which $\sin x = -\frac{3}{5}$ and $\cos x = -\frac{4}{5}$, $f(x)$ will have a minimum value and this minimum value

$$= 3(-\tfrac{3}{5}) + 4(-\tfrac{4}{5}) = -5$$

Exercise - 4 I

1. Find the lengths of the tangent, normal, subtangent and subnormal to the curve $y = \frac{1}{6}x^3(1+2x)$ at the point $x=1$.

2. Find the lengths of the tangent, normal, subtangent and subnormal to the curve $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$ at the point $\theta = \dfrac{\pi}{2}$.

3. Find the lengths of the tangent, normal, subtangent and subnormal to the curve $x = a(\cos t + t \sin t)$, $y = a(\sin t - t \cos t)$ at any point t.

4. Show that the subtangent to the curve $y = a^x$ is constant.

5. Find the maximum and minimum values of $y = x^3 - 9x^2 + 24x - 10$.

6. Find the maximum and minimum values of the following functions :

(i) $y = \sin x + \sin^2 x$, (ii) $y = \sin x + \cos^2 x$

(iii) $y = \cos^2 x - \cos x$.

7. Show that $+\sqrt{a^2+b^2}$ and $-\sqrt{a^2+b^2}$ are respectively the maximum and minimum values of $y = a \cos x + b \sin x$.

Miscellaneous Examples

Ex. 1. If $f(x)$ be a differentiable function and $y = \{f(x)\}^n$, show that $\dfrac{dy}{dx} = n\{f(x)\}^{n-1} f'(x)$. From the above formula find the differential coefficients of the following functions :

(i) $y=(x^2+4)^{10}$ (ii) $\sqrt[3]{x^2+1}$ (iii) $\dfrac{1}{(x+1)^2}$

(iv) $\sin^4 x$ (v) $(\log x)^{\frac{3}{2}}$ (vi) $(\sec^{-1}x)^3$.

$y=\{f(x)\}^n=u^n$, where $u=f(x)$

$\therefore \quad \dfrac{dy}{du}=nu^{n-1},\ \dfrac{du}{dx}=f'(x)$

$\therefore \quad \dfrac{dy}{dx}=\dfrac{dy}{du}\cdot\dfrac{du}{dx}=nu^{n-1}f'(x)=n\{f(x)\}^{n-1}f'(x).$

(i) $y=(x^2+4)^{10}$. Here $n=10$ and $f(x)=x^2+4$

$\therefore \quad f'(x)=2x. \quad \therefore \quad \dfrac{dy}{dx}=10(x^2+4)^9.2x=20x(x^2+4)^9.$

(ii) $y=(x^2+1)^{\frac{1}{3}} \quad \therefore \quad \dfrac{dy}{dx}=\frac{1}{3}(x^2+1)^{\frac{1}{3}-1}\ 2x=\dfrac{2x}{3(x^2+1)^{\frac{2}{3}}}$

(iii) $y=\dfrac{1}{(x+1)^2}=(x+1)^{-2}$

$\therefore \quad \dfrac{dy}{dx}=-2(x+1)^{-2-1}.1=\dfrac{-2}{(x+1)^3}.$

(iv) $y=\sin^4 x=(\sin x)^4$

$\therefore \quad \dfrac{dy}{dx}=4(\sin x)^{4-1}.\cos x=4\cos x\sin^3 x.$

(v) $\because \quad y=(\log x)^{\frac{3}{2}}, \quad \therefore \quad \dfrac{dy}{dx}=\dfrac{3}{2}(\log x)^{\frac{3}{2}-1}\dfrac{d}{dx}.(\log x)$

$$=\frac{3}{2}\cdot\frac{1}{x}\cdot\sqrt{\log x}$$

(vi) $\because \quad y=(\sec^{-1}x)^3, \quad \therefore \quad \dfrac{dy}{dx}=3(\sec^{-1}x)^{3-1}.\dfrac{d}{dx}(\sec^{-1}x)$

$$=3\,(\sec^{-1}x)^2.\frac{1}{x\sqrt{x^2-1}}.$$

Ex. 2. If $y=\log f(x)$, then show that $\dfrac{dy}{dx}=\dfrac{f'(x)}{f(x)}$, where $f(x)$ is a differentiable function. Hence find the derivatives of

(i) $y=\log\sin x$ (ii) $y=\log(x^2+x+1)$

(iii) $y=\log(\log x)$ (iv) $y=\log(\sec x+\tan x)$

(v) $y=\log(x+\sqrt{x^2+1})$ (vi) $y=\log(\tan^{-1}x)$.

$y = \log\{f(x)\} = \log u$ where $u = f(x)$.

$\therefore \quad \frac{dy}{du} = \frac{1}{u}$ and $\frac{du}{dx} = f'(x)$.

Now, $\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} = \frac{1}{u} \cdot f'(x) = \frac{1}{f(x)} f'(x) = \frac{f'(x)}{f(x)}$.

(i) $y = \log \sin x$, Here $f(x) = \sin x$. $\therefore \quad f'(x) = \cos x$

$\therefore \quad \frac{dy}{dx} = \frac{1}{\sin x} \cdot \cos x = \cot x$.

(ii) $y = \log(x^2 + x + 1)$,

$\therefore \quad \frac{dy}{dx} = \frac{1}{x^2 + x + 1} \cdot \frac{d}{dx}(x^2 + x + 1) \quad = \frac{2x+1}{x^2 + x + 1}$.

(iii) $y = \log(\log x)$, $\therefore \quad \frac{dy}{dx} = \frac{1}{\log x} (\log x)' = \frac{1}{x \log x}$.

(iv) $y = \log(\sec x + \tan x)$,

$\therefore \quad \frac{dy}{dx} = \frac{1}{\sec x + \tan x} (\sec x . \tan x + \sec^2 x) = \sec x$.

(v) $y = \log(x + \sqrt{x^2 + 1})$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{x + \sqrt{x^2+1}} \left\{ 1 + \tfrac{1}{2}(x^2+1)^{-\frac{1}{2}} . 2x \right\}$$

$$= \frac{1}{x + \sqrt{x^2+1}} \left\{ 1 + \frac{x}{\sqrt{x^2+1}} \right\} = \frac{1}{\sqrt{x^2+1}}$$

(vi) $y = \log(\tan^{-1} x)$, $\therefore \quad \frac{dy}{dx} = \frac{1}{\tan^{-1} x} \cdot \frac{1}{1+x^2}$.

Ex. 3. If $y = f(ax+b)$, where $f(x)$ is a differentiable function, show that $\frac{dy}{dx} = a \frac{d}{dz} \{f(z)\}$, where $z = ax + b$.

Using the above formula, find the derivatives of the following functions.

(i) $y = (3x+4)^{10}$ (ii) $y = \log(2x+3)$ (iii) $y = e^{-2x}$
(iv) $y = \sin 2x$ (v) $y = \sec(3 - 2x)$ (vi) $y = \tan^{-1} ax$

$y = f(ax+b) = f(z)$, where $z = ax + b$.

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{dz} \frac{dz}{dx} = \frac{dy}{dz} \frac{d}{dx}(ax+b) = a \frac{d}{dz} \{f(z)\}$$

(i) $\frac{dy}{dx}=3.\frac{d}{dz}z^{10}$, where $z=3x+4$

$=3.10\ z^9=30\ z^9=30(3x+4)^9$

(ii) $\frac{dy}{dx}=2\cdot\frac{1}{2x+3}=\frac{2}{2x+3}$

(iii) $\frac{dy}{dx}=-2e^{-2x}$ (iv) $\frac{dy}{dx}=2\cos 2x$

(v) $\frac{dy}{dx}=-2\sec(3-2x)\tan(3-2x)$

(vi) $\frac{dy}{dx}=\frac{a}{1+a^2x^2}$.

Ex. 4. Find $\frac{dy}{dx}$ if

(i) $\tan y=\frac{\cos x}{1+\sin x}$ (ii) $\tan \frac{1}{2}y=\sqrt{\frac{1-x}{1+x}}$.

(i) $\because\ \tan y=\frac{\cos x}{1+\sin x}$,

$$\therefore\ y=\tan^{-1}\frac{\cos x}{1+\sin x}=\tan^{-1}\frac{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)}$$

$$=\tan^{-1}\frac{\cos\frac{x}{2}-\sin\frac{x}{2}}{\cos\frac{x}{2}+\sin\frac{x}{2}}$$

$$=\tan^{-1}\frac{1-\tan\frac{x}{2}}{1+\tan\frac{x}{2}}=\tan^{-1}\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)=\frac{\pi}{4}-\frac{x}{2}.$$

$$\therefore\ \frac{dy}{dx}=\frac{d}{dx}\left(\frac{\pi}{4}\right)-\frac{d}{dx}\cdot\frac{x}{2}=0-\tfrac{1}{2}=-\tfrac{1}{2}.\ \left(\because\ \frac{\pi}{4}=\text{constant}\right)$$

(ii) $\tan\frac{1}{2}y=\sqrt{\frac{1-x}{1+x}}=\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}$, $[x=\cos\theta$ (say)$]$

$$=\sqrt{\frac{2\sin^2\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}}=\tan\frac{\theta}{2}$$

$\therefore\ \frac{1}{2}y=\frac{1}{2}\theta,\ \therefore\ y=\theta=\cos^{-1}x.$

$\therefore\ \frac{dy}{dx}=-\frac{1}{\sqrt{1-x^2}}.$

Ex. 5. If $x=3\tan^{-1}\dfrac{2t}{1-t^2}$, $y=2\sin^{-1}\dfrac{2t}{1+t^2}$, then find $\dfrac{dy}{dx}$.

Let $t=\tan\theta$

$\therefore\quad x=3\tan^{-1}\dfrac{2\tan\theta}{1-\tan^2\theta}=3\tan^{-1}\tan 2\theta=3.2\theta=6\theta$

and $y=2\sin^{-1}\dfrac{2\tan\theta}{1+\tan^2\theta}=2\sin^{-1}\sin 2\theta=2.2\theta=4\theta$

$\therefore\quad \dfrac{dx}{d\theta}=6$ and $\dfrac{dy}{d\theta}=4$.

$\therefore\quad \dfrac{dy}{dx}=\dfrac{dy}{d\theta}\Big/\dfrac{dx}{d\theta}=\dfrac{4}{6}=\dfrac{2}{3}$.

Ex. 6. Differentiate

(i) x^6 with respect to x^3

(ii) $x^{\tan^{-1}x}$ with respect to $\tan^{-1}x$

(iii) $\sin^{-1}\dfrac{2x}{1+x^2}$ with respect to $\cot^{-1}x$

(i) Let $y=x^6$, $z=x^3$.

$\therefore$ differential coefficient of x^6 with respect to x^3

$$=\frac{dy}{dz}=\frac{dy}{dx}\Big/\frac{dz}{dx}=\frac{6x^5}{3x^2}=2x^3.$$

(ii) Let $y=x^{\tan^{-1}x}$, $\quad\therefore\ \log y=\tan^{-1}x\log x$;

Differentiating with respect to x,

$$\frac{1}{y}\frac{dy}{dx}=\frac{1}{x}\tan^{-1}x+\frac{\log x}{1+x^2}$$

$$\therefore\quad \frac{dy}{dx}=x^{\tan^{-1}x}\left(\frac{1}{x}\tan^{-1}x+\frac{\log x}{1+x^2}\right)$$

Let $z=\tan^{-1}x$, $\quad\therefore\ \dfrac{dz}{dx}=\dfrac{1}{1+x^2}$.

Now derivative of $x^{\tan^{-1}x}$ with respect to $\tan^{-1}x$

$$=\frac{dy}{dz}=\frac{\dfrac{dy}{dx}}{\dfrac{dz}{dx}}$$

$$=x^{\tan^{-1}x}\left(\frac{1}{x}\tan^{-1}x+\frac{\log x}{1+x^2}\right)\Big/\frac{1}{1+x^2}$$

$$=x^{\tan^{-1}x}\left(\frac{1+x^2}{x}\tan^{-1}x+\log x\right).$$

(iii) Let $y=\sin^{-1}\dfrac{2x}{1+x^2}=\sin^{-1}\dfrac{2\tan\theta}{1+\tan^2\theta}$

[$x=\tan\theta$ (say)]

$=\sin^{-1}\sin 2\theta=2\theta \;\therefore\; \dfrac{dy}{d\theta}=2$

and $z=\cot^{-1}x=\cot^{-1}\tan\theta=\cot^{-1}\cot\left(\frac{\pi}{2}-\theta\right)=\frac{\pi}{2}-\theta$

$$\therefore\; \frac{dz}{d\theta}=-1$$

$\therefore$ Derivative of $\sin^{-1}\dfrac{2x}{1+x^2}$ with respect to $\cot^{-1}x$

$$=\frac{dy}{dz}=\frac{\frac{dy}{d\theta}}{\frac{dz}{d\theta}}=\frac{2}{-1}=-2.$$

Ex. 7. Find $\dfrac{dy}{dx}$ when

(i) $y=\log\sqrt{\left\{\dfrac{1+\sin x}{1-\sin x}\right\}}$ [C. U. '75]

(ii) $y=x^2\sqrt{\dfrac{x^2+x+1}{x^2-x+1}}.\sin x$

(iii) $y=\frac{1}{2}x.\;\sqrt{a^2-x^2}+\frac{1}{2}a^2\sin^{-1}\left(\frac{x}{a}\right)$

(iv) $y=(\sin x)^{\cos x}+(\cos x)^{\sin x}$, (v) $y=\log_x \sin x$.

(i) $y=\log\sqrt{\dfrac{1+\sin x}{1-\sin x}}=\frac{1}{2}\left\{\log(1+\sin x)-\log(1-\sin x)\right\}$

$$\therefore\; \frac{dy}{dx}=\frac{1}{2}\left[\frac{\cos x}{1+\sin x}-\frac{-\cos x}{1-\sin x}\right]$$

$$=\frac{1}{2}\cos x\;\frac{1-\sin x+1+\sin x}{(1+\sin x)(1-\sin x)}$$

$$=\frac{\cos x}{1-\sin^2 x}=\frac{1}{\cos x}=\sec x.$$

(ii) $y=x^2\sqrt{\dfrac{x^2+x+1}{x^2-x+1}}.\sin x,$

$$\therefore\; \log y=\log\left\{x^2\sqrt{\frac{x^2+x+1}{x^2-x+1}}.\sin x\right\}$$

$=2\log x+\frac{1}{2}\log(x^2+x+1)-\frac{1}{2}\log(x^2-x+1)+\log\sin x$

Differentiating both sides with respect to x.

$$\frac{1}{y}\frac{dy}{dx}=\frac{2}{x}+\frac{1}{2}\frac{2x+1}{x^2+x+1}-\frac{1}{2}\frac{2x-1}{x^2-x+1}+\frac{\cos x}{\sin x}$$

$$=\frac{2}{x}+\frac{1-x^2}{1+x^2+x^4}+\cot x.$$

$$\therefore \frac{dy}{dx}=x^2\sqrt{\frac{x^2+x+1}{x^2-x+1}}.\sin x\left\{\frac{2}{x}+\frac{1-x^2}{1+x^2+x^4}+\cot x\right\}.$$

(iii) $y=\frac{1}{2}x\sqrt{a^2-x^2}+\frac{1}{2}a^2\sin^{-1}\frac{x}{a}$

$$\therefore \frac{dy}{dx}=\frac{1}{2}.1.\sqrt{a^2-x^2}+\frac{1}{2}.x\frac{(-2x)}{2\sqrt{a^2-x^2}}+\frac{1}{2}a^2\frac{1}{\sqrt{1-\frac{x^2}{a^2}}}.\frac{1}{a}$$

$$=\frac{1}{2}\sqrt{a^2-x^2}-\frac{1}{2}\frac{x^2}{\sqrt{a^2-x^2}}+\frac{1}{2}\frac{a^2}{\sqrt{a^2-x^2}}$$

$$=\frac{1}{2}\sqrt{a^2-x^2}+\frac{1}{2}\frac{a^2-x^2}{\sqrt{a^2-x^2}}=\frac{1}{2}\sqrt{a^2-x^2}+\frac{1}{2}\sqrt{a^2-x^2}$$

$$=\sqrt{a^2-x^2}.$$

(iv) $y=(\sin x)^{\cos x}+(\cos x)^{\sin x}=u+v$, where

$u=(\sin x)^{\cos x}$, $\therefore$ $\log u=\cos x\log\sin x$

and $v=(\cos x)^{\sin x}$, $\therefore$ $\log v=\sin x\log\cos x$..

Differentiating $w.\ r\ t\ x$,

$$\frac{1}{u}.\frac{du}{dx}=-\sin x.\log\sin x+\cos x.\frac{\cos x}{\sin x}, \text{ and}$$

$$\frac{1}{v}\frac{dv}{dx}=\cos x\log\cos x+\sin x.\frac{-\sin x}{\cos x}$$

$$\therefore \frac{du}{dx}=(\sin x)^{\cos x}\left\{-\sin x\log\sin x+\cot x.\cos x\right\}$$

and $\dfrac{dv}{dx}=(\cos x)^{\sin x}\left\{\cos x.\log\cos x-\tan x.\sin x\right\}$

$$\therefore \frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx}$$

$$=(\sin x)^{\cos x}\{-\sin x\log\sin x+\cot x.\cos x\}$$

$$+(\cos x)^{\sin x}\{\cos x\log\cos x-\tan x.\sin x\}$$

(v) $y=\log_x \sin x=\dfrac{\log \sin x}{\log x}$; $\left[\because \ \log_b a=\dfrac{\log_e a}{\log_e b}\right]$

$$\therefore \quad \frac{dy}{dx}=\frac{\log x \dfrac{\cos x}{\sin x}-\log \sin x.\dfrac{1}{x}}{(\log x)^2}$$

$$=\frac{x \cot x.\log x-\log \sin x}{x(\log x)^2}.$$

Ex. 8. $f(x)$ and $g(x)$ are two differentiable functions. If for all values of x, in an interval

(i) $f'(x)=0$, then $f(x)=c$ in that interval, where c is a constant.

(ii) $f'(x)=g'(x)$, then $f(x)=g(x)+c$ where c is a constant.

(i) Let $y=f(x)$, $\therefore \dfrac{dy}{dx}=f'(x)=0$. Now $\dfrac{dy}{dx}$ is the gradient of the tangent to the curve $y=f(x)$ at the point (x, y). As $\dfrac{dy}{dx}=0$ at all points, so the tangents to the curve $y=f(x)$, at all points of the given interval, are parallel to the x-axis. This is possible if and only if the portion of the graph of the function corresponding to all values of x within the interval be parallel to the x-axis. Now the equation of a straight line parallel to the x-axis is $y=c$. $\therefore$ If $f'(x)=0$, then $f(x)=c$.

(ii) Let $\phi(x)=f(x)-g(x)$

$\therefore \quad \phi'(x)=f'(x)-g'(x)=0$ (according to given condition)

$\therefore \quad \phi(x)=c$, where c is a constant (by (i) above)

or, $f(x)-g(x)=c$, or, $f(x)=g(x)+c$.

Ex. 9. If a function be differentiable at a point, then it is continuous at that point. Is the converse of this proposition true ? Explain with the help of examples.

Let the function $f(x)$ be differentiable at the point $x=a$.

$\therefore \quad f'(a)=\lim\limits_{h\to 0} \dfrac{f(a+h)-f(a)}{h}$ has a definite value.

Let $a+h=x$ and so as $h\to 0$, then $x\to a$

and $f'(a)=\lim\limits_{x\to a} \dfrac{f(x)-f(a)}{x-a}$.

Now we can write

$$f(x)=\frac{f(x)-f(a)}{x-a}(x-a)+f(a)$$

$$\therefore \quad \lim_{x\to a} f(x)=\lim_{x\to a}\left\{\frac{f(x)-f(a)}{x-a}(x-a)+f(a)\right\}$$

($\because$ the limits on both sides exist)

$$=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}.\lim_{x\to a}(x-a)+\lim_{x\to a} f(a).$$

$$\left[\because \lim_{x\to a}\frac{f(x)-f(a)}{x-a} \text{ exists}\right]$$

$$=f'(a).0+f(a)=f(a).$$

Hence by definition, $f(x)$ is continuous at the point $x=a$. Hence if a function be differentiable at a point $x=a$, then it is continuos at that point.

But the converse of this proposition is not always true. A function may be continuous at a point but not differentiable at that point.

For example let $f(x)=|x|$. Here the function is continuous at the point $x=0$.

For, when $x>0$, then $f(x)=|x|=x$ and

$$\lim_{x\to 0+} f(x)=\lim_{x\to 0+} x=0.$$

and when $x<0$, then $f(x)=-x$.

So, $$\lim_{x\to 0-} f(x)=\lim_{x\to 0-}(-x)=0.$$

Again $f(0)=0$. $\therefore$ $$\lim_{x\to 0+} f(x)=\lim_{x\to 0-} f(x)=f(0).$$

But the function is not differentiable at the point $x=0$. For the right derivative of the function at the point $x=0$ is

$$f'(0+)=\lim_{h\to 0+}\frac{f(h)-f(0)}{h}=\lim_{h\to 0+}\frac{|h|-0}{h}$$

$$=\lim_{h\to 0+}\frac{h}{h}=1 \quad [\because h\to 0+, \therefore h>0 \text{ and } |h|=h]$$

The left derivative of the function at $x=0$ is

$$f'(0-)=\lim_{h\to 0-}\frac{f(h)-f(0)}{h}=\lim_{h\to 0-}\frac{|h|-0}{h}$$

$$=\lim_{h\to 0-}\frac{-h}{h}=-1 \quad [\because\ h\to 0-,\ \therefore\ h<0 \text{ and } |h|=-h]$$

$$\therefore\ f'(0+)\neq f'(0-)$$

Hence the function is not differentiable at the point, though continuous at the point.

Ex. 10. If $ax^2+2hxy+by^2=1$, then show that

$$\frac{dy}{dx}=-\frac{ax+hy}{hx+by} \text{ and } \frac{d^2y}{dx^2}=\frac{h^2-ab}{(hx+by)^3},$$

$ax^2+2hxy+by^2=1$, Differentiating both sides with respect to x we get

$$2ax+2h(x\cdot\frac{dy}{dx}+1.y)+2by\frac{dy}{dx}=0,$$

or, $\dfrac{dy}{dx}=-\dfrac{ax+hy}{hx+by}$. Differentiating both sides again with respect to x we get,

$$\frac{dy^2}{dx^2}=\frac{d}{dx}\left(-\frac{ax+hy}{hx+by}\right)$$

$$=-\frac{\left(a+h\frac{dy}{dx}\right)(hx+by)-(ax+hy)\left(h+b\frac{dy}{dx}\right)}{(hx+by)^2}$$

$$=-\frac{(ab-h^2)y+(h^2-ab)x.\frac{dy}{dx}}{(hx+by)^2}$$

$$=(h^2-ab)\frac{y+x\left(-\frac{dy}{dx}\right)}{(hx+by)^2}=(h^2-ab).\frac{y+x\frac{ax+hy}{hx+by}}{(hx+by)^2}$$

$$=(h^2-ab)\frac{y(hx+by)+x(ax+hy)}{(hx+by)^3},$$

$$=(h^2-ab).\frac{ax^2+2hxy+by^2}{(hx+by)^3}$$

$$=\frac{h^2-ab}{(hx+by)^3} \quad [\because\ ax^2+2hxy+by^2=1]$$

Ex. 11. Show that $\dfrac{d^2y}{dx^2}=-\dfrac{\dfrac{d^2x}{dy^2}}{\left(\dfrac{dx}{dy}\right)^2}$, and hence find $\dfrac{d^2y}{dx^2}$ when $\sin y=x\sin(a+y)$.

We know $\dfrac{dy}{dx}=\dfrac{1}{\dfrac{dx}{dy}}$

$$\therefore\quad \frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{1}{\dfrac{dx}{dy}}\right)=\frac{d}{dy}\left(\frac{1}{\dfrac{dx}{dy}}\right)\cdot\frac{dy}{dx}=-\frac{1}{\left(\dfrac{dx}{dy}\right)^2}\cdot\frac{d^2x}{dy^2}\cdot\frac{1}{\dfrac{dx}{dy}}$$

$$=-\frac{\dfrac{d^2x}{dy^2}}{\left(\dfrac{dx}{dy}\right)^3}.$$

Now, $\sin y=x\sin(a+y)$, or, $x=\dfrac{\sin y}{\sin(a+y)}$

$$\therefore\quad \frac{dx}{dy}=\frac{\sin(a+y)\cos y-\cos(a+y).\sin y}{\sin^2(a+y)}=\frac{\sin a}{\sin^2(a+y)},$$

$$\therefore\quad \frac{dy}{dx}=\frac{\sin^2(a+y)}{\sin a},\quad \frac{d^2x}{dy^2}=\frac{-2\sin a}{\sin^3(a+y)}.\cos(a+y).$$

$$\therefore\quad \frac{d^2y}{dx^2}=-\frac{\dfrac{d^2x}{dy^2}}{\left(\dfrac{dx}{dy}\right)^3}=-\frac{\dfrac{-2\sin a\cos(a+y)}{\sin^3(a+y)}}{\dfrac{\sin^3 a}{\sin^6(a+y)}}$$

$$=\frac{2\sin^3(a+y)\cos(a+y)}{\sin^2 a}.$$

Ex. 12. The actual radius of a sphere is 5 cms. But in measurement the error was ·5 mms. Find the error in volume. What is the percentage of error?

Let v be the volume of the sphere and r be its radius.

$\therefore\quad v=\frac{4}{3}\pi r^3.$

$\therefore\quad \dfrac{dv}{dr}=4\pi r^2.$ If for the increment δr of r, the change in volume be δv, then $\delta v=\dfrac{dv}{\delta r}\delta r=4\pi r^2\delta r.$

Now $r=5$, $\delta r=\cdot5\text{mm.}=\cdot05$ cm.

$\therefore$ error in volume of the sphere

$=\delta v=4.\pi.5^2(\cdot05)=5\pi=15\cdot7$ c.c. (nearly).

The approximate error per cent

$$=\frac{dv}{v}\times100=\frac{5\pi}{\frac{4}{3}\pi(5)^3}\times100=3\%.$$

Ex. 13. The time period of a pendulum of length l is given by $T=2\pi\sqrt{\dfrac{l}{g}}$. If the error in measuring the length be 1%, find the error in time period.

$$T=2\pi\sqrt{\frac{l}{g}},\quad\therefore\quad\log T=\log 2\pi+\tfrac{1}{2}\log l-\tfrac{1}{2}\log g$$

Differentiating both sides with respect to l we get,

$\dfrac{1}{T}\cdot\dfrac{dT}{dl}=\dfrac{1}{2}\,\dfrac{1}{l}$ [as π and g are constants]

$\therefore$ Percentage of error

$$=\frac{\delta T}{T}\times100=\frac{\frac{dT}{dl}.\delta l}{T}\times100=\frac{1}{T}\cdot\frac{dT}{dl}\cdot\delta l\times100$$

$=\dfrac{1}{2}\,\dfrac{\delta l}{l}\times100=\dfrac{1}{2}\times$ error in length per cent

$=\frac{1}{2}\times1\%=\frac{1}{2}\%.$

Ex. 14. Show that for all values of x, the function $x-\sin x$ is an increasing function. Hence prove that

when $x>0$, (i) $x-\sin x>0$

(ii) $\cos x-1+\frac{1}{2}x^2>0$

(iii) $\sin x-x+\frac{1}{6}x^3>0.$

Let $f(x)=x-\sin x$. $\therefore$ $f'(x)=1-\cos x$.

Now for all values of x, $\cos x\leqslant1$.

Hence for all values of x, $f'(x)\geqslant1-1=0$.

Hence $f(x)$ is increasing for all values of x.

(i) $\because$ $f(x)$ is increasing, if $x>0$, $f(x)>f(0)$

$\therefore$ $x-\sin x>0$, *i.e.*, $\sin x-x<0$, for all $x>0$.

(ii) Let $g(x)=\cos x-1+\frac{1}{2}x^2$

$\therefore$ $g'(x)=-\sin x+x=f(x)$

Now as $f(x)$ is increasing,

$\therefore$ $f(x)>f(0)$ when $x>0$

or, $g'(x) > f(0)$ when $x > 0$

$\therefore$ $g(x)$ is increasing when $x > 0$

$\therefore$ $g(x) > g(0)$ when $x > 0$

Now $g(0) = \cos 0 - 1 + 0 = 1 - 1 = 0$

$\therefore$ $\cos x - 1 + \frac{1}{2}x^2 > 0.$

(iii) Let $h(x) = \sin x - x + \frac{1}{6}x^3$

$\therefore$ $h'(x) = \cos x - 1 + \frac{1}{2}x^2 = g(x) > g(0) = 0$

when $x > 0$

$\therefore$ $h(x)$ is increasing when $x > 0$, $h(x) > h(0)$

Now $h(0) = 0$,

$\therefore$ $\sin x - x + \frac{1}{6}x^3 > 0$, when $x > 0$.

Ex. 15. Prove that if $x > 0$, then

$$x > \log(1+x) > x - \tfrac{1}{2}x^2.$$

Let $f(x) = x - \log(1+x)$

$$\therefore \quad f'(x) = 1 - \frac{1}{1+x} = \frac{x}{1+x} > 0 \text{ when } x > 0.$$

$\therefore$ $f(x)$ is increasing when $x > 0$,

$\therefore$ $f(x) > f(0)$ when $x > 0$

or, $x - \log(1+x) > 0$ [as $f(0) = 0 - \log(1+0) = 0 - \log 1 = 0$]

$\therefore$ $x > \log(1+x)$ ... $-[1]$

Next let $g(x) = \log(1+x) - x + \frac{1}{2}x^2$

$$g'(x) = \frac{1}{1+x} - 1 + x$$

$$= \frac{1 - 1 - x + x + x^2}{1+x} = \frac{x^2}{1+x} > 0 \text{ when } x > 0.$$

$\therefore$ $g(x)$ is increasing.

So, $g(x) > g(0)$ when $x > 0$.

or, $\log(1+x) - x + \frac{1}{2}x^2 > \log 1 - 0 + \frac{1}{2}.0^2$ when $x > 0$

or, $\log(1+x) - x + \frac{1}{2}x^2 > 0$

or, $\log(1+x) > x - \frac{1}{2}x^2$ ...(2)

From (1) and (2) we get

$$x > \log(1+x) > x - \tfrac{1}{2}x^2, \text{ when } x > 0.$$

Ex. 16. If $f(x) = x$, when $x < 1$

$= 2 - x$, when $1 \leqslant x \leqslant 2$

$= -2 + 3x - x^2$, when $x > 2$

then show that $f(x)$ is differentiable at $x = 2$, but not at $x = 1$.

When $x<1$, $\dfrac{f(x)-f(1)}{x-1}=\dfrac{x-1}{x-1}=1$,

$\therefore \lim\limits_{x\to1-} \dfrac{f(x)-f(1)}{x-1}=1.$

when $x>1$, $\dfrac{f(x)-f(1)}{x-1}=\dfrac{2-x-1}{x-1}=\dfrac{1-x}{x-1}=-1$

$\therefore \lim\limits_{x\to1+} \dfrac{f(x)-f(1)}{x-1}=-1.$

$\therefore \lim\limits_{x\to1-} \dfrac{f(x)-f(1)}{x-1}\neq\lim\limits_{x\to1+} \dfrac{f(x)-f(1)}{x-1}$

$\therefore$ $f'(1)$ does not exist and $f(x)$ is not differentiable at $x=1$.

Again, when $x<2$, $\dfrac{f(x)-f(2)}{x-2}=\dfrac{2-x-0}{x-2}=-1$

$\therefore \lim\limits_{x\to2-} \dfrac{f(x)-f(2)}{x-2}=-1.$

When $x>2$, $\dfrac{f(x)-f(2)}{x-2}=\dfrac{-2+3x-x^2-0}{x-2}$

$$=\frac{-(x-2)(x-1)}{x-2}=1-x$$

$\therefore \lim\limits_{x\to2+} \dfrac{f(x)-f(2)}{x-2}=\lim\limits_{x\to2+} (1-x)=1-2=-1$

$\therefore \lim\limits_{x\to2-} \dfrac{f(x)-f(2)}{x-2}=\lim\limits_{x\to2+} \dfrac{f(x)-f(2)}{x-2}$

Hence $f(x)$ is differentiable at $x=2$.

Ex. 17.. If $f x)=2-x$, when $x\leqslant2$

$=x-\frac{1}{2}x^2$, when $x>2$,

find the derivative of $f(x)$ at $x=2$.

When $x<2$, $\dfrac{f(x)-f(2)}{x-2}=\dfrac{2-x-0}{x-2}=-1$

$\therefore \lim\limits_{x\to2-} \dfrac{f(x)-f(2)}{x-2}=-1.$

When $x>2$, $\dfrac{f(x)-f(2)}{x-2}=\dfrac{x-\frac{1}{2}x^2}{x-2}=-\dfrac{x}{2}$

$\therefore \lim\limits_{x\to2+} \dfrac{f(x)-f(2)}{x-2}=-1$

$\therefore$ $f'(2)=f'(2+)=f'(2-)=-1.$

Ex. 18. Find the equation of the tangent to the circle $x^2+y^2=52$ parallel to the straight line $2x+3y=8$.

Let the tangent to the circle $x^2+y^2=52$ at the point (x_1,y_1) be parallel to the straight line $2x+3y=8$. Now the gradient of the tangent to the circle at the point (x_1,y_1) is $\left[\frac{dy}{dx}\right]_{(x_1,\, y_1)}$.

Now from the equation of the circle $2x+2y.\frac{dy}{dx}=0$

$$\therefore \quad \left[\frac{dy}{dx}\right]_{(x_1,\, y_1)}=-\frac{x_1}{y_1}.$$

Again the gradient of the straight line $2x+3y=8$ is $-\frac{2}{3}$. As the tangent is parallel to this straight line,

$$\therefore \quad \frac{-x_1}{y_1}=-\frac{2}{3}, \quad \text{or,} \quad \frac{x_1}{2}=\frac{y_1}{3}=k \text{ (say)}$$

$$\therefore \quad x_1=2k,\ y_1=3k. \quad \because \quad x_1{}^2+y_1{}^2=52$$

$$4k^2+9k^2=52 \quad \text{or,} \quad k^2=4 \qquad k=\pm 2$$

$\therefore$ The point of contact is the point $(\pm 4, \pm 6)$

The equation of the tangent at this point is $y\pm 6=-\frac{2}{3}(x\pm 4)$

or, $2x+3y=\pm 8\pm 18=\pm 26.$

Ex. 19. At which point of the curve $y=x^2-5x+8$ the subtangent and subnormal are equal ?

Here $\frac{dy}{dx}=2x-5$. As subtangent = subnormal,

$$\therefore \frac{y}{\frac{dy}{dx}}=y\frac{dy}{dx} \quad \text{or,} \quad \left(\frac{dy}{dx}\right)^2=1, \quad \text{or,} \quad \frac{dy}{dx}=\pm 1$$

or, $2x-5=\pm 1$, $\therefore$ $x=3, 2.$

When $x=3$, then $y=9-15+8=2$

When $x=2$, then $y=4-10+8=2$

Hence the subtangent and subnormal are equal at the two points $(3, 2)$ and $(2, 2)$.

Ex. 20. What should be the height and radius of the base of a right circular cylinder of given volume and closed at both ends, so that the surface area of the cylinder should be minimum ?

Let the radius of the base $=r$

height $=h$ and volume $=v$

If the total surface area $=A$,

then $A=2\pi r^2+2\pi rh$ [$\because$ the cylinder is closed at both ends]

$$=2\pi r^2+2\pi r\frac{v}{\pi r^2} \quad [\because \; v=\pi r^2 h]$$

$$=2\left(\pi r^2+\frac{v}{r}\right).$$

Now as v is known, so it is a constant and therefore A is a function of r.

Now $\frac{dA}{dr}=2\left(2\pi r-\frac{v}{r^2}\right)$ and $\frac{d^2A}{dr^2}=2\left(2\pi+\frac{2v}{r^3}\right)$

Now for minimum volume, $\frac{dA}{dr}=0$.

$\therefore \; 2\left(2\pi r-\frac{v}{r^2}\right)=0$, or, $r=\sqrt[3]{\frac{v}{2\pi}}=r_1$ (say)

$\therefore$ when $r=r_1$, $v=2\pi r_1^3$.

Again, $\left(\frac{d^2A}{dr^2}\right)_{r=r_1}=2(2\pi+4\pi)=12\pi>0$.

$\therefore$ for $r=r_1$, $\frac{dA}{dr}=0$ and $\frac{d^2A}{dr^2}>0$.

$\therefore$ when $r=r_1$, the value of A is a minimum.

Evidently, for all other values of r, the value of A is greater than the value of A at r_1, $\therefore$ The value of A is least when $r=r_1$

Now when $r=r_1$, $v=2\pi r_1^3$ and $h=\frac{v}{\pi r_1^2}$

$=\frac{2\pi r_1^3}{\pi r_1^2}=2r_1=$ diameter of the base of the cylinder.

Hence the total surface area of the cylinder of the given volume will be least when the height of the cylinder will be equal to the diameter of the base of the cylinder.

Ex 21. Show that the volume of a right circular cone of given total surface area will be greatest when its semivertical angle is $\sin^{-1}\frac{1}{3}$.

Let the height of the cone $=h$
radius of the base $=r$, slant height $=l$,
semi vertical angle $=\alpha$ and total surface area $=s$.

$\therefore \quad \frac{r}{l}=\sin\alpha \quad \therefore \quad l=r\operatorname{cosec}\alpha$ and $h=r\cot\alpha$.

Now, $s=\pi rl+\pi r^2=\pi r^2\operatorname{cosec}\alpha+\pi r^2=\pi r^2(1+\operatorname{cosec}\alpha)$.

and $v=\frac{1}{3}\pi r^2 h=\frac{1}{3}\,\frac{s}{1+\operatorname{cosec}\alpha}\,\frac{\sqrt{s}}{\sqrt{\pi}\sqrt{1+\operatorname{cosec}\alpha}}\cdot\cot\alpha$

$$=\frac{s\sqrt{s}}{3\sqrt{\pi}}\,\frac{\cot\alpha}{(1+\text{cosed})^{\frac{3}{2}}}.$$

$$\therefore \quad \frac{dv}{d\alpha}=\frac{s\sqrt{s}}{3\sqrt{\pi}}\times$$

$$\left\{\frac{-\operatorname{cosec}^2\alpha(1+\operatorname{cosec}\alpha)^{\frac{3}{2}}-\frac{3}{2}(1+\operatorname{cosec}\alpha)^{\frac{1}{2}}\times(-\operatorname{cosec}\alpha\cot\alpha)\cot\alpha}{(1+\text{cosee}\,\alpha)^3}\right\}$$

$$=\frac{s\sqrt{s}}{3\sqrt{\pi}}\cdot\frac{\operatorname{cosec}\alpha(\operatorname{cosec}\alpha-3)(\operatorname{cosec}\alpha+1)}{2(1+\operatorname{cosec}\alpha)^{\frac{5}{2}}}$$

$\therefore$ If $\frac{dv}{d\alpha}=0$, then $\operatorname{cosec}\alpha=0,\ 3$ or, -1

Now the value of $\operatorname{cosec}\alpha$ cannot be 0; also if $\operatorname{cosec}\alpha=-1$ then $\operatorname{cosec}\alpha+1=0$ and then $\frac{dv}{d\alpha}$ does not exist (for its denominator in this case vanishes).

$\therefore \quad \operatorname{cosec}\alpha\neq-1 \qquad \therefore \quad \operatorname{cosec}\alpha=3 \qquad$ or, $\quad \sin\alpha=\frac{1}{3}$

$\therefore$ The value of the cone will be greatest when $\sin\alpha=\frac{1}{3}$.

Ex. 22. Show that the largest rectangle that can be inscribed in a circle is a square.

Without any loss of generality we can take the equation of the circle to be $x^2+y^2=a^2$ and the co-ordinates of a vertex of an inscribed rectangle to be (x, y). Hence the lengths of the sides of the rectangle are $2x$ and $2y$.

$\therefore$ Area of the rectangle $=4xy$.

Now as (x, y) is a point of the circle,

$\therefore \quad x^2+y^2=a^2, \quad$ or, $\quad y=\sqrt{a^2-x^2}$.

Hence the area of the rectangle is

$$A=4x\sqrt{a^2-x^2}.$$

$$\therefore\quad \frac{dA}{dx}=4\sqrt{a^2-x^2}-4x\,\frac{1}{2}\cdot\frac{2x}{\sqrt{a^2-x^2}}$$

$$=4\sqrt{a^2-x^2}-4\,\frac{x^2}{\sqrt{a^2-x^2}}=\frac{4(a^2-2x^2)}{(a^2-x^2)^{\frac{1}{2}}}.$$

Also $\dfrac{d^2A}{dx^2}=\dfrac{8x^3-12xa^2}{(a^2-x^2)^{\frac{3}{2}}}$

For extreme values, $\dfrac{dA}{dx}=0$, or, $a^2-2x^2=0$

$$\therefore\quad x=\frac{a}{\sqrt{2}}.\quad \text{If } x=\frac{a}{\sqrt{2}},\ \frac{d^2A}{dx^2}=\frac{\dfrac{8a^3}{2\sqrt{2}}-\dfrac{12a^3}{\sqrt{2}}}{\left(\dfrac{a^2}{2}\right)^{\frac{3}{2}}}$$

$$=-16<0.$$

Hence for the value $x=\dfrac{a}{\sqrt{2}}$ the value of A is maximum.

Evidently, for all other values of x between 0 and a (*i.e.*, for all points on the circle), the value of A is less than the value of A when $x=\dfrac{a}{\sqrt{2}}$.

Now, when $x=\dfrac{a}{\sqrt{2}}$, $y=\sqrt{a^2-x^2}=\sqrt{a^2-\dfrac{a^2}{2}}=\dfrac{a}{\sqrt{2}}$

Hence the sides of the rectangle are equal. Hence the rectangle is a square Hence of all rectangles inscribed in a circle, the square is of the greatest area.

Ex. 23. Two particles start from the same point at the same instant along a given straight line, The distance between the particles at time t after start is $ut-\frac{1}{2}ft^2$ (u, f constants). Prove that the maximum distance between the particles before they meet again will be $\dfrac{u^2}{2f}$ at time $\dfrac{u}{f}$ after start.

Let at time t after start the distance between the particles be x.

$\therefore \quad x = ut - \frac{1}{2}ft^2$. When the particles will meet each other, then $x=0$, or $ut - \frac{1}{2}ft^2 = 0$.

$\therefore \quad t=0, \frac{2u}{f}$. So, particle will meet again at time $\frac{2u}{f}$ after start.

So, we are to determine the maximum value of x when $t<\frac{2u}{f}$

Now, $\frac{dx}{dt} = u - ft$ and $\frac{d^2x}{dt^2} = -f$

If $\frac{dx}{dt} = 0$, then $u - ft = 0$ or $t = \frac{u}{f} < \frac{2u}{f}$.

$\therefore$ When $t = \frac{u}{f}$, then $\frac{dx}{dt} = 0$ and $\frac{d^2x}{dt^2} < 0$.

$\therefore \quad x$ will be maximum when $t = \frac{u}{f}$.

When $t = \frac{u}{f}$, then $x = u\frac{u}{f} - \frac{1}{2}f\frac{u^2}{f^2} = \frac{u^2}{f} - \frac{1}{2}\frac{u^2}{f} = \frac{u^2}{2f}$.

So, before the particles meet again, the maximum distance between them will be $\frac{u^2}{2f}$ at time $\frac{u}{f}$ after start.

Ex. 24. A particle is moving along a straight line starting from a point O and at time t after start its distance from the point O is s. If $s = 3{\cdot}5t + 2{\cdot}1\,t^2 - 1{\cdot}4\,t^3$, then find the velocity and acceleration of the particle 1 sec. after start. When will the acceleration of the particle be zero.

$s = 3{\cdot}5t + 2{\cdot}1t^2 - 1{\cdot}4t^3$.

If v and f be respectively the velocity and acceleration at time t, then

v = rate of change of s with respect to $t = \frac{ds}{dt} = 3{\cdot}5 + 4{\cdot}2t - 4{\cdot}2t^2$

f = rate of change of v with respect to $t = \frac{dv}{dt} = 4{\cdot}2 - 8{\cdot}4t$.

$\therefore$ velocity of the particle after 1 second,

$= \left[v\right]_{t=1} = 3{\cdot}5 + 4{\cdot}2 - 4{\cdot}2 = 3{\cdot}5$

and acceleration at that time

$=\Big[f\Big]_{t=1}=4\cdot2-8\cdot4=-4\cdot2$

If the acceleration be 0, then

$4\cdot2-8\cdot4\,t=0 \quad \therefore \quad t=\frac{1}{2}$

Hence the acceleration will be zero at time $\frac{1}{2}$ sec. after start.

Ex. 25. The bottom of a tank is a square of sides 3 ft. If 9 cubic ft. of water is poured in the tank in every minute, find the rate of increase in water level in the tank.

Suppose when the height of water level in the tank is x ft., the volume of water in the tank is v cubic ft. $\therefore \quad v=3^2.x=9x.$

$\therefore \quad \frac{dv}{dt}=\frac{9dx}{dt}$ (where t is the time in minutes). Now it is given that

$\frac{dv}{dt}$ = rate of change of v with respect to $t=9$.

$$\therefore \quad \frac{dx}{dt}=\frac{1}{9}\frac{dv}{dt}=\frac{1}{9}\cdot 9=1.$$

Hence the rate of increase of water level in the tank is 1 ft /min.

Ex. 26. The radius of a circular plate when heated increases at the rate of ·0015 inch per second when the radius is $2\frac{1}{4}$ ft Find the rate of increase in the area of the plate at this moment.

Let the area of the plate be A sq. ft. and radius r. $\therefore$ $A=\pi r^2$. Now, if t denotes time, then

$\frac{dA}{dt}$ = rate of change of A with respect to t

= rate of change of area per second.

$\frac{dr}{dt}$ = rate of change of r with respect to to t

= rate of change of radius per second.

= ·0015 inch. (given),

Again, $\frac{dA}{dt}=\frac{dA}{dr}\cdot\frac{dr}{dt}=2\pi r.\frac{dr}{dt}$

When $r=2\frac{1}{2}$ ft. = 30 inches,

$$\frac{dA}{dt}=2\pi.30\times{}\cdot0015 \text{ sq. inch.}$$

$$=\cdot28 \text{ sq. inch.}$$

So, when the radius of the plate is $2\frac{1}{2}$ ft., its area will increase at the rate of ·23 sq. inch per second.

Ex 27. If a wheel rotates through θ radians per second, then $\theta=5t+\frac{3}{2}t^2-\frac{1}{3}t^3$. Show that in the first $1\frac{1}{2}$ seconds the angular velocity of the wheel increases. When will the wheel stop ?

$$\theta=5t+\frac{3}{2}t^2-\frac{1}{3}t^3.$$

$\therefore$ Angular velocity $\omega=\frac{d\theta}{dt}=5+3t-t^2$.

Now $\frac{d\omega}{dt}=3-2t>0$, so long as $t<\frac{3}{2}=1\frac{1}{2}$.

$\therefore$ $\frac{d\omega}{dt}$ is positive from $t=0$ to $t=1\frac{1}{2}$ secs.

Hence during this interval the angular velocity of the whee will continue to increase.

The wheel will stop when the angular velocity will be zero.

i e, when $\omega=5+3t-t^2=0$.

$$\therefore\quad t=\frac{3\pm\sqrt{9+20}}{2}=\frac{3\pm\sqrt{29}}{2}$$

Now, as t cannot be negative,

$$\therefore\quad t=\frac{3+\sqrt{29}}{2}=4\cdot19 \text{ seconds (nearly).}$$

Hence the wheel will stop after 4·19 seconds (nearly).

Ex 28. A ladder, 13 ft. long rests on a vertical wall. The lower end of the ladder is at a distance of 5 ft. from the bottom of the wall. The lower end of the ladder is moving away from the wall on the ground at the rate of 2 ft. per second. At what rate will the other end of the ladder move downwards along the wall ?

Let the distances of the lower and upper ends of the ladder from the line of intersection of the ground and the wall be x and y respectively. Hence the velocities of the lower and upper ends of the ladder are $\frac{dx}{dt}$ and $\frac{dy}{dt}$ respectively.

It is given that $\frac{dx}{dt}=2$.

Now as the length of the ladder is 13 ft.

$$x^2+y^2=13^2.$$

$$\therefore \quad 2x\frac{dx}{dt}+2y\frac{dy}{dt}=0$$

[Differentiating both sides with respect to t]

$$\text{or,} \quad \frac{dy}{dt}=-\frac{x}{y}\frac{dx}{dt}.$$

When $x=5$, then $y=\sqrt{13^2-5^2}=\sqrt{14}=12$.

$\therefore \quad \frac{dy}{dt}=-\frac{5}{12}\times 2$ ft /sec. $=-10$ inches per sec.

Hence the upper end of the ladder is moving downwards at the rate of 10 inches per second.

[Note that as t increases, y decreases. So, we have got negative value of $\frac{dx}{dt}$]

Ex. 29. The distance of a particle at time t from a point O is $a \cos nt+b \sin nt$, where a, b, n are constants. Show that the acceleration of the particle is proportional to its distance from the point O.

Let the distance of the particle from O at time t be x.

$$\therefore \quad x=a \cos nt+b \sin nt$$

$\therefore$ velocity, $v=\frac{dx}{dt}=-an \sin nt+bn \cos nt.$

and acceleration $f=\frac{dv}{dt}=-an^2 \cos nt-bn^2 \sin nt$

$$=-n^2(a \cos nt+b \sin nt)=-n^2x$$

$$\therefore \quad f \propto x \; (\because n^2=\text{canstant}).$$

Ex 30. If $x=a(t+\sin t$, $y=a(1-\cos t)$,

and $\rho=\frac{(1+y_1{}^2)^{\frac{3}{2}}}{y_2}$ where $y_1=\frac{dy}{dx}$, $y_2=\frac{d^2y}{dx^2}$,

then show that $\rho=4a \cos\frac{t}{2}$.

Here, $\dfrac{dx}{dt}=a(1+\cos t)$, $\dfrac{dy}{dt}=a\sin t$.

$$\therefore\ y_1=\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{a\sin t}{a(1+\cos t)}\quad \tan\frac{t}{2}$$

$$y_2=\tfrac{1}{2}\sec^2\frac{t\,dt}{2dx}=\tfrac{1}{2}\sec^2\frac{t}{2}\quad\frac{1}{a.2\cos^2\frac{t}{2}}=\frac{1}{4a}\cdot\frac{1}{\cos^4\frac{t}{2}}.$$

$$\therefore\ \rho=\frac{(1+y_1^2)^{\frac{3}{2}}}{y_2}=\frac{\left(1+\tan^2\frac{t}{2}\right)^{\frac{3}{2}}}{\frac{1}{4a}\cdot\frac{1}{\cos^4\frac{t}{2}}}=4a\cos\frac{t}{2}$$

Ex. 31. If $X=x-\dfrac{y_1(1+y_1^2)}{y_2}$ and $Y=y+\dfrac{1+y_1^2}{y_2}$, then show that when $y^2=4ax$, then $27aY^2=4(X-2a)^3$

Here $y^2=4ax$, $\therefore\ 2y\dfrac{dy}{dx}=4a$, or, $y_1=\dfrac{2a}{y}$

$$\therefore\ y_2=-\frac{2a}{y^2}\frac{dy}{dx}=-\frac{4a^2}{y^3}$$

$$\therefore\ X=x-\frac{\frac{2a}{y}\left(1+\frac{4a^2}{y^2}\right)}{-\frac{4a^2}{y^3}}=x+\frac{y^2+4a^2}{2a}=\frac{2ax+4ax+4a^2}{2a}$$

$$=3x+2a.$$

$$Y=y+\frac{1+\frac{4a^2}{y^2}}{-\frac{4a^2}{y^3}}=y-\frac{y(y^2+4a^2)}{4a^2}=-\frac{y^3}{4a^2}=-\frac{(y^2)^{\frac{3}{2}}}{4a^2}$$

$$=-\frac{(4ax)^{\frac{3}{2}}}{4a^2}=\pm\frac{2x^{\frac{3}{2}}}{a^{\frac{1}{2}}}.$$

$$\therefore\ Y^2=\frac{4x^3}{a}=\frac{4}{a}\cdot x^3=\frac{4}{a}\left(\frac{X-2a}{3}\right)^3=\frac{4(X-2a)^3}{27a}$$

or, $27aY^2=4(X-2a)^3$.

Ex. 32. If $(1+x)^n = c_0+c_1x+c_2x^2+\cdots+c_nx^n \ldots(1)$ then show that

(i) $c_1+2c_2+3c_3+\cdots+nc_n = n.2^{n-1}$,

(ii) $c_0+2c_1+3c_2+\cdots+(n+1)c_n = (n+2).2^{n-1}$

$(1+x)^n = c_0+c_1x+\cdots+c_nx^n$.

Differentiating both sides with respect to x we get

$n(1+x)^{n-1} = c_1.1+c_2.2x+c_3.3x^2+\cdots+nc_nx^{n-1}$

Now putting $x=1$ we have

$n.2^{n-1} = c_1+c_2.2+c_3.3.1^2+\cdots+c_n.n\,1^{n-1}$

or, $c_1+2c_2+3c_3+\ldots+nc_n = n.2^{n-1}\ldots\ldots$(i)

Putting $x=1$ in (i)

we get, $c_0+c_1+c_2+c_3+\cdots\cdots+c_n = 2^n\cdots\cdots$(ii)

Adding (i) and (ii) we get,

$c_0+2c_1+3c_2+\cdots+(n+1)c_n = n.2^{n-1}+2^n = (n+2).2^{n-1}$.

Ex. 33. $\sin x = 2^n \cos\frac{x}{2}\cdot\cos\frac{x}{2^2}\cdot\cos\frac{x}{2^3}\cdots\cos\frac{x}{2^n}\cdot\sin\frac{x}{2^n}$

From this identity show that

$$\frac{1}{2}\tan\frac{x}{2}+\frac{1}{2^2}\tan\frac{x}{2^2}+\cdots+\frac{1}{2^n}\tan\frac{x}{2^n} = \frac{1}{2^n}\cot\frac{x}{2^n}-\cot x.$$

From the given identity,

$$\cos\frac{x}{2}\cdot\cos\frac{x}{2^2}\cdot\cos\frac{x}{2^3}\cdots\cdots\cos\frac{x}{2^n} = \frac{\sin x}{2^n\sin\frac{x}{2^n}}$$

or, $\log\cos\frac{x}{2}+\log\cos\frac{x}{2^2}+\cdots\cdots+\log\cos\frac{x}{2^n}$

$= -\log\sin\frac{x}{2^n}+\log\sin x-\log 2^n$

$$\text{or,}\quad -\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}\cdot\frac{1}{2}-\frac{\sin\frac{x}{2^2}}{\cos\frac{x}{2^2}}\cdot\frac{1}{2^2}-\cdots\cdots-\frac{\sin\frac{x}{2^n}}{\cos\frac{x}{2^n}}\,\frac{1}{2^n}$$

$$= -\frac{\cos\frac{x}{2^n}}{\sin\frac{x}{2^n}}\,\frac{1}{2^n}+\frac{\cos x}{\sin x}$$ [Differentiating both sides with respect to x]

or, $\frac{1}{2}\tan\frac{x}{2}+\frac{1}{2^2}\tan\frac{x}{2^2}+\cdots+\frac{1}{2^n}\tan\frac{x}{2^n}$

$=\frac{1}{2^n}\cot\frac{x}{2^n}-\cot x.$

Ex. 34. If $f(b)=f(a)+(b-a)f'(c)$ then find c when $f(x)=\text{A}x^2+\text{B}x+\text{C}$, where A, B, C are constants.

Here, $f(x)=\text{A}x^2+\text{B}x+\text{C}$, $\therefore\ f'(x)=2\text{A}x+\text{B}$

and $f(a)=\text{A}a^2+\text{B}.a+\text{C}$, $f(b)=\text{A}b^2+\text{B}.b+\text{C}$,

$\therefore\ f(b)-f(a)=\text{A}(b^2-a^2)+\text{B}(b-a).$

From the given condition, $f(b)-f(a)=(b-a)f'(c)$,

or, $\text{A}(b^2-a^2)+\text{B}(b-a)=(b-a).(2\text{A}c+\text{B})$,

or, $\text{A}(b+a)+\text{B}=2\text{A}.c+\text{B}$,

$\therefore\ c=\frac{b+a}{2}.$

Ex 35. Show that if the curves $\frac{x^2}{a}+\frac{y^2}{b}=1$ and $\frac{x^2}{a_1}+\frac{y^2}{b_1}=1$ intersect each other orthogonally (i.e., at right angles), then $a-a_1=b-b_1$.

Let the curves intersect each other at (x_1, y_1)

$$\therefore\ \frac{{x_1}^2}{a}+\frac{{y_1}^2}{b}=1\ldots(1)\qquad \frac{{x_1}^2}{a_1}+\frac{{y_1}^2}{b_1}=1\ldots(2)$$

Now, differentiating both sides of

$\frac{x^2}{a}+\frac{y^2}{b}=1$, we get

$$\frac{2x}{a}+\frac{2y}{b}\cdot\frac{dy}{dx}=0,\quad\text{or,}\quad \frac{dy}{dx}=-\frac{bx}{ay}$$

$\therefore$ The gradient of the tangent to the curve at the point

(x_1, y_1) is $\left(\frac{dy}{dx}\right)_{(x_1,\, y_1)}=-\frac{bx_1}{ay_1}$

Similarly the gradient of the tangent at (x_1, y_1) of the curve

$\frac{x^2}{a_1}+\frac{y^2}{b_1}=1$ is $\frac{-b_1x_1}{a_1y_1}$

As the curves intersect each other orthogonally the tangents to the curves at the point of intersection (x_1, y_1) are perpendicular. Hence the product of the gradients of the two tangents at the point (x_1, y_1) is -1.

$$\therefore \quad -\frac{bx_1}{ay_1}\cdot\frac{-b_1x_1}{a_1y_1}=-1, \text{ or, } \frac{x_1^2}{y_1^2}=-\frac{aa_1}{bb_1} \quad \cdots \quad (3)$$

By subtracting (2) from (1) we obtain

$$x_1^2\left(\frac{1}{a}-\frac{1}{a_1}\right)+y_1^2\left(\frac{1}{b}-\frac{1}{b^1}\right)=0,$$

$$\text{or, } \frac{x_1^2}{y_1^2}=-\frac{(b_1-b)aa_1}{(a_1-a)bb_1} \quad \cdots \quad (4)$$

From (3) and (4) we get

$$-\frac{aa_1}{bb_1}=-\frac{(b_1-b)aa_1}{(a_1-a)bb_1}$$

$$\therefore \quad a_1-a=b_1-b \text{ or, } a-a_1=b-b_1$$

Ex. 36. (i) If $y=x^{x^{\cdots\cdots \text{to } \infty}}$, show that

$$\frac{dy}{dx}=\frac{y^2}{x(y\log x-1)}$$

(ii) If $y=\sqrt{x+\sqrt{x+\sqrt{x+\cdots\cdots \text{to } \infty}}}$, show that

$$\frac{dy}{dx}=\frac{1}{2y-1}.$$

(i) $y=x^{x^{x^{\cdots\cdots \text{to } \infty}}}=x^y$; $\therefore$ $\log y=y\log x.$

Differentiating both sides w. r. t. x,

$$\frac{1}{y}\frac{dy}{dx}=\frac{dy}{dx}\cdot\log x-\frac{y}{x}.$$

$$\text{or, } \frac{dy}{dx}\left(\log x-\frac{1}{y}\right)=\frac{y}{x}, \quad \therefore \quad \frac{dy}{dx}=\frac{y^2}{x(y\log x-1)}.$$

(ii) $y=\sqrt{x+\sqrt{x+\sqrt{x+\cdots\cdots\text{to }\infty}}}=\sqrt{x+y}$,

$\therefore\ y^2=x+y$, differentiating both sides w. r. t. x,

$$2y\cdot\frac{dy}{dx}=1+\frac{dy}{dx},\quad\therefore\quad\frac{dy}{dx}=\frac{1}{2y-1}.$$

Ex. 37. (i) Eliminate a and b, from $y=a\log x+b$.

(ii) Eliminate a and b, from $y=\frac{a}{x}+b$,

(i) Differentiating both sides of $y=a\log x+b$,

$$\frac{dy}{dx}=\frac{a}{x},\quad\text{or,}\quad x.\frac{dy}{dx}=a.$$

Differentiating again, $1\cdot\frac{dy}{dx}+x\cdot\frac{d^2y}{dx^2}=0$

$\therefore$ Required a, b eliminant is $\frac{dy}{dx}+x\frac{d^2y}{dx^2}=0.$

(ii) Differentiating $y=\frac{a}{x}+b$, w. r. t. x,

$$\frac{dy}{dx}=-\frac{a}{x^2},\quad\text{or,}\quad x^2\frac{dy}{dx}=-a.$$

Differentiating again, $2x\cdot\frac{dy}{dx}+x^2\frac{d^2y}{dx^2}=0.$

or, $2\frac{dy}{dx}+x\frac{d^2x}{dx^2}=0$ is the required a, b, eliminant.

Ex. 38. The abscissa of a particle moving at an axis, is given at time t by $x=2a\sin nt+a\sin 2nt$, where a and n are constants. Show that, in the interval $0\leqslant t\leqslant\frac{\pi}{n}$, the particle moves from the centre O to another point A, and then return back to the point O. Determine the abscissa of the point A and the acceleration of the particle at A.

$\because\ x=2a\sin(nt)+a\sin(2nt)$,

$\therefore\ \frac{dx}{dt}=2an\cos nt+2an\cos 2nt$,

an $\quad\frac{d^2x}{dt^2}=-2an^2\sin nt-4an^2\sin 2nt.$

$$\text{when } t=0,\ x=0,\ \frac{dx}{dt}=4an \text{ and } \frac{d^2x}{dt^2}=0.$$

$\therefore$ At time $t=0$, the particle starts from the origin O with a velocity $4an$, moving along the x-axis Now, $\frac{dx}{dt}=0$ gives

$$\cos nt+\cos 2nt=0; \text{ or, } \cos\frac{3nt}{2}\cdot\cos\frac{nt}{2}=0$$

$$\therefore \frac{3nt}{2}=\frac{\pi}{6}, \text{ or } \frac{nt}{2}=\frac{\pi}{2} \quad \therefore t=\frac{\pi}{9n} \text{ or, } \frac{\pi}{n}$$

$\therefore$ At time $t=\frac{\pi}{3n}$, the velocity of the particle is zero and at this time acceleration $=-3\sqrt{3}an^2$, when $t=\frac{\pi}{3n}$, $x=\frac{3\sqrt{3}}{2}a$

$\therefore$ At the distance $\frac{3\sqrt{3}}{2}a$ the velocity of the particle is zero and as the acceleration is negative, the particle will move towards O. At time $t=\frac{\pi}{n}$, $x=0$, $\frac{dx}{dt}=0$, $\frac{d^2x}{dt^2}=0$, $\therefore$ After reaching the point O, the velocity and acceleration of the particle are both zero. So, the particle will remain at rest. Taking the point $x=\frac{3\sqrt{3}}{2}a$ as the point A, we get that the particle moves from the point O to the point A and again comes back from the point A to the point O. The abscissa of the point A is $x=\frac{3\sqrt{3}}{2}a$ and the acceleration at A is $3\sqrt{3}an^2$ towards O.

Exercise 4

1. If $y=e^{f(x)}$, then show that $\frac{dy}{dx}=e^{f(x)}f'(x)$.

Hence find $\frac{dy}{dx}$ when

(i) $y=e^{x^3+4x+1}$ (ii) $y=e^{\sin x}$ (iii) $y=e^{\tan^{-1}x}$

(iv) $y=e^{\sqrt{x+1}}$ (v) $y=(e)^{e^x}$

2. If $y=\sin\{f(x)\}$, then show that

$\frac{dy}{dx}=\cos\{f(x)\}.f'(x)$ and hence find $\frac{dy}{dx}$ if

(i) $y=\sin(x^2+x+1)$ (ii) $y=\sin(\log x)$

(iii) $y=\sin(e^x)$ (iv) $y=\sin(\tan^{-1}x)$.

(v) $y=\sin(x^2\cos x)$.

3. Show that

(i) If $y=\sin^{-1}f(x)$, then $\frac{dy}{dx}=\frac{1}{\sqrt{1-f^2(x)}}f'(x)$

(ii) If $y=\tan^{-1}f(x)$, then $\frac{dy}{dx}=\frac{1}{1+f^2(x)}.f'(x)$

(iii) If $y=\frac{1}{f(x)}$, then $\frac{dy}{dx}=-\frac{1}{f^2(x)}f'(x)$ where $f^2(x)=f\{f\,x)\}$.

Use the above formulas to determine $\frac{dy}{dx}$ when

(iv) $y=\sin^{-1}(x^2+x-1)$ (v) $y=\sin^{-1}(\cos x)$

(vi) $y=\tan^{-1}(e^x)$ (vii) $y=\tan^{-1}(\sqrt{x})$

(viii) $y=\frac{1}{x^2+a^2}$ (ix) $y=\frac{1}{x+e^x\log x}$

(x) $y=\frac{1}{\sin^{-1}(x^2+4)}$.

4. Find $\frac{dy}{dx}$ when

(i) $y=\log\sqrt{\frac{x^2+x+1}{x^2-x+1}}$ (ii) $y=\frac{1}{\sqrt{x+a}-\sqrt{x+b}}$

(iii) $y=\sqrt{\frac{1-\tan x}{1+\tan x}}$ (iv) $y=\frac{x^{\frac{1}{3}}(1-x)^{\frac{2}{3}}}{(2-3x)^{\frac{3}{4}}(3-4x)^{\frac{2}{5}}}$

(v) $y=\sin x.\,e^x.\log x.\,x^x$

(vi) $y=(\tan x)^{\cot x}+(\cot x)^{\tan x}$

(vii) $y=e^{(x)^x}$ (viii) $y=(1+x^{-1})^x$

(ix) $y=\log\left(\frac{1+x}{1-x}\right)^{\frac{1}{4}}-\frac{1}{2}\tan^{-1}x.$ (x) $(x^x)^x$

(xi) $x^{(x^x)}$ (xii) $y=\tan^{-1}\left(\dfrac{x\sin\alpha}{1-x\cos\alpha}\right)$

(xiii) $y=\log_{\sin x} x$ (xiv) $y=\log_{e^x} x$ (xv) $\log_{x_9} 4$

(xvi) $y=x^{\log x}+(\log x)^x$

(xvii) $y^x+x^y=a^b$.

5. Prove that if

(i) $y=\frac{1}{2}x\sqrt{x^2+a^2}+\dfrac{a^2}{2}\log(x+\sqrt{x^2+a^2})$, then

$$\frac{dy}{dx}=\sqrt{x^2+a^2}$$

(ii) $y=-\sqrt{(8+x-x^2)}+\frac{1}{2}\sin^{-1}\dfrac{2x-1}{\sqrt{33}}$, then

$$\frac{dy}{dx}=\frac{x}{\sqrt{(8+x-x^2)}}$$

(iii) $y=-\frac{1}{5}\cot^5 x+\frac{1}{3}\cot^3 x-\cot x-x$, then $\dfrac{dy}{dx}=\cot^6 x$.

(iv) $y=\frac{1}{6}[(x^6+1)\tan^{-1}x^3-x^3]$, then $\dfrac{dy}{dx}=x^5\tan^{-1}x^3$.

(v) $y=\log\dfrac{1+x}{1-x}+\dfrac{1}{2}\log\dfrac{1+x+x^2}{1-x+x^2}+\sqrt{3}\tan^{-1}\dfrac{x\sqrt{3}}{1-x^2}$, then

$$\frac{dy}{dx}=\frac{6}{1-x^6}.$$

(vi) $y=\dfrac{1}{1+x^{l-m}+x^{n-m}}+\dfrac{1}{1+x^{m-n}+x^{l-n}}+\dfrac{1}{1+x^{n-l}+x^{m-l}}$,

then $\dfrac{dy}{dx}=0$.

6. Find $\dfrac{dy}{dx}$ when

(i) $y=\tan^{-1}\dfrac{\cos x-\sin x}{\cos x+\sin x}$ (ii) $y=\sin^{-1}\dfrac{2x}{1+x^2}$

(iii) $y=\sin^{-1}(3x-4x^3)$ (iv $y=\tan^{-1}\sqrt{\dfrac{1+\cos x}{1-\cos x}}$

(v) $y=\tan^{-1}\dfrac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}$

tangent will be normal to the curve at the point $(m^2, -m^3)$ if $m = \pm \frac{\sqrt{2}}{3}$.

36. (i) Show that the portion of the tangent to the curve $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$, intercepted between the axes of co-ordinates is of constant length.

(ii) Show that the sum of the intercepts on the axes of co-ordinates of the tangent to the curve $\sqrt{x} + \sqrt{y} = \sqrt{a}$ is constant.

37. Show that the subtangent at every point of the curve $y = b e^{\frac{x}{a}}$ is constant and the subnormal at the point (x, y) is $\frac{y^2}{a}$.

[Pat. '37, '33]

38. Show that the subtangent at every point of the curve $x^m y^n = a^{m+n}$ is proportional to the abscissa of the point. [Bihar]

39. Prove that the length of the perpendicular from the foot of the ordinate on the tangent to the curve

$y = c \dfrac{e^{\frac{x}{c}} + e^{-\frac{x}{c}}}{2}$ is the same at all points of the curve.

40. Prove that the product of the abscissa of every point of the curve $y = \frac{h}{\sqrt{\pi}} e^{-h^2 x^2}$ and the subtangent at the point is constant.

41. Show that the sum of the subnormal and the subtangent of the curve $x = a + b \log (b + \sqrt{b^2 - y^2}) - \sqrt{b^2 - y^2}$ is constant at all points of the curve.

42. Show that the square of the subtangent of the curve $by^2 = (x+a)^3$ is proportional to the subnormal. [Pat. '32]

43. Divide the number 12 into two parts so that the product of the parts is the greatest

44. Divide 10 into two parts such that the sum of two times of one part and the square of the other part becomes least.

45. Find the positive integer, which when added to its reciprocal, yields the least sum.

46. Which cylinder with top open and given volume will have the least surface area ?

47. What should be the length and height of a conical tent of given volume, so that least amount of canvas will be required in building the tent.

48. Show that of all rectangles inscribed in a circle, the square has the greatest perimeter.

49. Find the length and breadth of the greatest rectangle inscribed in the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

50. Prove that of all isosceles triangles inscribed in a circle, the one with greatest area is an equilateral triangle.

51. The length of the hypotenuse of a right-angled triangle is a. Determine the lengths of the other two sides of the triangle so that (i) sum of the lengths of the other two sides will be greatest and (ii) the area of the triangle will be greatest.

52. If $s^2=(x-l_1)^2+(x-l_2)^2+\cdots+(x-l_n)^2$, where $l_1, l_2 \cdots, l_n$ are constants, show that the value of s^2 will be the least when

$$x=\frac{l_1+l_2+\ldots+l_n}{n}.$$

53. According to postal rules the sum of the length and diameter of a parcel should not exceed 6 ft. If the parcel be cylindrical, show that its volume will be the greatest when the length and diameter of the base will be 2ft. and 4ft. respectively.

54. From a square tin plate of sides 16 cms, four squares each of side x cms. are cut off from the four corners and a box, open at the top is prepared by folding the sides. Determine the greatest volume of the box.

55. The consumption of petrol in a ship is proportional to the cube of its velocity. Show that if the ship moves in a direction opposite to that of the current flowing at the rate of c miles per hour, then the consumption for moving through a definite distance will be minimum if the velocity of the ship be $\frac{3}{2}c$ miles per hour.

56. The displacement of a particle from a point O at time t after start is given by $s=36t+6t^2-t^3$. Find the velocity and acceleration of the particle after 1 second. At what distance from O, the acceleration of the particle will be zero ?

57. If h be the height of an object at time t, then $h=ut-\frac{1}{2}gt^2$ (u and g are constants). Determine the velocity and accelera-

tion of the particle at time t and also the maximum height attained by the particle.

58. The vertex of a conical water pot is downwards. If the height and radius of the base of the cone be 15 inches and 5 inches respectively, find the volume of water in the pot when the height of the water level is x inches. Find also the rate of increase of volume of the water with respect to height when the height of the water level is 10 inches.

59. Of two tanks connected by a pipe one is above the other. The base of the upper tank is a rectangle of length 6 ft. and width 4 ft. and the base of the lower tank is a rectangle of length 7 ft and width $3\frac{1}{2}$ ft. If 25 cubic ft. of water is poured in the upper tank, compare the rate of increase of water level in the lower tank with the rate of decrease of water level in the upper tank.

60. A circular plate expands when heated. The radius of the plate when 3ft. in length, increases at the rate of ·0012 inch per second. Determine the rate of change of area of the plate at this instant.

61. The radius of a sphere when $2\frac{1}{2}$ ft. in length increases at the rate of ·0015 inch at being heated. Find the rate of increase in volume at this instant.

62. If P and V be the pressure and volume of a gas, then the adiabatic law of expansion of the gas is $PV^{1 \cdot 4} = \text{Constant}$. If when the volume of the gas is 10 cubic metres and the pressure of the gas be 25 kg. per square centimetre, the volume of the gas is increased by 2 cubic metres per second, then find the rate of change of the pressure.

63. If a wheel rotates through an angle θ in t seconds, then $\theta = 5\pi t(4 - 3t + t^2)$. Show that the least angular velocity of the wheel is 5π and also find the angular velocity and acceleration of the wheel after 2 seconds.

64. The distances of a particle from a fixed point after time t is given by $s = 4t - \frac{1}{2}t^2 + \frac{1}{3}t^3$. Find the velocity of the particle. What is the least velocity of the particle and what is the distance of the particle from the fixed point when the least velocity is attained ?

65. If s be the distance of a particle from a fixed point due to rectilinear motion, then $s=ae^{-kt}$, where a and k are constants Find the initial velocity and acceleration of the particle.

66. In the above example if $s=\frac{a}{2}(e^{t}-e^{-t})$, show that the acceleration at any instant is equal to s.

67. The distance of a particle at time t from a fixed point O is $\cos\frac{\pi}{2}t+\sin\frac{\pi}{2}t$. Determine the greatest acceleration of the particle. When is this greatest acceleration attained? Show that at that time the velocity of the particle is zero.

68. Show that if $\rho=\frac{(1+y_1^2)^{\frac{3}{2}}}{y_2}$, then

(i) $\rho=\frac{y^2}{c}$ when $y=c.\frac{e^{\frac{x}{c}}+e^{-\frac{x}{c}}}{2}$

(ii) $\rho=at$ when $x=a(\cos t+t\sin t)$, $y=a(\sin t-t\cos t)$.

(iii) $\rho=3(axy)^{\frac{1}{3}}$ when $x^{\frac{2}{3}}+y^{\frac{3}{3}}=a^{\frac{2}{3}}$.

69. If $r=f(\theta)$ and $\rho=\frac{(r^2+r_1^2)^{\frac{3}{2}}}{r^2+2r_1^2-rr_2}$, then show that

(i) $\rho=\frac{1}{2}a$ when $r=a\cos\theta$.

(ii) $\rho=\frac{a^2}{3r}$ when $r^2=a^2\cos 2\theta$.

(iii) at $\theta=\pi$, $\rho=l$ when $r=\frac{l}{1+e\cos\theta}$.

(iv) at $\theta=0$, $\rho=a$ when $r=a(\theta+\sin\theta)$,

70. If $X=x-\frac{y_1(1+y_1^2)}{y_2}$, $Y=y+\frac{1+y_1^2}{y_2}$, then

(i) $(aX)^{\frac{2}{3}}+(bY)^{\frac{2}{3}}=(a^2-b^2)^{\frac{2}{3}}$ when $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

(ii) $(X+Y)^{\frac{2}{3}}+(X-Y)^{\frac{2}{3}}=2a^{\frac{2}{3}}$ when $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$.

(iii) $X^2+Y^2=a^2$ when $x=a(\cos t+t\sin t)$

and $y=a(\sin t-t\cos t)$.

71. From the formula $1+x+x^2+\cdots+x^n=\dfrac{1-x^{n+1}}{1-x}$,

find the sum of $1+2x+3x^2+\cdots+nx^{n-1}$.

72. If $\sin x \sin\left(\dfrac{\pi}{n}+x\right) \sin\left(\dfrac{2\pi}{n}+x\right) \cdots \sin\left(\dfrac{n-1}{n}\pi+x\right)$

$=\dfrac{\sin nx}{2^{n-1}}$, then show that

$$\cot x+\cot\left(\frac{\pi}{n}+x\right)+\cot\left(\frac{2\pi}{n}+x\right)+\cdots$$

$$+\cot\left(\frac{n-1}{n}\pi+x\right)=n\cot nx$$

73. From the formula,

$$\sin x+\sin(x+h)+\cdots+\sin(x+nh)=\frac{\sin\left(x+\dfrac{nh}{2}\right)\sin(n+1)\dfrac{h}{2}}{\sin\dfrac{h}{2}}$$

find the sum of $\cos x+\cos(x+h)+\cdots+\cos(x+nh)$.
From these two series find the sums of

(i) $\sin x+2\sin 2x+\cdots+n\sin nx$

and (ii) $\cos x+2\cos 2x+\cdots+n\cos nx$.

74. If $f(a+h)=f(a)+hf'(a+\theta h)$, then find θ if

(i) $a=1,\ h=3,\ f(x)=\sqrt{x}$

(ii) $a=0,\ h=3,\ f(x)=\frac{1}{3}x^3-\frac{3}{2}x^2+2x$. [C. U.]

75. If $f(h)=f(0)+hf'(0)+\dfrac{h^2}{2!}f''(\theta h)$, find θ,

when $f(x)=(1-x)^{\frac{5}{2}}$, $h=1$. [C. U.]

76. Show that the curves $x^3-3xy^2=-2$ and $3x^2y-y^3=2$ intersect each other orthogonally.

77. If the curves $ax^3+by^3=1$ and $a'x^3+b'y^3=1$ intersect each other orthogonally, then show that

$$aa'(b-b')^{\frac{4}{3}}+bb'(a-a')^{\frac{4}{3}}=0.$$

78. Show that the straight line $x\cos\alpha+y\sin\alpha=p$ will touch the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \text{ if } a^2\cos^2\alpha+b^2\sin^2\alpha=p^2.$$

79. (i) The straight line $lx+my=1$ will touch the curve $(ax)^n+(by)^n=1$ if $\left(\frac{l}{a}\right)^{\frac{n}{n-1}}+\left(\frac{m}{b}\right)^{\frac{n}{n-1}}=1.$

(ii) The straight line $lx+my=1$ will be a normal to the parabola $y^2=4ax$ if $al^3+2alm^2=m^2$.

80. Show that the curves $x^{\frac{2}{3}}+y^{\frac{2}{3}}=c^{\frac{2}{3}}$ and $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ will touch each other if $c=a+b$.

81. If $\frac{ds}{dx}=\sqrt{1+\left(\frac{dy}{dx}\right)^2}$, then show that

(i) $ds=\sqrt{1+36x^2}\,dx$, when $y=3x^2$,

(ii) $ds=\sqrt{\frac{(x+2a)}{3a}}\,dx$, when $27ay^2=4(x-a)^3$,

(iii) $ds=\left(\frac{a}{x}\right)^{\frac{1}{3}}dx$, when $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$.

82. If $\frac{ds}{dt}=\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}$, show that

(i) $ds=a\,dt$, when $x=a\cos t$, $y=a\sin t$,

(ii) $ds=\frac{3}{2}\sin 2t$, when $x=\cos^3 t$, $y=\sin^3 t$.

83. Two particles start at the same instant from a point and move in the same direction along a straight line. The first has a uniform velocity of 40 ft./sec. while the second starts with an initial velocity of 16 ft./sec. and has a uniform acceleration of 6 ft./sec^2. Find when, before the particles meet again, the distance between them is maximum and what is the maximum distance ?

84. Two particles start simultaneously from the same point and move along two straight lines at an angle α, one with uniform velocity u, and the other from rest with uniform acceleration f. Show that their relative velocity is least after a time $\frac{u\cos\alpha}{f}$ and that the least relative velocity is $u\sin\alpha$.

85. A battle-ship sailing North at a speed of 30 kilometres an hour, observes a sea-plane carrier due East of itself at a distance of 20 kilometres, the latter steaming due West with a speed of 40

kilometres an hour. After what time are they at the least distance from each other? Also find the least distance.

86. A ship is 21 km. East of a point O and moving West at 28 km/h.; at that time another ship is 84 km. South of O and moving North at 21 km./h.

(i) After one hour, at what rate the ships are approaching or separating each other?

(ii) After 3 hours what is this rate?

(iii) When are they nearest to each other?

87. Displacement s of a particle from a fixed point O after time t is given by $s=t^3-9t^2+24t$.

At what interval of time the particle will move towards O? When will its velocity decrease? When will it increase?

88. Gas is escaping from a spherical balloon at the rate of 900 c.c. per sec. How fast is the surface area shrinking when the radius is 450 cms.

89. Find the maximum and minimum values of $y=x^4+2x^3-3x^2-4x+4$. On which intervals y is increasing and decreasing?

90. Two sides of a triangle are 6 cms. and 8 cms. long, the angle included between the sides is $60°$. If the included angle increases at the rate of $1°$ per sec., find

(i) at what rate is the third side increasing?

(ii) at what rate is the area increasing?

91. The sides a and b of a triangle ABC are respectively 25 cm and 16 cm., the angle C is $60°$. If a and b are measured correctly and the error in measuring c is $\frac{1}{2}°$, find the error in the area of the triangle.

92. Four bars of length a, b, c, d are hinged together to form a quadrilateral. Show that the area of the quadrilateral is greatest when it is cyclic.

93. (i) If $y=(\sin x)^{(\sin x)^{(\sin x)\cdots\cdots\text{to } \infty}}$, then

show that $\dfrac{dy}{dx}=\dfrac{y^2 \cot x}{1-y\log\sin x}$.

(ii) If $y=\sqrt{\sin x+\sqrt{\sin x+\sqrt{\sin x+\sqrt{\ldots\ldots\text{to} \infty}}}}$,

show that $\dfrac{dy}{dx}=\dfrac{\cos x}{2y-1}$.

94. (i) If $y=\sqrt{\log x+\sqrt{\log x+\sqrt{\ldots\ldots \text{to } \infty}}}$,

show that $x\dfrac{dy}{dx}=\dfrac{1}{2y-1}$.

(ii) If $y=(\log x)^{(\log x)^{(\log x)\cdots\cdots\text{to }\infty}}$, then show that

$x.\log x.\dfrac{dy}{dx}=\dfrac{y^2}{1-y\log(\log x)}$.

95. Eliminate c from the following equations :—

(i) $y+1=x+c.e^{-x}$, (ii) $y^{-1}=cx-x\log x$,

(iii) $y=cx+\dfrac{a}{c}$. (iv) $c^3x-c^3y-1=0$,

(v) $y^2=2cx+c^2$, (vi) $(2y-3x^2-c)(2y+x^2-c)=0$.

96. Eliminate a and b from the following equations :—

(i) $y=ax+b$ (ii) $ax^2+by^2=1$

(iii) $y=a+b\cos x$ (iv) $y=ax^2+bx^3+\frac{1}{2}x$.

97. (i) If $ax^2+2hxy+by^2+2gx+2fy=0$, show that

$\left(\dfrac{d^2y}{dx^2}\right)_{(0,\,0)}=-\dfrac{af^2+bg^2-2fgh}{f^3}$.

(ii) If $ax^2+2hxy+by^2+2gx+2fy+c=0$, show that

$\dfrac{d^2y}{dx^2}=\dfrac{abc+2fgh-af^2-bg^2-ch^2}{(hx+by+f)^3}$.

98. The consumption of coal in running a locomotive is proportional to the square of its speed and is 2 quintals per hour for a speed of 32 kmh. If one quintal coal costs Rs. 10, and other cost per hour is Rs. 11·25, find the minimum cost for a journey of 100 km.

99. The horizontal velocity with which water comes out of a hole made in the surface of a cylindrical vessel full of water is proportional to the height of the water over the hole. Show that water is projected to the maximum distance in the horizontal plane through the bottom of the vessel if the hole is made at a height of one-third the height of the water in the vessel from the bottom of the vessel.

ANSWERS

Exercise 1

1. (i) $x=0$; (ii) $x=0$; (iii) $x=n\frac{\pi}{2}$ (n is a positive integer) ; (iv) $x=\pm 2$.

2. (i) $x=0$; (ii) $x=+a$. (iii) $\sqrt{x}=-1$. (iv) $x=n\pi$ (n is a positive integer.)

4. $2\cdot6457\cdots$ 5. $1\cdot2599$.

7. $\frac{134}{165}, \frac{136}{165}, \frac{138}{165}, \frac{140}{165}, \frac{142}{165}, \frac{144}{165}, \frac{146}{165}, \frac{148}{165}$

8. $\sqrt{2}+\cdot06$, $\sqrt{2}+\cdot12$; $\sqrt{2}+\cdot18$, $\sqrt{2}+\cdot24$.

11. (i) Infinitely many. (ii) Yes (iii) No

12. (i) rational number ; rational number (ii) non-recurring and non-terminating (iii) rational.

15. (i) $\sqrt{8}, \sqrt[3]{10}, \sqrt{3}$, (ii) $\sqrt{3}, \sqrt[4]{7}, \sqrt[3]{2}$ (iii) $\sqrt[3]{8}, \sqrt[4]{9}, \sqrt[6]{25}$.

Exercise 2

1. $f(0)=1$; $f(1)=4$; $f(2\cdot1)=9\cdot841$; $f(-2)=-23$;
$f(a)=a^3-2a^2+4a+1$; $f(-x)=-x^3-2x^2-4x+1$
$f(a+h)=a^3-2a^2+4a+1+(3a^2-4a+4)h+(3a-2)h^2+h^3$.

2. $f(0)=3$; $f\left(\frac{\pi}{2}\right)=1$; $f\left(\frac{\pi}{3}\right)=2$; $f\left(\frac{\pi}{4}\right)=\sqrt{2}+1$;
$f\left(\frac{\pi}{2}+h\right)=-2\sin h+1$; $f(x)=2\cos x+1$.

3. $f(2)=4$; $f(2\cdot1)=4\cdot41$; $f(2\cdot01)=4\cdot0401$
$f(2\cdot001)=4\cdot004001$; $\dfrac{f(2\cdot0001)-f(2)}{\cdot0001}=4\cdot0001$.

4. $f(3)=0$ $f(0)=-4$, $f(-1)=0$, $f(4)=0$.
$f(-x)=\dfrac{(x+3)(x+4)(1-x)}{(2x-3)(x+1)}$; $f\left(\dfrac{1}{x}\right)=\dfrac{(1-3x)(1-4x)(1+x)}{x(2+3x)(1-x)}$;
$f(1)$ is undefined ; 5. $f(0)=1$; $f(\log_a x)=x$;
$\dfrac{f(x+h)-f(x)}{h}=a^x\dfrac{a^h-1}{h}$; 6. $f(0)=-\frac{1}{2}$;

$f(-1)=-\frac{2}{3}$; $f(2x)=\dfrac{2x-1}{4x^2+2}$; $f\left(\dfrac{1}{x}\right)=\dfrac{x(1-x)}{2x^2+1}$;

$f(x+h)=\dfrac{(x+h)-1}{(x+h)^2+2}$.

15. Odd ; Odd ; 17. (i) $-1 \leqslant x \leqslant 1$ (ii) $-4 \leqslant x \leqslant 5$.

(iii) $-\infty < x < \infty$, (iv) all real numbers other than a.

(v) $0 < x < \infty$. (vi) $-\frac{1}{2} < x < \infty$. 20. $x=0$.

22. $V=4x(8-x)^2$, $0<x<8$; 288.

23. $A=\dfrac{\sqrt{3}}{4}x^2$. 24. $a=2, b=1$. 25. $\dfrac{13}{2}$

26. (i) $x=\sqrt{y^2-1}$ (ii) $x=\frac{1}{2}\left(y-\dfrac{1}{y}\right)$

(iii) $x=\frac{1}{2}(\sqrt{y+2}+\sqrt{(\frac{1}{2})}=x$ (iv) $x=\frac{1}{2}(\sqrt{y-3}+\sqrt{y+1})$.

27. (i) $-\dfrac{a+cy}{b+dy}$ (ii) $2\tan^{-1}\sqrt{y}$

28. $y=1{\cdot}8$ when $0<x\leqslant 1$.

$=1{\cdot}8+(k-1)\times{\cdot}18$, when $1+(k-1)x\times{\cdot}1<x<1+k\times{\cdot}1$
where k is any positive integer.

29. $y=50-x$, $0\leqslant x\leqslant 30$
$=20+(x-30)$, $30\leqslant x\leqslant 60$,
$=50+2(x-60)$, $60\leqslant x\leqslant 120$
$=170$, $x\geqslant 120$.

Exercise 3A

2. $2{\cdot}99983<x<3{\cdot}00017$.
3. $\lim\limits_{x\to 0} |x|=0$, 4. $\lim\limits_{x\to 0} f(x)$ is not defined.

Exercise 3C (Page 65)

1. (a) (i) 2 ; (ii) 16 ; (iii) 1 ; (iv) 1.

(b) (i) $x=2$, (ii) $x=1, 2$ (iii) $x=(2n+1)\frac{\pi}{2}$
$[n=0, \pm 1, \pm 2, \cdots]$

2. 0 3. (i) $x=0$; (ii) continuous ; (iii) continuous at $x=0$; discontinuous at $x=\frac{3}{2}$ 4. No point of discontinuity.

5. 6 ; is removable discontinuity.

6. $x=1, x=2$; Not removable. 7. 4.

Exercise 3D

1. (i) -3 ; (ii) 9 ; (iii) 12 ; (iv) 64 ; (v) $-\frac{1}{8}$; (vi) 1 ; (vii) 4 ; (viii) 4 ; (ix) 1 ; (x) e^2 ; (xi) $\frac{22}{13}$; (xii) 2.

Exercise 3E

2. (i) 256 ; (ii) 12 ; (iii) $\frac{7}{5}$; (iv) 32 ; (v) 1 ; (vi) $\frac{2}{3}$; (vii) 1 ; (viii) e^a.

Exercise 3C (Page 92)

1. (i) -3 ; (ii) 9 ; (iii) 12 ; (iv) 64 ; (v) $-\frac{1}{8}$; (vi) 1 ; (vii) 4 ; (viii) 4 ; (ix) 1 ; (x) e^2 ; (xi) $\frac{22}{13}$; (xii) 2.

2. (i) 4 ; (ii) $\frac{\pi}{2}$; (iii) $1-e$; (iv) -4 ; (v) 9 ; (vi) $\frac{2}{\pi}$; (vii) 10 ; (viii) $\frac{c}{f}$.

3. (i) 1 ; (ii) 4 ; (iii) 2 ; (iv) $\frac{1}{2}$; (v) $\frac{-3}{4}$; (vi) 2 ; (vii) 3 ; (viii) 1 ; (ix) $\frac{2\sqrt{2}}{3}$; (x) $\frac{1}{4}$.

6. (i) $\frac{1}{6}$; (ii) $\frac{1}{2}$.

8. (i) 3 ; (ii) $\frac{5}{2}$; (iii) $\frac{b^2-a^2}{2}$; (iv) $\frac{m}{n}$; (v) $\frac{3}{5}$; (vi) 2 ; (vii) $\frac{1}{2}$; (viii) 2 ; (ix) $\frac{2}{3}$; (x) $\sin a - a\cos a$; (xi) $\frac{3}{4}$; (xii) $\frac{\alpha}{\beta}$; (xiii) 2 ; (xiv) 2 ; (xv) 2 ; (xvi) 1.

12. (i) 1 ; (ii) e^a.

15. (i) $2x$; (ii) $-\operatorname{cosec}^2 x+2$; (iii) $\sec x \tan x$; (iv) $-\frac{2}{x^3}$.

16. (i) $-\frac{1}{9}$; (ii) e ; (iii) 1.

17. $\underset{x\to 1}{\text{Lt}}\, f(x)$ does not exist ; $\underset{x\to 2}{\text{Lt}}\, f(x)=3$.

19. (i) 0 ; (ii) 1 ; (iii) 0 ; (iv) 2 ; (v) 0 ; (vi) 2 ; (vii) 0.

Exercise 4A

1. (i) $02x - \cdot 0299$ (ii) $\dfrac{3x - \cdot 75}{x^2(x - \cdot 5)^2}$ (iii) 0.

2. (i) $-\cdot 1$; $-2\cdot 37, 23\cdot 7$; (ii) $1\cdot 1, 4\cdot 957, 4\cdot 506$;
(iii) $2, \dfrac{98}{3375}, \dfrac{49}{3375}$

3. (i) $\triangle x\{3x^2 + 3x.\triangle x + (\triangle x)^2\}$

(ii) $\dfrac{-\triangle x}{(x-1)(x+\triangle x-1)}$; (iii) $\dfrac{2(x-1)\triangle x + (\triangle x)^2}{(x-1)^2(x+\triangle x-1)^2}$

(iv) $a.\triangle x - \dfrac{b\triangle x}{x(x+\triangle x)}$

(v) $3.\triangle x\{4x^3 + 2x + 6x^2\triangle x + \triangle x + 4x(\triangle x)^2 + (\triangle x)^3\}$

(vi) $x\dfrac{\sin\triangle x}{\sin x.\sin(x+\triangle x)} + \triangle x.\tan(x+\triangle x) + 2x.\triangle x + (\triangle x)^2$

(vii) $-\dfrac{2x\triangle x + (\triangle x)^2}{(x+\triangle x)^{\frac{2}{3}}x^{\frac{2}{3}}\{x^{\frac{4}{3}} + x^{\frac{2}{3}}(x+\triangle x)^{\frac{2}{3}} + (x+\triangle x)^{\frac{4}{3}}\}} + 4.\triangle x$

Exercise 4A

1. (i) 2 ; (ii) $8x$ (iii) x^2 ; (iv) $2ax$;
(v) $-\dfrac{1}{(x-1)^2}$ (vi) $-\dfrac{a}{(ax+b)^2}$; (vii) $-\dfrac{3}{x^4}$
(viii) $3x^2+1$ (ix) $\dfrac{1}{2\sqrt{x}}$ (x) 0.

2. (i) $\frac{1}{4}$ (ii) -2. 3. (i) 2 ; (ii) 0. (iii) 2.
4. (i) -2 ; (ii) $-\frac{10}{9}$.
(iii) $-\dfrac{2(1+a^2)}{(a^2-1)^2}$. 5. $2x.e^{x^2}$. 6. $\dfrac{1}{(1-x)\sqrt{1-x^2}}$

7. (i) $2ax+b$ (ii) $\dfrac{x}{\sqrt{x^2+1}}$ (iii) $-\dfrac{2}{x^3}$ (iv) $4(2x+3)$
(v) $-\dfrac{1}{(x+1)^2} + e^x$ (vi) $\dfrac{ax}{\sqrt{ax^2+b}}$ (vii) 0.

8. $6t+4$. 9. $\cos t + 2t$.
10. (*a*), (*b*) continuous and not differentiable.
11. some as 0.
13. $f(x)=0$ when $0 \leqslant x \leqslant a$; $-\dfrac{1}{3}x + \dfrac{1}{3}\dfrac{a^3}{x^2}$ when $a < x < b$.

$f(b) = -\dfrac{1}{3b^2}(b^3 - b^3)$.

15. (i) $2x$ (ii) $99x^{98}$ (iii) $1000x^{999}$ (iv) $2\cdot5x^{1\cdot5}$

(v) $1\cdot5x^{\cdot5}$ (vi) $-\dfrac{3}{x^4}$ (vii) $-1\cdot5x^{-2\cdot5}$ (viii) $-\dfrac{3}{x^4}$

(ix) $\dfrac{20}{x^{21}}$ (x) $-\dfrac{2\cdot1}{x^{3\cdot2}}$.

(xi) $\dfrac{1}{2\sqrt{x}}$ (xii) $\dfrac{3}{2}\sqrt{x}$ (xiii) $\dfrac{1}{n}x^{\frac{1-n}{n}}$ (xiv) $-\dfrac{4}{3\sqrt[3]{x^7}}$

(xv) $\dfrac{5}{3}x^{\frac{2}{3}}$ (xvi) ex^{e-1} 16. (i) $12x^2$ (ii) $\dfrac{1}{\sqrt{x}}$

(iii) $-5\sin x$ (iv) $-\dfrac{9}{4x^4}$ (v) $\sqrt{x}$ (vi) $-\dfrac{4}{x^3}$.

(vii) $2^{x+2}\log_e 2$.

17. (i) $8x+5$ (ii) $-\dfrac{3}{x^2}+4x$ (iii) $6x+1$ (iv) $2ax+b$

(v) $\dfrac{3}{2}\sqrt{x}-\dfrac{2}{2x^2\sqrt{x}}-\sec x\tan x$. (vi) $\dfrac{2}{\sqrt{x}}-\dfrac{1}{2}\operatorname{cosec}^2 x$

(vii) $2x+3$ (viii) $2nx^{2n-1}-2nx$ (ix) $-\dfrac{1}{2x\sqrt{x}}-\dfrac{1}{2\sqrt{x}}$

(x) $\dfrac{5}{6\sqrt[6]{x}}+\dfrac{1}{12}\dfrac{1}{\sqrt[12]{x^{11}}}$ (xi) $\dfrac{3}{2}\sqrt{x}+\dfrac{1}{2\sqrt{x}}-\dfrac{1}{2\sqrt{x^3}}$

(xii) $\cos x-2\sin x+3\sec^2 x-4\operatorname{cosec}^2 x+5\sec x\tan x$
$-6\operatorname{cosec} x\cot x-e^x$.

(xiii) $1-\sqrt{\dfrac{a}{x}}$ (xiv) $-\dfrac{1}{x^2}+\dfrac{3}{2\sqrt{x^3}}-\dfrac{1}{2\sqrt{x}}-\operatorname{cosec} x\cot x$.

18. (i) $6-4x-12x^2$ (ii) $45x^4-45x^2-12x$ (iii) $4x^3+2x$

(iv) $2x\sec x+x^2\sec x\tan x$ (v) $1-\dfrac{1}{x^2}$

(vi) $e^x(\sin x+\cos x)+\sin x$

(vii) $\sec^3 x+\sec x\tan^2 x=\sec x(1+2\tan^2 x)$.

(viii) $\sec^2 x-\operatorname{cosec}^2 x=-4\operatorname{cosec} 2x\cot 2x$

(ix) $2\tan x\sec^2 x$ (x) $4x^3+\frac{9}{2}x^3\sqrt{x}$

(xi) $\operatorname{cosec} x-x\operatorname{cosec} x\cot x$ (xii) $n.x^{n-1}\cot x-x^n\operatorname{cosec}^2 x$

(xiii) $\operatorname{cosec}^2 x-2x\operatorname{cosec}^2 x\cot x=\operatorname{cosec}^2 x(1-2x\cot x)$

(xiv) $2xe^x\cos x+x^2e^x\cos x-x^2e^x\sin x$

(xv) $5x^4+16x^3+8x+16-2x(3x+8)\cos x+2x^2(x+4)\sin x$
$-e^x(x^3+3x^2+4+2x\sin x-2x\cos x-2\cos x)$

(xvi) $\operatorname{cosec} x - x \operatorname{cosec} x \cot x$ (xvii) $-\left(\dfrac{\sin x}{x} + \dfrac{\cos x}{x^2}\right)$

(xviii) $2x \cot x - x^2 \operatorname{cosec}^2 x$.

20. (i) $\dfrac{x \sec^2 x - \tan x}{x^2}$ (ii) $\dfrac{2}{(x+1)^2}$ (iii) $\dfrac{x^2 - 8x}{(x-4)^2}$

(iv) $\dfrac{1}{\sqrt{x}(\sqrt{x}+1)^2}$ (v) $\dfrac{-x - 3\sqrt{x} + \dfrac{1}{2\sqrt{x}}}{(x^{\frac{3}{2}}+1)^2}$ (vi) $\dfrac{ad - bc}{(cx+d)^2}$

(vii) $\dfrac{4b(c - ax^2)}{(ax^2 - 2bx + c)^2}$ (viii) $-\dfrac{2}{(2x+1)^2}$ (ix) $-\dfrac{a}{(ax+b)^2}$

(x) $-\dfrac{3x^2}{(x^3-1)^2}$ (xi) $\dfrac{-e^{2x} - e^x\{(x+1)\cos x + x \sin x - 1\} - \sin x}{(xe^x+1)^2}$

(xii) $\dfrac{x(x^3 + 3x + 2)}{(x^2+1)^2}$

(xiii) $-\dfrac{x^4 + 2x^3 + 5x^2 + 2x - 2 + (x^2 + x)\cot x + (x^2 - x + 1)\cot^2 x}{(x^2 - x + 1)^2 e^x}$

(xiv) $-\dfrac{2(x^2+1)}{(x^2-1)^2}$

(xv) $\dfrac{1}{2\sqrt{x}(x^2+1)}\left(2x \cos x + \sin x - \dfrac{1}{x}\right) - \dfrac{2\sqrt{x}}{(x^2+1)^2}\left(1 + x \sin x\right)$

Exercise 4C

1. (i) $10x(x^2+1)^4$ (ii) $-\dfrac{4x}{(x^2-1)^3}$ (iii) $20x(x^2+a^2)^9$

(iv) $6(x+1)\sqrt{2x^2+4x-1}$

(v) $n(2ax+b)(ax^2+bx+c)^{n-1}$ (vi) $\dfrac{2x-3}{2\sqrt{x^2-3x+7}}$

(vii) $2 \cos 2x$ (viii) $-3 \sin 3x$ (ix) $-5 \operatorname{cosec}^2 5x$

(x) $n \sec nx \tan nx$ (xi) $\frac{1}{2}\sqrt{\cot x}\ \cos x$

(xii) $3x^2 \cos(x^3)$ (xiii) $\sec x . \operatorname{cosec} x$ (xiv) $-\cot x$

(xv) $-\tan x$ (xvi) $-\dfrac{1}{x}\sin(\log x)$ (xvii) $-\sec x$

(xviii) $\dfrac{1}{e^x+1}$ (xix) $\dfrac{1}{x \log x}$.

2. (i) $-\dfrac{3x}{(x^2+a^2)^{\frac{5}{2}}}$ (ii) $\dfrac{1+x^2}{\sqrt{2x}(1+x^2)^{\frac{3}{2}}}$ (iii) $\dfrac{a}{(x+a)\sqrt{x^2-a^2}}$

(iv) $\frac{b}{3a^2}\sec^4\theta \operatorname{cosec}\theta$ (v) $\frac{1}{a(1+\cos t)^2}$ (iv) $\frac{2(1+t^2)^3}{3a(1-t^2)^3}$

5. $y+2x-4=0$; $2y-x-3=0$
6. $ty=x+at^2$; $y+tx=2at+at^3$.
7. $ax-by=a^2-b^2$
11. (i) (*a*) $-2<x<4$ (*b*) $1<x<2$ (ii) $1<x<3$.

Exercise 4I

1. $\frac{5}{8}, \frac{5}{8}$; $\frac{3}{8}, \frac{2}{3}$ 2 $a\sqrt{2}, a\sqrt{2}$; a, a.
3. $a(1-t\cot t)$, $a(\tan t-t)$,
$a\cos.t(1-t\cot t)$, $a(\tan t-t)$.
5. Maximum at $x=2$; the maximum value is 10.
Minimum value 6 at $x=4$.
6. (i) maximum values at $x=(2n+1)\frac{\pi}{2}$; minimum values at
$$x=nx+(-1)^n\frac{7\pi}{6}$$
(ii) maximum values at $x=(2n+1)\frac{\pi}{2}$, minimum values at
$$x=n\pi+(-1)^n\frac{\pi}{6}.$$
(iii) maximum values at $x=n\pi$, minimum values at
$$x=2n\pi\pm\frac{\pi}{3}.$$

Exercise 4

1. (i) $e^{x^3+4x+1}.(3x^2+4)$ (ii) $e^{\sin x}.\cos x$
(iii) $e^{\tan^{-1}x}.\frac{1}{1+x^2}$ (iv) $e^{\sqrt{x+1}}\frac{1}{2\sqrt{x+1}}$ (v) e^{e^x+x}
2. (i) $(2x+1)\cos(x^2+x+1)$ (ii) $\frac{1}{x}\cos(\log x)$
(iii) $e^x\cos(e^x)$ (iv) $\frac{1}{1+x^2}\cos(\tan^{-1}x)$
(v) $(2x\cos x-x^2\sin x)\cos(x^2\cos x)$.
3. (iv) $\frac{2x+1}{\sqrt{2x+x^2-2x^3-x^4}}$ (v) -1 (vi) $\frac{e^x}{1+e^{2x}}$

(vii) $\dfrac{1}{2\sqrt{x}(1+x)}$ (viii) $\dfrac{2x}{-(x^2+a^2)^2}$

(ix) $-\dfrac{1}{(x+e^x \log x)^2}\cdot\left(1+e^x \log x+\dfrac{1}{x}e^x\right)$

(x) $\dfrac{-1}{\{\sin^{-1}(x^2+4)\}^2}\cdot \quad \dfrac{2x}{\sqrt{1-(x^2+4)^2}}.$

4. (i) $\dfrac{1-x^2}{1+x^2+x^4}$ (ii) $\dfrac{1}{2(a-b)}\left[\dfrac{1}{\sqrt{x+a}}+\dfrac{1}{\sqrt{x+b}}\right]$

(iii) $\dfrac{-1}{(\cos x+\sin x)\sqrt{\cos 2x}}$

(iv) $y\left[\dfrac{1}{2x}-\dfrac{2}{3(1-x)}+\dfrac{9}{4(2-3x)}+\dfrac{16}{5(3-4x)}\right]$

(v) $\sin x.e^x.\log x.x^x\left[\cot x+\dfrac{1}{x\log x}+\log x+2\right]$

(vi) $(\tan x)^{\cot x}.\operatorname{cosec}^2 x \log(e\cot x)-(\cot x)^{\tan x}$
$\sec^x x \log(e\tan x).$

(vii) $e^{(x)^x}.x^x.\log ex.$

(viii) $(1+x^{-1})^x\left[\log(1+x^{-1})-(1+x)^{-1}\right]$

(ix) $\dfrac{x^2}{1-x^4}$ (x) $x^{x^2+1}.\log ex^2$

(xi) $x^{(x^x)}.x^{x-1}\left[1+x\log x\log ex\right]$

(xii) $\dfrac{\sin\alpha}{1-2x\cos\alpha+x^3}$ (xiii) $\dfrac{\log\sin x-x\cot x\log x}{x(\log\sin x)^2}$

(xiv) $\dfrac{1-\log x}{x^2}$ (xv) $-\dfrac{\log 4}{x(\log x)^2}$

(xvi) $2x^{\log x-1}.\log x+(\log x)^x\left\{\log\left(\log x+\dfrac{1}{\log x}\right)\right\}$

(xvii) $-\dfrac{y^x\log y+yx^{y-1}}{x^y\log x+xy^{x-1}}.$

6. (i) -1 (ii) $\dfrac{2}{1+x^2}$ (iii) $\dfrac{3}{\sqrt{1-x}}$

(iv) $-\dfrac{1}{2}$ (v) $\dfrac{1}{2\sqrt{1-x^2}}$ (vi) -1

(vii) $\dfrac{2}{\sqrt{1-x^2}}$ (viii) $\dfrac{-x}{\sqrt{1-x^2}}$ (ix) $\dfrac{-\sqrt{b^2-a^2}}{b+a\cos x}(b>a)$

(x) $\dfrac{(a^2-b^2)\sin x}{(a^2+b^2)(1+\cos^2 x)+4ab\cos x}$

(xi) $\frac{1}{2}\times\dfrac{1}{1+x^2}$ (xii) $\dfrac{2a^2x}{a^4+x^4}$

(xiii) 1. (xiv) $\frac{1}{2}$.

7. (i) $\frac{2}{3}$, (ii) $-\frac{1}{2}$, (iii) $-\frac{1}{2}$, (iv) $-\frac{4}{3}$, (v) 1.

8. (i) $\frac{8}{3}x^5$ (ii) $\frac{3}{4}\cdot\frac{1}{x^2}$ (iii) $\frac{1}{2}$ (iv) 1.

(v) $x^{\sin^{-1}x}\left\{\log x+\dfrac{\sqrt{1-x^2}}{x}\sin^{-1}x\right\}$

14. 44 sq. mm. ·3% (nearly) 15. 1·39 inches (nearly)

16. ·33075 c. c., 29% (nearly) 17. 2·2%

18. ·02 (nearly) 23. continuous and not differentiable.

25. $f'(x)=0$, in $0\leqslant x\leqslant a$

$$=\frac{x^3-a^3}{x^2}, \text{ in } a<x\leqslant b=-\frac{b^3-a^3}{3x^2}, \text{ in } x>b$$

27. $xx_1^{-\frac{1}{3}}+yy_1^{-\frac{1}{3}}=a^{\frac{2}{3}}$, 28. (i) $(1, -1), (-1, 1)$

(ii) $\left(\sqrt{2}, -\dfrac{1}{\sqrt{2}}\right), \left(-\sqrt{2}, \dfrac{1}{\sqrt{2}}\right)$ 29. $(3, 0), (\frac{7}{3}, \frac{4}{27})$

31. $x+y=2$, $x-y=0$.

32. (i) $(0, 0), (\frac{1}{3}.2^{\frac{1}{3}}a^{\frac{2}{3}}b^{\frac{1}{3}};\ \frac{1}{3}.2^{\frac{2}{3}}a^{\frac{1}{3}}.b^{\frac{2}{3}})$,

$(0, 0), (\frac{1}{3}.2^{\frac{2}{3}}a^{\frac{2}{3}}b^{\frac{1}{3}};\ \frac{1}{3}2^{\frac{1}{3}}a^{\frac{1}{3}}b^{\frac{2}{3}})$.

(ii) The pair of points intersected by $ax+hy=0$; the two points intersected by $hx+by=0$.

33. $y=0$ and $y-\frac{4}{3}x=-\frac{4}{9}$, $y-\frac{4}{3}x=-\frac{16}{27}$

34. (2, 4) 43. 6, 6 44. 9, 1 45. 1

46. If height=radius 48. height$=\sqrt{2}\times$radius

49. $a\sqrt{2}$, $b\sqrt{2}$ 51. $\dfrac{a}{\sqrt{2}}$, $\dfrac{a}{\sqrt{2}}$ 54. $\dfrac{8192}{27}$

56. 45, 6, 88 57. $u-gt$, $-g$, $\dfrac{u^2}{2g}$ 58. $\frac{1}{27}\pi x^3$; 34·9

59. 49 : 48 60. ·27 sq. in./sec 61. 17 cu. in /sec

62. 7 kg./sq. cm. will be reduced. 63. 20π, 30π

64. $4-t+t^2$, $3\frac{3}{4}$, $\frac{23}{12}$ 65. $-ak$, ak^2

67. $\dfrac{\pi^2}{2\sqrt{2}}$, $(2n+\frac{1}{2})$ second   71. $\dfrac{1-(n+1)x^n+nx^{n+1}}{(1-x)^2}$

73. $\dfrac{\cos\left(x+\frac{nh}{2}\right).\sin\left(n+1\right)\frac{h}{2}}{\sin\frac{h}{2}}$

(i) $\dfrac{(n+1)\sin nx - n\sin(n+1)x}{4\sin^2\frac{1}{2}x}$

(ii) $\dfrac{(n+1)\cos nx - n\cos(n+1)x - 1}{4\sin^2\frac{1}{2}x}$

74. (i) $\frac{5}{12}$ (ii) $\frac{1}{6}(3\pm\sqrt{3})$ 75. $\frac{9}{25}$

83. 4 sec, 48 feet.   85. $19\frac{1}{5}$ min, 12 kms.

86. (i) nearer, $161\sqrt{82}$ km./hr.

(ii) further, $4{\cdot}2\sqrt{10}$ km./hr.   (iii) 1 hr. 55 min.

87. $2<t<4$; $t<3$; $t>3$.   88. 4 sq. cms.

89. Min. value $y=0$, when $x=-2$ and $x=1$;
maximum value $y=\frac{81}{16}$ when $x=-\frac{1}{2}$;
decreasing in $x<-2$ and $-\frac{1}{2}<x<1$,
increasing in $-2<x<-\frac{1}{2}$ and $x>1$.

90. (i) $\dfrac{\pi}{5\sqrt{39}}$ cms./sec.   (ii) $\dfrac{\pi}{15}$ sq. cm./sec.   91. $\dfrac{5\pi}{18}$ sq. cm.

95. (i) $y_1+y=x$ (ii) $xy_1+y=xy^2$,

(iii) $y=xy_1+\dfrac{a}{y_1}$ (iv) $y_1{}^3.x-y_1{}^2.y-1=0$

(v) $y=yy_1{}^2+2xy_1$; (vi) $y_1{}^2-2xy_1-3x^2=0$

96. (i) $y_2=0$ (ii) $x(yy_2+y_1{}^2)=yy_1$;

(iii) $y_2=y_1\cot x$;   (iv) $x^2y_2-4xy_1+6y=x$.

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INTEGRAL CALCULUS AND DIFFERENTIAL EQUATIONS

CHAPTER ONE
INDEFINITE INTEGRAL

§ 1·1. Aim of Integral Calculus

The term Integration means "finding the sum of". The aim of integral calculus is to find the sum of special types of those infinite series, each term of which tends to zero as the number of terms of the series tends to zero. In fact the subject of integral calculus came into being in the attempt of finding the area enclosed by closed curves. In such an attempt, finding the sum of above mentioned infinite series was required. Integral calculus has another view-point. This view-point is to find the original function (primitive) when its differential coefficient or derivative is known. In this respect the process of integration can be regarded as the inverse process of differentiation. These two viewpoints have been bridged by the *Fundamental Theorem of Integral Calculus* to be discussed in Chapter Four.

Integral Calculus has various applications in Applied Mathematics, Physics and other branches of science. You shall find applications of Integral Calculus in the Mechanics part of this book.

Historically Integral Calculus was discovered from the first view point. But we shall, in this book, discuss first, the second view point i.e., we shall first discuss integration as the inverse process of differentiation.

§ 1·2. Integration as the inverse process of differentiation : Indefinite Integration.

You know that $\frac{d}{dx}(x^2)=2x$. This is also written as $\int 2xdx=x^2$ which is read as integral $2xdx$ is equal to x^2 or integral of $2x$ with respect to x is x^2. You know that differentiation of x^2 gives $2x$. The inverse process of getting x^2 from $2x$ is called the process of *integration* with respect to x. $2x$ is called the *Integrand* and $\int 2xdx$ or x^2 is the *Indefinite Integral* with respect to x of the Integrand $2x$. In $\int 2xdx$, the quantity dx is the differential of x. As the operation of differentiation with respect to x is indicated by the symbol $\frac{d}{dx}$, so is the process of integration with respect to x indicated by the symbol $\int dx$.

Example. As, $\frac{d}{dx}(\sin x)=\cos x$,

$\therefore \int \cos x \, dx=\sin x$

Def. If $\frac{d}{dx}\{f(x)\}=g(x)$, then $\int g(x)dx=f(x)$.

Hence by definition, if $\int g(x)dx=f(x)$ then,

$\frac{d}{dx}\{f(x)\}=g(x)$, or, $\frac{d}{dx}\{\int g(x)dx\}=g(x)$.

§ 1·3. Integral of a function with respect to a variable is not unique : Constant of Integration :

Let $\int f(x)dx=g(x)$. $\therefore \frac{d}{dx}\{g(x)\}=f(x)$.

Again, $\frac{d}{dx}\{g(x)+c\}=f(x)$ [c, is any constant]

$\therefore$ By definition, $\int f(x)dx=g(x)+c$.

So, $g(x)+c$, is an integral of $f(x)$ with respect to x.

Hence integrals of $f(x)$ with respect to x are more than one. Again, if $g(x)$ and $h(x)$, be two integrals of $f(x)$ with respect to x, then $g(x)$ and $h(x)$ differ by a constant.

For, since $\int f(x)dx=g(x)$ and $\int f(x)dx=h(x)$

$\therefore \frac{d}{dx}\{g(x)\}=f(x)$ and $\frac{d}{dx}\{h(x)\}=f(x)$.

$\therefore \frac{d}{dx}\{g(x)\}-\frac{d}{dx}\{h(x)\}=0$, or, $\frac{d}{dx}\{g(x)-h(x)\}=0$

$\therefore g(x)-h(x)$ is a constant [See Ex. 7(ii) Exercise 4 Differential Calculus].

For, you have learnt in differential calculus that the derivative with respect to a variable or the rate of change of a function cannot be zero ; if the function be not a constant.

Now, it is clear from the above discussion that if $g(x)$ be an integral of $f(x)$ with respect to x, then $g(x)+c$; where c is a constant, will also be an integral of $f(x)$ with respect to x. For different values of c, one can obtain different integrals of $f(x)$. When $c=0$, the integral $g(x)$ is obtained. Hence $g(x)+c$ is the general form of integral of $f(x)$ with respect to x. The constant c is called an arbitrary constant of integration. The general form of the indefinite

integral of a function with respect to a variable is expressed by adding with the integral of the function, an arbitrary constant of integration.

From the above discussion it is now evident that every function possesses more than one integral with respect to a variable. So, no function can be definitely called that, it is the only integral of another function with respect to a variable and hence the name *Indefinite Integral.*

Note. For convenience of printing we have not added in this book, the arbitrary constant of integrations on many a occasion. But you must always remember its presence. To find the indefinite integral is to find the general form of the integral. If the constant of integration is absent, then one cannot get the general form.

§ 1·4. **Standard Forms :**

(1) $\int x^n dx = \frac{x^{n+1}}{n+1} + c \ (n \neq -1)$.

Proof. $\frac{d}{dx}\left(\frac{x^{n+1}}{n+1} + c\right) = \frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) + \frac{d}{dx}(c)$

$$= \frac{1}{n+1}\frac{d}{dx}\left(x^{n+1}\right) + 0 = \frac{1}{n+1}\left(n+1\right)x^n = x^n.$$

$\therefore \int x^n dx = \frac{x^{n+1}}{n+1} + c \ (n \neq -1)$.

Examples.

1. $\int x^5 dx = \frac{x^{5+1}}{5+1} + c = \frac{x^6}{6} + c$.

2. $\int \sqrt{x}\,dx = \int x^{\frac{1}{2}} dx = \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + c = \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + c = \frac{2}{3}x^{\frac{3}{2}} + c$.

3. $\int dx = \int 1.dx = \int x^0 dx = \frac{x^{0+1}}{0+1} + c = \frac{x^1}{1} + c = x + c$.

(2) $\int x^{-1} dx$ or, $\int \frac{dx}{x} = \log x + c$.

Proof. $\frac{d}{dx}\left(\log x + c\right) = \frac{1}{x}$, $\therefore \int \frac{dx}{x} = \log x + c$.

(3) $e^{ax} dx = \frac{e^{ax}}{a} + c$.

Proof. $\frac{d}{dx}\left(\frac{e^{ax}}{a} + c\right) = \frac{d}{dx}\left(\frac{e^{ax}}{a}\right) + \frac{d}{dx}(c)$

$$= \frac{1}{a}\frac{d}{dx}\left(e^{ax}\right) + 0 = \frac{1}{a}\left(a.e^{ax}\right) = e^{ax}. \quad \therefore \int e^{ax} dx = \frac{e^{ax}}{a} + c.$$

Examples :

1. $\int e^{3x}dx=\frac{e^{3x}}{3}+c.$

2. $\int e^{-5x}dx=\frac{e^{-5x}}{-5}+c=-\frac{e^{-5x}}{5}+c.$

(4) $\int a^{mx}dx=\frac{1}{m}a^{mx}/\log_e^a+c,$

Proof. $\frac{d}{dx}\left(\frac{1}{m}a^{mx}/\log_3^a+c\right)$

$=\frac{1}{m}\log_3^a\frac{d}{dx}\left(a^{nx}\right)+\frac{d}{dx}(c)=\frac{1}{m}\log_8^a.m.\frac{a^{mx}}{\log_e^a}$

$=a^{mx}.$

$\therefore$ By definition, $\int a^{nx}dx\ \frac{a^{nx}}{m\log_e^a}+c.$

Examples.

1. $\int 2^{5x}dx=\frac{1}{5}2^{5x}/\log_e^2+c.$

(5) **Integration of Trigonometric functions :**

(i) $\int\frac{d}{dx}\left(\frac{\sin ax}{a}\right)=\frac{1}{a}\frac{d}{dx}\left(\sin ax\right)=\frac{1}{a}a\cos ax=\cos ax,$

$\therefore$ $\int\cos ax\,dx=\frac{\sin ax}{a}+c.$

(ii) $\frac{d}{dx}\left(-\frac{\cos ax}{a}\right)=-\frac{1}{a}\frac{d}{dx}\left(\cos ax\right)$

$=-\frac{1}{a}\left(-a\sin ax\right)=\sin ax.$

$\therefore$ $\int\sin ax\,dx=-\frac{\cos ax}{a}.$

(iii) $\frac{d}{dx}\left(\frac{\tan ax}{a}\right)=\frac{1}{a}\frac{d}{dx}\left(\tan ax\right)=\frac{1}{a}.a\sec^2ax=\sec^2ax.$

$\therefore$ $\int\sec^2ax\,dx=\frac{\tan ax}{a}+c.$

(iv) $\frac{d}{dx}\left(-\frac{\cot ax}{a}\right)=-\frac{1}{a}\frac{d}{dx}\left(\cot ax\right)$

$=-\frac{1}{a}\left(-a\,\text{cosec}^2ax\right)=\text{cosec}^2ax.$

$\therefore$ $\int\text{cosec}^2ax\,dx=-\frac{\cot ax}{a}+c.$

(v) $\frac{d}{dx}\left(-\frac{\text{cosec } ax}{a}\right)=-\frac{1}{a}\frac{d}{dx}\left(\text{cosec } ax\right)$

$=-\frac{1}{a}\left(-a \text{ cosec } ax \cot ax\right)=\text{cosec } ax \cot ax.$

$\therefore \quad \int \text{cosec } ax \cot ax \, dx=-\frac{\text{cosec } ax}{a}+c.$

(vi) $\frac{d}{dx}\left(\frac{\sec ax}{a}\right)=\frac{1}{a}\frac{d}{dx}\left(\sec ax\right)$

$=\frac{1}{a}\left(a \sec ax \tan ax\right)=\sec ax \tan ax.$

$\therefore \quad \int \sec ax \tan ax \, dx=\frac{\sec ax}{a}+c.$

Cor. Putting $a=1$ in each of the above formulas,

$\int \cos x \, dx=\sin x+c$; $\int \sin x \, dx=-\cos x+c.$

$\int \sec^2 x \, dx=\tan x+c$; $\int \text{cosec}^2 x \, dx=-\cot x+c.$

$\int \sec x \tan x \, dx=\sec x+c$; $\int \text{cosec } x \cot x \, dx=-\text{cosec } x+c.$

Examples.

1. $\int \cos 5x \, dx=\frac{\sin 5x}{5}+c.$

2. $\int \sin(-3x)dx=-\frac{\cos(-3x)}{-3}+c=\frac{\cos 5x}{3}+c.$

Examples 1A

Ex. 1. Integrate :

1. (i) $\int \frac{dx}{\sqrt[3]{x}}$ (ii) $\int x^{-6}dx.$

(i) $\int \frac{dx}{\sqrt[3]{x}}=\int x^{-\frac{1}{3}}dx=\frac{x^{-\frac{1}{3}+1}}{-\frac{1}{3}+1}+C=\frac{x^{\frac{2}{3}}}{\frac{2}{3}}+C=\frac{3}{2}x^{\frac{2}{3}}+C.$

(ii) $\int x^{-6} \, dx=\frac{x^{-6+1}}{-6+1}+C=\frac{x^{-5}}{-5}+C=-\frac{1}{5x^5}+C.$

Ex. 2. Integrate :

(i) $\int \frac{dx}{e^{6x}}$ (ii) $\int \sqrt[3]{e^x} \, dx.$ (iii) $\int \frac{dx}{\sqrt{e^x}}.$

(i) $\int \frac{dx}{e^{6x}}=\int e^{-6x} \, dx=\frac{e^{-6x}}{-6}+C=-\frac{1}{6e^{6x}}+C.$

(ii) $\int \sqrt[3]{e^x} \, dx=\int e^{\frac{x}{3}} \, dx=\frac{e^{\frac{x}{3}}}{\frac{1}{3}}+C=3e^{\frac{x}{3}}+C=3\sqrt[3]{e^x}+C$

(iii) $\int \frac{dx}{\sqrt{e^x}} = \int e^{-\frac{x}{2}}\, dx = \frac{e^{-\frac{x}{2}}}{-\frac{1}{2}} + C = -2\frac{1}{\sqrt{e^x}} + C.$

Ex. 3. Integrate :

(i) $\int 7^x\, dx$ (ii) $\int e^x\, dx.$

(i) $\int 7^x\, dx = \frac{7^x}{\log^7{}_e} + C$

(ii) $\int e^x\, dx = \frac{e^x}{\log^e{}_e} + C = e^x + C\ [\because\ \log^e{}_e = 1\ ].$

Ex. 4. Integrate :

(i) $\int \sec^2 4x\, dx$ (ii) $\int \frac{\sin\theta}{\cos^2\theta}\, d\theta$

(i) $\int \sec^2 4x\, dx = \frac{\tan 4x}{4} + C$

(ii) $\int \frac{\sin\theta}{\cos^2\theta} d\theta = \int \sec\theta \tan\theta\, d\theta = \sec\theta + C.$

Exercise 1A

1. Integrate :

(i) $\int x^{100}\, dx$ (ii) $\int x^7\, dx$ (iii) $\int \frac{dx}{x^2}$ (iv) $\int x^{-3}\, dx$

(v) $\int \frac{dx}{\sqrt[4]{x^5}}$ (vi) $\int x^2\sqrt{x}\, dx$ (vii) $\int \frac{dx}{\sqrt{x}}$ (viii) $\int \frac{dx}{x^{\frac{n-1}{n}}}$

2. Integrate :

(i) $\int e^{2x}\, dx$ (ii) $\int e^{17x}\, dx$ (iii) $\int e^{cx}\, dx$ (iv) $\int e^{\frac{4}{5}x} dx.$

(v) $\int \sqrt{e^x}\, dx$ (vi) $\int e^{-10x}\, dx$ (vii) $\int \frac{dx}{\sqrt[5]{e^x}}$ (viii) $\int e^{\log x}\, dx.$

3. Integrate :

(i) $\int 3\, dx$ (ii) $\int (\frac{1}{2})^x\, dx$ (iii) $\int a^x\, dx$ (iv) $\int 6^{2x}\, dx.$

(v) $\int 10\, dx$ (vi) $6^{10x}\, dx.$

4. Integrate :—

(i) $\int \sin 7x\, dx$. (ii) $\int \sin(-2x)\, dx$. (iii) $\int \cos 6x\, dx$.

(iv) $\int \cos(-4x)\, dx$. (v) $\int \operatorname{cosec}^2 3x\, dx$.

(vi) $\int -\operatorname{cosec} 2x \cot 2x\, dx$.

§ 1·5. General rules of Integration :

(1) $\int a.f(x)dx = a.\int f(x)dx$. ($a$ is a constant)

Proof. Let $\int f(x)dx = g(x) + c$,

$\therefore \quad \frac{d}{dx}\Big(g(x)+c\Big) = f(x)$, or, $\frac{d}{dx}\Big(g(x)\Big) + \frac{d}{dx}(c) = f(x)$,

or, $\frac{d}{dx}\Big(g(x)\Big) = f(x) \quad \left[\because \frac{d}{dx}(c) = 0\right]$

Now, $\frac{d}{dx}\Big(a\{g(x)\} + ac\Big) = \frac{d}{dx}\Big(a\{g(x)\Big) + \frac{d}{dx}(ac)$

$= a\frac{d}{dx}\Big(g(x)\Big) + 0 = a.f(x)$

$\therefore \quad \int a(x)dx = a.g(x) + a.c = a.\{g(x) + c\} = a\int(x)dx$.

Cor. Let, $f(x) = 1$, $\therefore \quad \int adx = a\int dx = ax + c$.

(2) $\int\{f(x) \pm g(x)\}dx = \int f(x)dx \pm \int g(x)dx$.

Proof. Let, $\int f(x)dx = h_1(x)$ and $\int g(x)dx = h_2(x)$

$\therefore \quad \frac{d}{dx}\Big(h_1(x)\Big) = f(x)$ and $\frac{d}{dx}\Big(h_2(x)\Big) = g(x)$.

$\therefore \quad \frac{d}{dx}\Big(h_1(x) \pm h_2(x)\Big) = \frac{d}{dx}h_1(x) \pm \frac{d}{dx}h_2(x) = (f)x \pm g(x)$

$\therefore \quad \int\{f(x) \pm g(x)\}dx = h_1(x) \pm h_2(x) = \int f(x)dx \pm \int g(x)dx$.

Cor. 1. By repeated application of the above rule, it can be proved that if each of $\int f_1(x)dx$, $\int f_2(x)dx, \cdots, \int f_n(x)dx$ can be determined.

$$\int\{\pm f_1(x) \pm f_2(x) \pm f_3(x) \pm \cdots \pm f_n(x)\}dx.$$

$$= \pm\int f_1(x)dx \pm \int f_2(x)dx \pm \int f_3(x)dx \pm \cdots \pm \int f_n(x)dx.$$

[n is a finite positive integer]

Cor. 2. From Cor. 1 and rule 1 we get that if each of $\int f_1(x)dx, \int f_2(x)dx, \cdots, \int f_n(x)dx$, can be determined and if a_1, $a_2, \cdots a_n$ be constants (n is a finite positive integer).

$$\int\{\pm a_1 f_1(x) \pm a_2 f_2(x) \pm \cdots \pm a_n f_n(x)\}dx$$
$$= \pm a_1 \int f_1(x)dx \pm a_2 \int f_2(x)dx \pm \cdots \pm a_n \int f_n(x)dx.$$

Examples :

1. $\int(x^2+e^x)dx = \int x^2 dx + \int e^x dx = \dfrac{x^4}{4} + e^x + c.$

2. $\int(x^3+\cos x)dx = \int x^3 dx + \int \cos x\, dx = \dfrac{x^4}{4} + \sin x + c.$

3. $\int 2x\, dx = 2\int x dx = 2.\ \dfrac{x^2}{2} + c = x^2 + c.$

4. $\int \sin(-2x)dx = \int -\sin 2x dx = -\int \sin 2x dx$
$$= -\left(-\frac{\cos 2x}{2}\right) + c = \frac{\cos 2x}{2} + c.$$

§ 1·6. Determination of integrals of powers and products of sine and co-sine functions of a variable by reducing it to functions of multiple angles.

From the formula (i) $1+\cos 2x = 2\cos^2 x$ and (ii) $1-\cos 2x = 2\sin^2 x$, $\cos^2 x$ and $\sin^2 x$ can be expressed in the forms $\frac{1}{2}(1+\cos 2x)$ and $\frac{1}{2}(1-\cos 2x)$ respectively.

Hence $\int \cos^2 x\, dx = \int \frac{1}{2}(1+\cos 2x)dx = \frac{1}{2}\int dx + \frac{1}{2}\int \cos 2x dx$
$$= \tfrac{1}{2}x + \frac{\sin 2x}{4} + c.$$

Similarly, $\int \sin^2 x\, dx = \frac{1}{2}x - \dfrac{\sin 2x}{4} + c.$

(ii) $\sin 3x = 3\sin x - 4\sin^3 x$

$\therefore$ $\sin^3 x = \frac{1}{4}(3\sin x - \sin 3x)$

or, $\int \sin^3 x dx = \int \frac{1}{4}(3\sin x - \sin 3x)dx = -\dfrac{3}{4}\cos x + \dfrac{\cos 3x}{12} + c.$

(iii) Again, $\cos 3x = 4\cos^3 x - 3\cos x$ gives

$\cos^3 x = \frac{1}{4}(\cos 3x + 3\cos x).$

or, $\int \cos^3 x dx = \frac{1}{4}\int(\cos 3x + 3\cos x)dx = \dfrac{\sin 3x}{12} + \dfrac{3}{4}\sin x + c.$

(iv) Products of sines and co-sines can be integrated by expressing them as sums of sines and cosines by the formulac $2\sin A \cos B = \sin(A+B) + \sin(A-B)$ etc. Follow the illustrations carefully.

Examples :

1. $\int \cos 2x \cos 4x dx = \int \frac{1}{2}.2 \cos 4x \cos 2x dx$

$= \frac{1}{2}\int(\cos 6x + \cos 2x)dx = \frac{1}{2}\int \cos 6x dx + \frac{1}{2}\int \cos 2x dx$

$= \frac{1}{2}.\frac{\sin 6x}{6} + \frac{1}{2}.\frac{\sin 2x}{2} + c = \frac{\sin 6x}{12} + \frac{\sin 2x}{4} + c.$

2. $\int 4 \sin 2x.\cos 3x dx = \int 2.2 \cos 3x \sin 2x dx$

$= 2\int(\sin 5x - \sin x)dx = 2\int \sin 5x dx - 2\int \sin x\, dx$

$= 2\left(-\frac{\cos 5x}{5}\right) - 2(-\cos x) + c = 2\left(\cos x - \frac{\cos 5x}{5}\right) + c.$

Examples 1B

Ex. 1. Integrate :

(i) $\int(x^2 - \sin x)dx$ (ii) $\int(x+2)^2 dx$ (iii) $\int(2+x)^3 dx.$

(iv) $\int(x+2)(x+3)dx$ (v) $\int \frac{1}{\sqrt{x}}(x^4 + 7\sqrt{x})dx$

(i) $\int(x^2 - \sin x)dx = \int x^2 dx - \int \sin x\, dx$

$= \frac{x^3}{3} - (-\cos x) + c = \frac{x^3}{3} + \cos x + c.$

(ii) $\int(x+2)^2 dx = \int(x^2 + 4x + 4)dx = \int x^2 dx + \int 4x\, dx$

$+ \int 4dx = \int x^2 dx + 4\int x dx + 4\int dx = \frac{x^3}{3} + 4.\frac{x^2}{3} + 4x + c$

$= \frac{x^3}{3} + 2x^2 + 4x + c.$

(iii) $\int(2+x)^3 dx = \int(8 + 12x + 6x^2 + x^3)dx$

$= \int 8dx + \int 12x\, dx + \int 6x^2 dx + \int x^3 dx$

$= 8 \int dx + 12\int x dx + 6\int x^2 dx + \int x^3 dx$

$= 8x + 12.\frac{x^2}{2} + 6.\frac{x^3}{3} + \frac{x^4}{4} + c$

$= 8x + 6x^2 + 2x^3 + \frac{x^4}{4} + c.$

(iv) $\int(x+2)(x+3)dx = \int(x^2 + 5x + 6)dx$

$= \int x^2 dx + \int 5x\, dx + \int 6dx = \int x^2 dx + 5\int x\, dx + 6\int dx$

$= \frac{x^3}{3} + 5.\frac{x^2}{2} + 6.x + c = \frac{1}{3}x^3 + \frac{5}{2}x^2 + 6x + c$

(v) $\int \frac{1}{\sqrt{x}}(x^4 + 7\sqrt{x})dx = \int(x^{\frac{7}{2}} + 7)dx = \int x^{\frac{7}{2}} dx + \int 7dx$

$= \frac{x^{\frac{9}{2}}}{\frac{9}{2}} + 7x + c = \frac{2}{9}x^{\frac{9}{2}} + 7x + c.$

Ex. 2. Integrate : $\int \tan^2 x dx$

$$\int \tan^2 x dx = \int (\sec^2 x - 1) dx = \int \sec^2 x dx - \int dx = \tan x - x + c.$$

Ex. 3. $\int \left(e^{\frac{x}{2}} + e^{-\frac{x}{2}} \right)^2 dx = \int \left(e^x + 2 + e^{-x} \right) dx$

$$= \int e^x dx + 2\int dx + \int e^{-x} dx = e^x + 2x - e^{-x} + c.$$

Ex. 4. $\int \frac{e^{4x} + e^{6x}}{e^x + e^{-x}} dx = \int \frac{e^{5x}(e^{-x} + e^x)}{e^x + e^{-x}} dx = \int e^{5x} dx = \frac{e^{5x}}{5} + c.$

Ex. 5. $\int e^{n \log x} dx = \int e^{\log x^n} dx = \int x^n dx = \frac{x^{n+1}}{n+1} + c. (n \neq -1).$

$$= \log x + c \text{ (when } n = -1)$$

Ex. 6. $\int \sin x^0 dx = \int \sin \frac{\pi x}{180} dx = -\frac{\cos \frac{\pi x}{180}}{\frac{\pi}{180}} + A$

$$= -\frac{180}{\pi} \cos \frac{\pi x}{180} + A = -\frac{180}{\pi} \cos x^0 + A$$

Ex. 7. $\int \frac{\cot x}{\tan x} dx = \int \cot^2 x dx = \int (\operatorname{cosec}^2 x - 1) dx$

$$= \int \operatorname{cosec}^2 x dx - \int dx = -\cot x - x + c.$$

Ex. 8. $\int \frac{2 \sin^3 x + 3 \cos^3 x}{\sin^2 x \cos^2 x} dx = \int \left(\frac{2 \sin^3 x}{\sin^2 x \cos^2 x} + \frac{3 \cos^3 x}{\sin^2 x \cos^2 x} \right) dx$

$$= \int (2 \sec x \tan x + 3 \cot x \operatorname{cosec} x) dx$$
$$= 2\int \sec x \tan x dx + 3\int \cot x \operatorname{cosec} x dx$$
$$= 2 \sec x - 3 \operatorname{cosec} x + c.$$

Ex. 9. $\int \sec^2 x \operatorname{cosec}^2 x dx = \int \frac{dx}{\sin^2 x \cos^2 x}$

$$= \int \frac{\sin^2 x + \cos^2 x}{\sin^2 x \cos^2 x} dx = \int (\sec^2 x + \operatorname{cosec}^2 x) dx$$

$$= \int \sec^2 x dx + \int \operatorname{cosec}^2 x dx = \tan x - \cot x + c.$$

Ex. 10. $\int \frac{\cos x + \sin x}{\cos x - \sin x} \left(1 - \sin 2x \right) dx$

$$\int \frac{\cos x + \sin x}{\cos x - \sin x} \left(\sin^2 x + \cos^2 x - 2 \sin x \cos x \right) dx$$

$$= \int \frac{\cos x + \sin x}{\cos x - \sin x} \left(\cos x - \sin x \right)^2 dx$$

$$= \int (\cos x + \sin x)(\cos x - \sin x) dx$$

$$= \int \left(\cos^2 x - \sin^2 x \right) dx = \int \cos 2x \, dx = \frac{\sin 2x}{2} + c.$$

Ex. 11. $\int \frac{dx}{1-\sin x}=\int \frac{1+\sin x}{(1-\sin x)(1+\sin x)}\,dx=\int \frac{1+\sin x}{\cos^2 x}dx$

$=\int \left(\frac{1}{\cos^2 x}+\frac{\sin x}{\cos^2 x}\right)dx=\int \left(\sec^2 x+\sec x\tan x\right)dx$

$=\int \sec^2 x\,dx+\int \sec x\tan x\,dx=\tan x+\sec x+c.$

Ex. 12. $\int \frac{9^{1+x}+3^{1+x}}{3^x}\,dx=\int \frac{9.9^x+3.3^x}{3^x}dx$

$=\int \frac{9.3^{2x}+3.3^x}{3^x}dx=\int \frac{3^x(9.3^x+3)dx}{3^x}=\int \left(9.3^x+3\right)dx$

$=9\int 3^x dx+3\int dx=9.\frac{3^x}{\log_e 3}+3x+c.$

Ex. 13. $\int e^{x\log 2}dx=\int e^{\log 2^x}dx=\int 2^x dx=\frac{2^x}{\log_e 2}+c.$

Ex. 14. $\int \frac{\cos^4 x}{\sin^2 x}dx=\int \frac{(\cos^2 x)^2}{\sin^2 x}\,dx=\int \frac{(1-\sin^2 x)^2}{\sin^2 x}dx$

$=\int \frac{1-2\sin^2 x+\sin^4 x}{\sin^2 x}dx=\int (\operatorname{cosec}^2 x-2+\sin^2 x)dx$

$=\int \operatorname{cosec}^2 x\,dx-2\int dx+\frac{1}{2}\int (1-\cos 2x)dx$

$=-\cot x-2x+\frac{1}{2}x-\frac{\sin 2x}{4}+c.$

$=-\cot x-\frac{3}{2}x-\frac{\sin 2x}{4}+c.$

Ex. 15. $\int \cos x.\cos 2x\cos 3x\,dx$

$=\frac{1}{2}\int (2\cos x\cos 2x)\cos 3x\,dx$

$=\frac{1}{2}\int (\cos 3x+\cos x)\cos 3x\,dx$

$=\frac{1}{2}\int (\cos^2 3x+\cos 3x\cos x)dx$

$=\frac{1}{4}\int 2\cos^2 3x+\frac{1}{4}\int 2\cos 3x\cos x\,dx$

$=\frac{1}{4}\int (1+\cos 6x)\,dx+\frac{1}{4}\int (\cos 4x+\cos 2x)dx$

$=\frac{1}{4}x+\frac{\sin 6x}{24}+\frac{\sin 4x}{16}+\frac{\sin 2x}{8}+c.$

Exercise IB

Integrate :

1. (a) (i) $\int (x^2+3^x)\,dx$ (ii) $\int (2+3x)^3 dx.$ (iii) $\int \frac{x^4+x^2+1}{x^2+x+1}\,dx$

(iv) $\int (1+e^{2x})\,dx$ (v) $\int (x^7+e^x+a^x)\,dx$ (vi) $\int \left(\frac{x-1}{\sqrt{x}}-\frac{e^x}{e^{-x}}\right)dx$

(vii) $\int \cot^2 x dx.$ (viii) $\int (2\cos x+\tan^2 x)dx.$

(b) (i) $\int \sin x\sin 2x dx$ (ii) $\int \sin 10x\cos 6x\,dx$

(iii) $\int 2\cos 6x \cos 4x dx$ (iv) $\int \sin^2 x\, dx$. (v) $\int \cos^2 2x dx$.

(vi) $\int \cos^2 \frac{x}{2}\, dx$ (vii) $\int \cos^3 x dx$. (viii) $\int \sin^3 3x dx$

(ix) $\int \cos x \cos 2x \cos 3x\, dx$ (x) $\int \sin x \sin 2x \sin 4x dx$.

2\. (i) $\int (2x-1)(x-2)dx$ (ii) $\sqrt[3]{x}\left(x^6-\frac{2}{x^2}\right)dx$.

(iii) $\int \left(x+\frac{2}{x}\right)\left(2x+\frac{1}{x}\right)dx$ (iv) $\int \frac{(1+x)^3}{x^5}dx$. [C.U. Int. '60]

3\. (i) $\int \frac{x^2-7x+12}{x-4}dx$ (ii) $\int \frac{x^3+3x^2-4x-12}{x+2}\, dx$.

4\. (i) $\int \frac{(e^x+1)^2}{e^{2x}}dx$ (ii) $\int \frac{e^{3x}-4e^x+1}{e^{2x}}dx$.

5\. $\int \frac{e^{6x}-1}{e^{2x}-1}$. 6. $\int \cos x^0 dx$. 7. $\int \sin^2 ax\, dx$.

8\. $\int \cos^2 6x\, dx$. 9. $\int \sin^2 \frac{x}{3} dx$. 10. $\int \cot^2 2x\, dx$.

11\. $\int \frac{1-\tan^2 x}{1+\tan^2 x}dx$. 12. $\int \frac{\cos\theta}{\sin^2\theta}d\theta$.

13\. $\int 2\sin 3x \sin 4x\, dx$. 14. $\int \cos 4x \cos 5x\, dx$.

15\. $\int \sin mx \cos nx\, dx$. 16. $\int \sin^3 \frac{x}{2}\, dx$.

17\. $\int \frac{\tan x}{\cot x}dx$. 18. $\int \frac{\sin^3 x-\cos^3 x}{\sin^2 x \cos^2 x}dx$.

19\. (i) $\int (\tan^2 x+2)dx$. (ii) $\int (\tan^2 x+2)(\cot^2 x+3)dx$.

20\. $\int \frac{1+\sin 2x}{\sin x+\cos x}dx$.

21\. (i) $\int \frac{(a^x+1)^2}{a^x}dx$ (ii) $\int \frac{a^{3x}+a^x}{a^{2x}}dx$.

22\. (i) $\int \frac{dx}{1+\cos 2x}$ (ii) $\int \frac{dx}{1-\cos x}$ (iii) $\int \frac{dx}{1+\sin x}$

23\. $\int \frac{\sin x-\cos 2x}{1-\sin x}dx$

24\. (i) $\int \sqrt{1+\sin 2x}\, dx$ (ii) $\int \sqrt{1+\cos 2x}\, dx$.

25\. $\int \frac{\cos 2x-\cos 2\theta}{\cos x-\cos\theta}dx$; where θ is a constant.

26\. $\int (\sin^6 x+\cos^6 x)dx$. 27. (i) $\int \cos^4 x dx$ (ii) $\int \sin^4 x\, dx$.

28\. $\int \sin ax \sin bx \cos cx\, dx$.

29\. $\int \sin 2x(1+\cos 2x)dx$. [C. U. '60 Int.]

30\. $\int (\cos 3x-2\sin x+x^2)dx$. [C. U. '61 Int.]

CHAPTER TWO

INTEGRATION BY SUBSTITUTION OF VARIABLES

§ 2·1. You have seen in Differential Calculus that there are certain general rules for differentiation of functions (of course, which are differentiable and most of the elementary functions are differentiable). But in case of integration, there is no definite rule for integration of integrable functions. The processes of integration is mainly tentative. So integration is more difficult than differentiation. In chapter one you have seen that many functions which are not of the standard forms have been integrated by expressing the integrand as the sum or difference of more than one integrands of standard forms with the help of Trigonometric and Algebraic formulas. But all integrands cannot be reduced to standard forms with the help of those formulas. In those cases different other methods are followed. In this book we shall discuss about two methods, viz., (i) Integration by substitution and (ii) Integration by parts. The subject matter of the present chapter is integration by substitution of variables.

§ 2·2. Let $\int f(x)dx = g(x)$

$$\therefore \quad \frac{d}{dx}\{g(x)\} = f(x).$$

Now if $x = F(z)$, then $\dfrac{dx}{dz} = F'(z)$

$$\therefore \quad \frac{d}{dz}\{g(x)\} = \frac{d}{dx}\{g(x)\}.\ \frac{dx}{dz} = f(x)F'(z) = f\{F(z)\}F'(z)$$

$\therefore$ By definition, $g(x) = \int f\{F(z)\}F'(z)dz$

or, $\int f(x)dx = \int f\{F(z)\}F'(z)dz.$

Now the variable of integration is z instead of x. Hence the integral (if it can be determined) will be expressed in terms of z. Express the final result in terms of x from the relation of x and z.

Note : You know that $\dfrac{dx}{dz}$ is the ratio of the two differentials dx and dz. $\therefore$ $\dfrac{dx}{dz} = F'(z)$ or $dx = F'(z)dz$ and use of this form is more convenient.

Ex. 1. Integrate : $\int(2x+3)^5 dx.$

Let $I=\int(2x+3)^5 dx.$

$$\therefore \quad \frac{dI}{dx}=(2x+3)^5.$$

Let, $2x+3=z$ or, $x=\frac{z}{2}-\frac{3}{2}$, $\therefore \quad \frac{dx}{dz}=\frac{1}{2}$

Now, $\frac{dI}{dz}=\frac{dI}{dx}\cdot\frac{dx}{dz}=(2x+3)^5\cdot\frac{1}{2}=\frac{1}{2}z^5$

$$\therefore \quad I=\int\tfrac{1}{2}z^5 dz=\tfrac{1}{2}\int z^5 dz=\tfrac{1}{2}\frac{z^6}{6}+c=\tfrac{1}{12}(2x+3)^6+c.$$

Ex. 2. Integrate : $\int\frac{dt}{t\sqrt{t^2-1}}$.

Let $t=\sec\theta \quad \therefore \quad dt=\frac{dt}{d\theta}d\theta=\sec\theta\tan\theta\, d\theta$

and $t\sqrt{t^2-1}=\sec\theta\sqrt{\sec^2\theta-1}=\sec\theta\tan\theta$

$$\therefore \quad \int\frac{dt}{t\sqrt{t^2-1}}=\int\frac{\sec\theta\tan\theta\, d\theta}{\sec\theta\tan\theta}=\int d\theta=\theta+c$$

Now, $\because \sec\theta=t, \quad \therefore \quad \theta=\sec^{-1}t$

$$\therefore \quad \int\frac{dt}{t\sqrt{t^2-1}}=\sec^{-1}t+c.$$

Sometimes variables are substituted in the form $\phi(x)=z$.

$$\therefore \quad \phi(x)=z, \quad \frac{dz}{dx}=\phi'(x), \quad \therefore \quad dx=\frac{dz}{\phi'(x)}.$$

Now from $\phi(x)=z$, express $f(x)$ in terms of z and also $dx=\frac{dz}{\phi'(x)}$.

$\therefore \quad \int f(x)dx$ will be of the form $\int g(z)dz$.

Ex. 3. $\int\sin^2\theta\cos\theta\, d\theta$.

Let $\sin\theta=z, \quad \therefore \quad \cos\theta\, d\theta=dz.$

and $\int\sin^2\theta\cos\theta\, d\theta=\int z^2 dz=\frac{z^3}{3}+c=\frac{\sin^3\theta}{3}+c.$

§ 2·3. Rules of substitution : There is no general rule for substitution of variables for integration of functions. Variables are generally substituted by inspection. In the next few articles we discuss some convenient rules of substitution of variables.

§ 2·4. **Integration of integrals of the form $\int f(ax+b)dx$.**

Ex. 1. Integrate : $\int \sin(ax+b)dx$.

Let $ax+b=z$. $\therefore$ $adx=dz$ or, $dx=\dfrac{dz}{a}$.

$$\therefore \quad \int \sin(ax+b)dx=\int \frac{1}{a}\sin z dz=\int \frac{1}{a}\sin z\, dz=-\frac{\cos z}{a}+c.$$

$$=-\frac{\cos(ax+b)}{a}+c.$$

Ex. 2. Integrate : $\displaystyle\int \frac{dx}{3x+4}$.

Let $3x+4=t$. $\therefore$ $3dx=dt$ or, $dx=\dfrac{dt}{3}$.

$$\therefore \quad \int \frac{dx}{3x+4}=\int \frac{dt}{3t}=\frac{1}{3}\int \frac{dt}{t}=\frac{1}{3}\log t+c$$

$$=\tfrac{1}{3}\log(3x+4)+c.$$

§ 2·5. **The form : $\int \{f(x)\}^n f'(x)dx$**

To integrate $\int \{f(x)\} f'(x)dx$ let, $f(x)=z$

$\therefore$ $f'(x)=\dfrac{dz}{dx}$ or, $f'(x)dx=dz$.

Hence given integral$=\displaystyle\int z^n dz=\frac{z^{n+1}}{n+1}+c$ [If $n\neq -1$]

and$=\log z+c$ [If $n=-1$]

Hence given integral$=\dfrac{\{f(x)\}^{n+1}}{n+1}+c$ [If $n\neq -1$]

and$=\log\{f(x)\}+c$ [If $n=-1$]

Example 1. For, integration of $\int (ax^2+bx+c)^3(2ax+b)dx$ notice that if $f(x)=(ax^2+bx+c)$, then $f'(x)=2ax+b$. Hence the given integral is of the form

$$\int \{f(x)\}^3 f'(x)dx=\frac{\{f(x)\}^4}{4}+c=\frac{(ax^2+bx+c)^4}{4}+c.$$

§ 2·6. **The form : $\int \phi\{f(x)\}f'(x)dx$**

If $\int \phi(x)dx=g(x)$, then, $\int \phi\{f(x)\}.\, f'(x)dx=g\{f(x)\}$

Proof. $\int \phi\{f(x)\}.\, f'(x)dx=\int \phi(z)dz$, [putting $z=f(x)$, $dz=f'(x)dx$.]

$=g(z)$, [$\because$ $\int \phi(x)dx=g(x)$]

$=g\{f(x)\}$.

Hence if the integrand be the product of a function of an integrable function of the form $\phi\{f(x)\}$ and the derivative $f'(x)$ of the second function, then substitute z for the second function $f(x)$.

Note. The form $\int \{f(x)\}^n f'(x)\,dx$ discussed in the last article is a special form of $\int \phi\{f(x)\} f'(x)dx$.

Example 1. Integrate : $\int \frac{e^{\tan^{-1} x}}{1+x^2} dx$ [C. U. 1939]

Let, $\tan^{-1} x = z \quad \therefore \quad \frac{dx}{1+x^2} = dz.$

$\therefore$ Given integral$=\int e^z \, dz = e^z + c = e^{\tan^{-1} x} + c.$

Examples 2A

Ex. 1. Integrate : $\int (4-3x)^{100} dx$

Let $4-3x=z. \quad \therefore \quad -3dx=dz \quad \therefore \quad dx=-\frac{dz}{3}.$

$$\therefore \int (4-3x)^{100} dx = -\int z^{100} \frac{dz}{3}$$

$$= -\tfrac{1}{3}\frac{z^{101}}{101} + c = -\frac{(4-3x)^{101}}{303} + c.$$

Ex. 2. Integrate : $\int \frac{dx}{x^2-5x+6} = \int \frac{dx}{(x-2)(x-3)}$

$$= \int \left(\frac{1}{x-3} - \frac{1}{x-2}\right) dx = \int \frac{dx}{x-3} - \int \frac{dx}{x-2}.$$

Now, let $x-3=u$ and $x-2=v$

$\therefore \quad dx=du$ and $dx=dv$ (respectively)

$$\therefore \text{ Given integral} = \int \frac{du}{u} - \int \frac{dv}{v} = \log u - \log v + c$$

$$= \log \frac{u}{v} + c = \log \frac{x-3}{x-2} + c.$$

Ex. 3. Integrate :

(a) $\int \frac{dx}{ax+b}$ (b) $\int \frac{dx}{b-ax}$ (c) $\int \frac{a'x+b'}{ax+b} dx.$

(a) Let $ax+b=z \quad \therefore \quad adx=dz$ or $dx=\frac{dz}{a}$

$\therefore \quad \int \frac{dx}{ax+b} = \int \frac{1}{z} \frac{dz}{a} = \frac{1}{a}\int \frac{dz}{z} = \frac{1}{a}\log z + c.$

$= \frac{1}{a}\log(ax+b) + c.$

(b) Let $b - ax = z \quad \therefore \quad -adx = dz$ or, $dx = -\frac{dz}{a}$.

So, $\int \frac{dx}{b-ax} = \int \frac{1}{z}\left(-\frac{dz}{a}\right) = -\frac{1}{a}\int \frac{dz}{z} = -\frac{1}{a}\log z + c$

$= -\frac{1}{a}\log(b-ax) + c.$

(c) Let $ax + b = z \quad \therefore \quad adx = dz$ and $x = \frac{z-b}{a}$

$$\therefore \quad \int \frac{a'x+b'}{ax+b}dx = \int \frac{a'\frac{z-b}{a}+b'}{z}\cdot\frac{dz}{a}$$

$$= \frac{1}{a^2}\int \frac{a'z+ab'-a'b}{z}dz = \frac{a'}{a^2}\int dz + \frac{ab'-a'b}{a^2}\int \frac{dz}{z}$$

$$= \frac{a'}{a^2}z + \frac{ab'-a'b}{a^2}\log z + c'$$

$$= \frac{a'}{a^2}(ax+b) + \frac{ab'-a'b}{a^2}\log(ax+b) + c'$$

$$= \frac{a'}{a}x + \frac{ab'-a'b}{a^2}\log(ax+b) + c$$

$$\left[\frac{a'b}{a^2} + c' = c \text{ (say)}\right]$$

Alt. method : $\int \frac{a'x+b'}{ax+b}dx = a'\int \frac{x+\frac{b'}{a'}}{ax+b}dx$

$$= \frac{a'}{a}\int \frac{ax+\frac{ab'}{a'}}{ax+b}dx = \frac{a'}{a}\int \frac{ax+b+\frac{ab'}{a'}-b'}{ax+b}dx$$

$$= \frac{a'}{a}\int \frac{ax+b}{ax+b}dx + \frac{a'}{a}\left(\frac{ab'-a'b}{a'}\right)\int \frac{dx}{ax+b}$$

$$= \frac{a'}{a}\int dx + \frac{ab'-a'b}{a}\int \frac{dx}{ax+b}$$

$$= \frac{a'}{a}x + \frac{ab'-a'b}{a^2}\log(ax+b) + c$$ [See (i) above]

Ex. 4. Integrate : $\int \frac{\cos x dx}{\cos (x+a)}$.

Let $x+a=z$ $\therefore$ $dx=dz$ and $x=z-a$

$$\therefore \int \frac{\cos x dx}{\cos (x+a)}=\int \frac{\cos (z-a)dz}{\cos z}=\int \frac{\cos z \cos a+\sin z \sin a}{\cos z}dz$$

$$=\cos a \int dz+\sin a \int \frac{\sin z}{\cos z}dz$$

$$=\cos a \int dz-\sin a \int \frac{d(\cos z)}{\cos z} \text{ [as } d(\cos z)=-\sin z dz \text{]}$$

$$=z \cos a-\sin a \log (\cos z)+c'$$
$$=(x+a) \cos a+\sin a \log \sec (x+a)+c'$$
$$=x \cos a+\sin a \log \sec (x+a)+c$$

[as $a \cos a$ is constant, $a \cos a+c'$=constant=c (say)]

Ex. 5. Integrate : $\int \frac{e^x dx}{e^x+1}$

Let $e^x+1=z$ $\therefore$ $e^x dx=dz$

$$\therefore \int \frac{e^x dx}{e^x+1}=\int \frac{dz}{z}=\log z+c=\log (e^x+1)+c.$$

Ex. 6. Integrate : $\int x^2\sqrt{x^3+1}\, dx$.

Let $x^3+1=z^2$ $\therefore$ $3x^2 dx=2z dz$

or, $x^2 dx=\frac{2}{3}z dz$.

$$\therefore \int x^2 \sqrt{x^3+1}\, dx=\int \tfrac{2}{3} z dz.z=\tfrac{2}{3}\int z^2 dz$$

$$=\tfrac{2}{3} \frac{z^3}{3}+c=\tfrac{2}{9}(x^3+1)^{\frac{3}{2}}+c.$$

Ex. 7. Integrate : $\int \tan x \sec^2 x\, dx$

Let $\tan x=z$. $\therefore$ $\sec^2 x\, dx=dz$

$$\therefore \int \tan x \sec^2 x\, dx=\int z\, dz=\frac{z^2}{2}+c=\frac{\tan^2 x}{2}+c.$$

Ex. 8. Integrate : $\int \frac{\tan^{-1}x\, dx}{1+x^2}$

Let $\tan^{-1}x=z$ $\therefore$ $\frac{dx}{1+x^2}=dz$

$$\therefore \int \frac{\tan^{-1}x}{1+x^2}=\int z\, dz=\frac{z^2}{2}=\frac{(\tan^{-1}x)^2}{2}+c$$

Ex. 9. Integrate :

(i) $\int \frac{1+\cos x}{\sqrt{x+\sin x}} dx$ (ii) $\int \frac{1+\cos x}{x+\sin x} dx.$

Let $x+\sin x=z$ $\therefore$ $(1+\cos x)\,dx=dz.$

Now (i) $\int \frac{1+\cos x}{\sqrt{x+\sin x}} dx=\int \frac{dz}{\sqrt{z}}=2\sqrt{z}+c=2\sqrt{x+\sin x}+c.$

(ii) $\int \frac{1+\cos x}{x+\sin x} dx=\int \frac{dz}{z}=\log z+c=\log (x+\sin x)+c.$

Ex. 10. Integrate : (i) $\int \tan x\, dx$ (ii) $\int \cot x dx$ (iii) $\int \sec x dx$ and (iv) $\int \operatorname{cosec} x dx$.

(i) $\int \tan x dx=\int \frac{\sin x}{\cos x} dx$

Now let $\cos x=z$ $\therefore$ $-\sin x dx=dz$ or $\sin x dx=-dz.$

$\therefore$ Given integral$=-\int \frac{dz}{z}=-\log (z)+c=-\log (\cos x)+c$

$=\log (\cos x)^{-1}+c=\log (\sec x)+c.$

(ii) $\int \cot x\, dx=\int \frac{\cos x}{\sin x} dx=\int \frac{dz}{z}$

[where, $z=\sin x$ $\therefore$ $dz=\cos x\, dx$]
$=\log z+c=\log (\sin x)+c.$

(iii) $\int \sec x\, dx=\int \frac{\sec x(\sec x+\tan x)}{\sec x+\tan x} dx$

Now let $\sec x+\tan x=z$ $\therefore$ $(\sec x \tan x+\sec^2 x)\, dx$
$=dz$ or $\sec x(\sec x+\tan x)dx=dz.$

$\therefore$ $\int \sec x\, dx=\int \frac{dz}{z}=\log z+c=\log (\sec x+\tan x)+c$

(iv) $\int \operatorname{cosec} x dx=\int \frac{dx}{\sin x}=\int \frac{dx}{2 \sin \frac{x}{2} \cos \frac{x}{2}}=\int \frac{\sec^2 \frac{x}{2}}{2 \tan \frac{x}{2}} dx$

Let $\tan \frac{x}{2}=z$ $\therefore$ $\frac{1}{2} \sec^2 \frac{x}{2}\, dx=dz$

$\therefore$ Given integral$=\int \frac{dz}{z}=\log z+c=\log \left(\tan \frac{x}{2}\right)+c.$

[Note. Remember the integrals of this example as formulas.]

Ex. 11. Integrate : $\int \frac{dx}{e^x+1}$

$$\int \frac{dx}{e^x+1} = \int \frac{e^{-x}dx}{1+e^{-x}}$$

Now Let $1+e^{-x}=t \quad \therefore \quad -e^{-x}dx=dt$ or $e^{-x}dx=-dt$

$$\therefore \quad \int \frac{dx}{e^x+1} = -\int \frac{dt}{t} = \log t + c = -\log(e^{-x}+1)+c$$

Ex. 12. Integrate : $\int \frac{\sec x\, dx}{\log(\sec x+\tan x)}$

Let $\log(\sec x+\tan x)=z$

$$\therefore \quad \frac{1}{\sec x+\tan x}(\sec x\tan x+\sec^2 x)\, dx=dz$$

or, $\frac{1}{\sec x+\tan x}\sec x(\tan x+\sec x)\, dx=dz$

or, $\sec x\, dx=dz$.

$\therefore$ Given integral $=\int \frac{dz}{z}=\log z+c$

$=\log\{\log(\sec x+\tan x)\}+c$.

Ex. 13. Integrate :

(i) $\int \frac{dx}{(\sqrt{1-x^2})\sin^{-1}x}$ (ii) $\int \frac{\sec^2 x}{1+\tan x}dx$.

(i) Let $\sin^{-1}x=z \quad \therefore \quad \frac{dx}{\sqrt{1-x^2}}=dz$

$\therefore$ Given integral $=\int \frac{dz}{z}=\log(z)+c=\log(\sin^{-1}x)+c$.

(ii) Let $1+\tan x=z \quad \therefore \quad \sec^2 x\, dx=dz$

$$\therefore \quad \int \frac{\sec^2 x\, dx}{1+\tan x}=\int \frac{dz}{z}=\log z+c=\log(1+\tan x)+c.$$

Ex. 14. Integrate : $\int \frac{1}{\sqrt{x}}\sin\sqrt{x}\, dx$ [B. H. U. '50]

Let $\sqrt{x}=z \quad \therefore \quad \frac{1}{2}\frac{1}{\sqrt{x}}dx=dz$ or, $\frac{dx}{\sqrt{x}}=2dz$

$$\therefore \quad \int \frac{1}{x}\sin\sqrt{x}\, dx=\int 2\sin z\, dz=-2\cos z+c$$

$=-2\cos(\sqrt{x})+c$

Ex. 15. Integrate : $\int \frac{e^x(1+x)}{\cos^2(xe^x)}dx.$

Let, $xe^x = z$. $\therefore$ $(e^x + xe^x)\,dx = dz$

or, $e^x(1+x)\,dx = dz$,

$\therefore$ $\int \frac{e^x(1+x)}{\cos^2(xe^x)}dx = \int \frac{dz}{\cos^2 z} = \int \sec^2 z\,dz$

$=$ $\tan z + c = \tan(xe^x) + c.$

Exercise 2A

Integrate :

1. (i) $\int (ax+b)^{11}\,dx$ (ii) $\int (4x-5)^6 dx$ (iii) $\int \frac{dx}{(a-x)^2}$

(iv) $\int \frac{dx}{a-bx}.$

2. (i) $\int \cos(ax+b)dx$ (ii) $\int \sec^2(2x+3)dx$

(iii) $\int \sin^2(2t+3)dt$ (iv) $\int \cot^2(2-3t)dt.$

3. $\int a^{p+qt}dt.$

4. (i) $\int \frac{dx}{x^2-9}$ (ii) $\int \frac{dx}{4-x^2}$ (iii) $\int \frac{dx}{4x^2-25}$

5. $\int \frac{dx}{\sqrt{x+a}+\sqrt{x+b}}$

Hints : $\int \frac{dx}{\sqrt{x+a}+\sqrt{x+b}} = \int \frac{\sqrt{x+a}-\sqrt{x+b}}{(x+a)-(x+b)}dx$

$= \frac{1}{a-b}\left\{\int \sqrt{x+a}\,dx - \int \sqrt{x+b}\,dx\right\}$].

6. $\int \frac{xdx}{a+bx}.$

[Hints : $\int \frac{xdx}{a+bx} = \frac{1}{b}\int \frac{bxdx}{a+bx} = \frac{1}{b}\int \frac{a+bx-a}{a+bx}dx$

$= \frac{1}{b}\left\{\int \frac{a+bx}{a+bx}dx - \int \frac{a}{a+bx}dx\right\} = \frac{1}{b}\left[\int dx - \frac{a}{b}\int \frac{dx}{a+bx}\right]$

7. (i) $\int \frac{dx}{x^2-10x+24}$ (ii) $\int \frac{dx}{x^2-7x+12}$

8. (i) $\int \frac{2ax+b}{ax^2+bx+c}dx$ (ii) $\int (x^3+6x^2+5x+2)(3x^2+12x+5)\,dx$

9. $\int \frac{e^x-e^{-x}}{e^x+e^{-x}}dx$ [Allahabad, '59] 10. $\int \frac{2x\,dx}{1+x^2}.$

11. (i) $\int \frac{\sin^{-1}x}{\sqrt{1-x^2}}dx$ [C. P. 1933] (ii) $\int \frac{dx}{(1+x^2)\tan^{-1}x} dx.$

12. (i) $\int x^3\sqrt{x^4+a^4}\, dx$ (ii) $\int \frac{xdx}{\sqrt{2x^2+3}}$ 13. $\int \frac{\cos x \sin x}{1+\sin^2 x} dx.$

14. $\int(\tan x+\sin x)^2 (\sec^2 x+\cos x)\, dx.$

15. (i) $\int \frac{\sec^2 x}{(1+\tan x)^2} dx$ (ii) $\int \frac{\text{cosec}^2 x}{1+\cot x} dx$

(iii) $\int \frac{dx}{\cos^2 x\sqrt{\tan x-1}}$

16. (i) $\int \frac{xdx}{\sqrt{x^2-a^2}}$ (ii) $\int \frac{4x^3}{1+x^4} dx$ (iii) $\int \frac{ax^{n-1}}{x^n+b^n} dx.$

17. (i) $\int \frac{dx}{x+x \log x}$ [Agra '63] (ii) $\int \frac{dx}{x \log x \log (\log x)}$

18. (i) $\int \frac{\cot x dx}{\log (\sin x)}$ (ii) $\int \frac{\tan x dx}{\log (\sec x)}$

19. $\int \frac{\sin x+x \cos x}{x \sin x} dx$ 20. $\int \frac{\sec x \,\text{cosec}\, x}{\log \tan x} dx.$

21. $\int e^x \sec^2(e^x)dx.$ 22. $\int \frac{a^{\sin^{-1}x}}{\sqrt{1-x^2}} dx.$

23. $\int(\cot e^x)\, e^x dx.$ 24. $\int \frac{1}{\sqrt{x}} \cos \sqrt{x}\, dx.$

25. $\int e^{\sin^{-1}x} \cdot \frac{dx}{\sqrt{1-x^2}}.$ 26. $\int \cos (\log x) \frac{dx}{x}.$

27. $\int e^{\sin x} \cos x dx$

28. $\int x^2 \cos x^3 dx.$ 29. $\int x^{-1} \cos x^n\, dx.$

30. (i) $\int x(a+bx^2)dx$ (ii) $\int x^{n-1} \sin (a+bx^n)dx.$

31. $\int(\tan x-x) \tan^2 x dx.$ 32. $\int x^{m-1}(2x^m+11)^{100} dx.$

§ 2·7. A few standard forms :

In this article we shall discuss integrations of integrands of the forms $\frac{1}{x^2+a^2}$, $\frac{1}{x^2-a^2}$ and $\frac{1}{a^2-x^2}$ and their square roots with respect to x. These integrals are taken as standard forms and are to be used as formulas.

Now as $1+\tan^2\theta=\sec^2\theta$, or, $\sec^2\theta-1=\tan^2\theta$ and $1-\cos^2\theta=\sin^2\theta$, or, $1-\sin^2\theta=\cos^2\theta$, so if the integrand be a power of x^2+a^2 put $x=a \tan \theta$, if it be a power of x^2-a^2, put $x=a \sec \theta$

and if it be a power of a^2-x^2, put $x=a\sin\theta$. In many cases the alternative methods are useful.

(i) $\int\frac{dx}{a^2+x^2}$.

Let, $x=a\tan\theta$. $\therefore$ $dx=a\sec^2\theta\,d\theta$.

and $a^2+x^2=a^2+a^2\tan^2\theta=a^2(1+\tan^2\theta)=a^2\sec^2\theta$

$\therefore$ $\int\frac{dx}{a^2+x^2}=\int\frac{a\sec^2\theta\,d\theta}{a^2\sec^2\theta}=\frac{1}{a}\int d\theta=\frac{1}{a}\theta+c.$

Now, $\because$ $x=a\tan\theta$, $\therefore$ $\tan\theta=\frac{x}{a}$, or, $\theta=\tan^{-1}\frac{x}{a}$.

$\therefore$ $\int\frac{dx}{a^2+x^2}=\frac{1}{a}\tan^{-1}\frac{x}{a}+c.$

Ex. 1. $\displaystyle\int\frac{dx}{x^2+\frac{3}{4}}=\int\frac{dx}{x^2+\left(\frac{\sqrt{3}}{2}\right)^2}=\frac{1}{\frac{\sqrt{3}}{2}}\tan^{-1}\frac{x}{\frac{\sqrt{3}}{2}}$

$$=\frac{2}{\sqrt{3}}\tan^{-1}\frac{2x}{\sqrt{3}}$$

(ii) $\int\frac{dx}{x^2-a^2}$ $(x>a)$.

Let $x=a\sec\theta$, $\therefore$ $dx=a\sec\theta\tan\theta\,d\theta$.

and $x^2-a^2=a^2\sec^2\theta-a^2=a^2(\sec^2\theta-1)=a^2\tan^2\theta$.

$\therefore$ $\int\frac{dx}{x^2-a^2}=\int\frac{a\sec\theta\tan\theta\,d\theta}{a^2\tan^2\theta}=\frac{1}{a}\int\frac{\sec\theta}{\tan\theta}d\theta.$

$$=\frac{1}{a}\int\left(\frac{1}{\cos\theta}\cdot\frac{\cos\theta}{\sin\theta}\right)d\theta=\frac{1}{a}\int\left(\frac{1}{\sin\theta}\right)d\theta=\frac{1}{a}\int\operatorname{cosec}\theta\,d\theta.$$

$=\frac{1}{a}\log\left(\tan\frac{\theta}{2}\right)+c.$ [see § 2·5 Ex. 7(iv)]

$$=\frac{1}{2a}\log\left(\tan^2\frac{\theta}{2}\right)+c=\frac{1}{2a}\log\frac{x-a}{x+a}+c.$$

$$\left[\because\ \cos\theta=\frac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}},\right.$$

$$\left.\therefore\ \frac{x}{a}=\sec\theta=\frac{1+\tan^2\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}},\ \text{or,}\ \frac{x-a}{x+a}=\tan^2\frac{\theta}{2}.\right]$$

Alternative method.

$$\int\frac{dx}{x^2-a^2}=\int\left\{\frac{1}{2a}\left(\frac{1}{x-a}-\frac{1}{x+a}\right)\right\}dx=\frac{1}{2a}\left\{\int\frac{dx}{x-a}-\int\frac{dx}{x+a}\right\}$$

$$=\frac{1}{2a}\,[\log u-\log v]+c\ [x-a=u \text{ and } x+a=v \text{ (say)}]$$

$$=\frac{1}{2a}\left[\log\frac{u}{v}\right]+c=\frac{1}{2a}\log\frac{x-a}{x+a}+c.$$

(iii) $\displaystyle\int\frac{dx}{a^2-x^2}\quad (a>x).$

Let $x=a\sin\theta$; $\therefore\ dx=a\cos\theta\,d\theta$.

and $a^2-x^2=a^2-a^2\sin^2\theta=a^2(1-\sin^2\theta)=a^2\cos^2\theta$.

Now, $\displaystyle\int\frac{dx}{a^2-x^2}=\int\frac{a\cos\theta\,d\theta}{a^2\cos^2\theta}=\frac{1}{a}\int\sec\theta\,d\theta.$

$$=\frac{1}{a}\log(\sec\theta+\tan\theta)+c$$

$$=\frac{1}{a}\log\left(\frac{1+\sin\theta}{\cos\theta}\right)+c=\frac{1}{a}\log\sqrt{\frac{a+x}{a-x}}+c=\frac{1}{2a}\log\frac{a+x}{a-x}+c$$

$$\left[\frac{1+\sin\theta}{\cos\theta}=\sqrt{\frac{(1+\sin\theta)^2}{\cos^2\theta}}=\sqrt{\frac{(1+\sin\theta)^2}{(1-\sin\theta)(1+\sin\theta)}}\right.$$

$$=\sqrt{\frac{1+\sin\theta}{1-\sin\theta}}=\sqrt{\frac{1+\frac{x}{a}}{1-\frac{x}{a}}}\left(\because\ x=a\sin\theta,\ \therefore\ \sin\theta=\frac{x}{a}\right)$$

$$\left.=\sqrt{\frac{a+x}{a-x}}\right].$$

Alternative method :

$$\int\frac{dx}{a^2-x^2}=\int\frac{1}{2a}\left\{\frac{1}{a+x}+\frac{1}{a-x}\right\}dx$$

$$=\frac{1}{2a}\left\{\int\frac{dx}{a+x}+\int\frac{dx}{a-x}\right\}=\frac{1}{2a}\left\{\int\frac{du}{u}-\int\frac{dv}{v}\right\}\ [\text{ putting } x+a=u \text{ and } a-x=v\ ]$$

$$=\frac{1}{2a}\left\{\log u-\log v\right\}+c=\frac{1}{2a}\log\frac{u}{v}+c=\frac{1}{2a}\log\frac{a+x}{a-x}+c.$$

Note : 1. As, $\dfrac{dx}{a^2-x^2}=-\dfrac{dx}{x^2-a^2}$, so it may appear that the

integrals $\int \frac{dx}{a^2-x^2}$ and $\int \frac{dx}{x^2-a^2}$ are of the same form. But as logarithms of negative number are not defined, so if $x>a$, then $\int \frac{dx}{x^2-a^2}$ and if $x<a$ then $\int \frac{dx}{a^2-x^2}$ can be integrated. For this the two integrations are performed separately.

2. In integrating the above two integrals the formula of integration of $\int \operatorname{cosec}\theta\, d\theta$ and $\int \sec\theta\, d\theta$ have been used. In the alternative methods these two integrals have not been used. Again $\int \sec\theta\, d\theta$ and $\int \operatorname{cosec}\theta\, d\theta$ are often evaluated with the help of the integrals $\int \frac{dx}{x^2-a^2}$ and $\int \frac{dx}{a^2-x^2}$ [See Ex. 4 below]. But there is nothing wrong in any case. For, in the integration of $\int \frac{dx}{x^2-a^2}$ and $\int \frac{dx}{a^2-x^2}$ by the alternative method $\int \sec\theta\, d\theta$ and $\int \operatorname{cosec}\theta\, d\theta$ are not used. Again, in Ex. 7 (iii), (iv) § 2·5 $\int \frac{dx}{x^2-a^2}$ or $\int \frac{dx}{a^2-x^2}$ have not been used in the integration of $\int \sec\theta\, d\theta$ and $\int \operatorname{cosec}\theta\, d\theta$.

Ex. 1. $\int \frac{dx}{x^2-16}=\int \frac{dx}{x^2-(4)^2}=\frac{1}{8}\log\frac{x-4}{x+4}+c.$ $[x>4]$

Ex. 2. $\int \frac{dx}{9-x^2}=\int \frac{dx}{(3)^2-x^2}=\frac{1}{6}\log\frac{3+x}{3-x}+c$ $[x<3]$

(iv) $\int \frac{dx}{\sqrt{x^2+a^2}}$.

Let $x=a\tan\theta$, $\therefore$ $dx=a\sec^2\theta\, d\theta$

and $\sqrt{x^2+a^2}=\sqrt{a^2\tan^2\theta+a^2}=\sqrt{a^2(1+\tan^2\theta)}=a\sec\theta$.

$$\therefore \int \frac{dx}{\sqrt{x^2+a^2}}=\int \frac{a\sec^2\theta\, d\theta}{a\sec\theta}=\int \sec\theta\, d\theta$$

$$=\log(\sec\theta+\tan\theta)+c$$

Now $\because$ $x=a\tan\theta$, $\therefore$ $\tan\theta=\frac{x}{a}$

and $\sec\theta=\sqrt{1+\tan^2\theta}=\sqrt{1+\left(\frac{x}{a}\right)^2}=\frac{1}{a}\sqrt{x^2+a^2}$

$$\therefore \int \frac{dx}{\sqrt{x^2+a^2}}=\log(\sec\theta+\tan\theta)+c$$

$$=\log\left(\frac{\sqrt{x^2+a^2}}{a}+\frac{x}{a}\right)=\log\left(\frac{x+\sqrt{x^2+a^2}}{a}\right)+c.$$

$=\log(x+\sqrt{x^2+a^2})-\log a+c$

$=\log(x+\sqrt{x^2+a^2})+k$ [$k=c-\log a$, is a constant]

(v) $\int\frac{dx}{\sqrt{x^2-a^2}}$ [$x>a$]

Let, $x=a\sec\theta$ $\therefore$ $dx=a\sec\theta\tan\theta\,d\theta$.

and $\sqrt{x^2-a^2}=\sqrt{a^2\sec^2\theta-a^2}=\sqrt{a^2(\sec^2\theta-1)}=a\tan\theta$.

$\therefore$ $\int\frac{dx}{\sqrt{x^2-a^2}}=\int\frac{a\sec\theta\tan\theta\,d\theta}{a\tan\theta}=\int\sec\theta\,d\theta$.

$=\log(\sec\theta+\tan\theta)=\log\left(\frac{x}{a}+\frac{\sqrt{x^2-a^2}}{a}\right)+c$

$\left[\because\ x=a\sec\theta.\ \ \therefore\ \sec\theta=\frac{x}{a}\text{ and }\tan\theta=\frac{\sqrt{x^2-a^2}}{a}\right]$

$=\log(x+\sqrt{x^2-a^2})-\log a+c$

$=\log(x+\sqrt{x^2-a^2})+k.$

Note. Putting $x+\sqrt{x^2\pm a^2}=z$, the two integrals (iv) and (v) can also be integrated.

(vi) $\int\frac{dx}{\sqrt{a^2-x^2}}$.

Let $x=a\sin\theta$. $\therefore$ $dx=a\cos\theta\,d\theta$.

and $\sqrt{a^2-x^2}=\sqrt{a^2-a^2\sin^2\theta}=a\cos\theta$.

$\therefore$ $\int\frac{dx}{\sqrt{a^2-x^2}}=\int\frac{a\cos\theta\,d\theta}{a\cos\theta}=\int d\theta=\theta+c$

$=\sin^{-1}\frac{x}{a}+c.$ [$\because$ $x=a\sin\theta$,

$\therefore$ $\sin\theta=\frac{x}{a}$, or, $\theta=\sin^{-1}\frac{x}{a}$]

Ex. 1. $\int\frac{dx}{\sqrt{x^2+3}}=\log(x+\sqrt{x^2+3})+c.$

§ 2·8. Integration of integrals of the forms $\int\frac{dx}{ax^2+bx+c}$ and $\int\frac{(px+q)dx}{ax^2+b^2+c}$.

(i) To integrate $\int\frac{dx}{ax^2+bx+c}$, if ax^2+bx+c can be resolved into factors, then follow the methods of Ex 4 § 2·4. Note carefully the following examples.

If ax^2+bx+c cannot be resolved into factors, then express $\int\frac{dx}{ax^2+bx+c}$ in one of the forms (i), (ii) and (iii) of § 2·7 depending on the values of a, b and c.

(ii) $\int\frac{px+q}{ax^2+bx+c}dx.$

You know $\frac{d}{dx}\left(ax^2+bx+c\right)=2ax+b.$

Now, $\int\frac{px+q}{ax^2+bx+c}dx$

$$=\frac{p}{2a}\int\frac{2ax+\frac{2aq}{p}}{ax^2+bx+c}dx=\frac{p}{2a}\int\frac{\left(2ax+b\right)+\frac{2aq}{p}-b}{ax^2+bx+c}\,dx$$

$$=\frac{p}{2a}\int\frac{2ax+b}{ax^2+bx+c}dx+\frac{2aq-bp}{2a}\int\frac{dx}{ax^2+bx+c}=\frac{p}{2a}I_1+\frac{2aq-bq}{2a}I_2$$

Now, $I_1=\log\,(ax^2+bx+c)$ and the form of I_2 has been discussed in (i) above.

Note. If ax^2+bx+c can be resolved into factors, then integration can also be performed by expressing $\frac{px+q}{ax^2+bx+c}$ in the form $\frac{k_1x+k_2}{e_1x+e_2}+\frac{k_3x+k_4}{e_3x+e_4}$.

§ 2·9. (i) $\int\frac{dx}{\sqrt{(x-\alpha)(x-\beta)}}$ and (ii) $\int\frac{dx}{\sqrt{(x-\alpha)(\beta-x)}}(\beta>\alpha)$.

(i) Let $x-\alpha=z^2$ $\therefore$ $dx=2zdz$

or, $\frac{dx}{z}=2dz$ or, $\frac{dx}{\sqrt{x-\alpha}}=2dz.$

Again, $x-\alpha=z^2$ $\therefore$ $x=\alpha+z^2$, or, $x-\beta=z^2+\alpha-\beta.$

$\therefore$ Given integral$=\int\frac{2dz}{\sqrt{z^2+\alpha-\beta}}$

$$=2\log\,(z+\sqrt{z^2+\alpha-\beta})=2\log\,(\sqrt{x-\alpha}+\sqrt{x-\beta}).$$

§ 2·10. (i) $\int\frac{dx}{\sqrt{ax^2+bx+c}}$ and (ii) $\int\frac{px+q}{\sqrt{ax^2+bx+c}}dx$

(i) $\int\frac{dx}{\sqrt{ax^2+bx+c}}$

Here there are two possibilities.

(*a*) ax^2+bx+c can be expressed as the product of two linear factors with real coefficients.

or, (*b*) Factorisation is not possible. In case (*a*) follow the method of § 2·9. In case (*b*), follow the following general method. This general method can also be followed in case (*a*).

If a be positive,

$$\int\frac{dx}{\sqrt{ax^2+bx+c}}=\frac{1}{\sqrt{a}}\int\frac{dx}{\sqrt{x^2+\frac{b}{a}x+\frac{c}{a}}}$$

$$=\frac{1}{\sqrt{a}}\int\frac{dx}{\sqrt{\left(x+\frac{b}{2a}\right)^2+\frac{4ac-b^2}{4a^2}}}$$

$$=\frac{1}{\sqrt{a}}\int\frac{dx}{\sqrt{\left(x+\frac{b}{2a}\right)^2\pm\alpha^2}}$$

[If $4ac-b^2$ be positive, then take the '+' sign. If $4ac-b^2$ be negative, then take the '$-$' sign]

Now, $$\int\frac{dx}{\sqrt{\left(x+\frac{b}{2a}\right)^2\pm\alpha^2}}=\int\frac{dt}{\sqrt{t^2\pm\alpha^2}}\quad\left(t=x+\frac{b}{2a},\ \text{say}\right)$$

and this is of the form (iv) or (v) of § 2·7.

If a be negative let $a=-d$ (d positive)

$$\therefore\quad\int\frac{dx}{\sqrt{ax^2+bx+c}}=\int\frac{dx}{\sqrt{c+bx-dx^2}}$$

$$=\frac{1}{\sqrt{d}}\quad\frac{dx}{\sqrt{\frac{4dc+b^2}{4d^2}-\left(x-\frac{b}{2d}\right)^2}}$$

$$=\frac{1}{\sqrt{d}}\int\frac{dt}{\sqrt{\alpha^2-t^2}}\qquad\left[t=x-\frac{b}{2d},\ \alpha^2=\frac{4dc+b^2}{4d^2},\ \text{say}\right]$$

This form has been discussed in § 2·7 (vi).

§ **2·11.** Integration of integrals of the forms :

(i) $\displaystyle\int\frac{dx}{(ax+b)\sqrt{cx+d}}$.

and (ii) $\displaystyle\int\frac{dx}{(ax+b)\sqrt{px^2+qx+r}}$

(i) If we put $cx+d=t^2$, then $\int \frac{dx}{(ax+b)\sqrt{cx+d}}$ will be reduced to one of the forms (i), (ii) and (iii) of § 2·7.

If we put $ax+b=\frac{1}{t}$, $\int \frac{dx}{(ax+b)\sqrt{px^2+qx+r}}$ will be reduced to forms discussed in § 2·10.

Examples 2B

Integrate :

1. $\int \frac{dx}{1+a^2x^2}$.

Let $ax=z$. $\therefore$ $adx=dz$ and $dx=\frac{dz}{a}$

$$\therefore \int \frac{dx}{1+a^2x^2}=\int \frac{dz}{a(1+z^2)}=\frac{1}{a}\int \frac{dz}{1+z^2}=\frac{1}{a}\tan^{-1}z=\frac{1}{a}\tan^{-1}(ax).$$

2. $\int \frac{e^x dx}{e^{2x}+1}$.

Let, $e^x=z$. $\therefore$ $e^x dx=dz$ and $e^{2x}=(e^x)^2=z^2$.

So, $\int \frac{e^x dx}{e^{2x}+1}=\int \frac{dz}{z^2+1}=\tan^{-1}z=\tan^{-1}(e^x)$.

3. $\int \frac{\cos x\, dx}{1+\sin^2 x}$. Let, $\sin x=z$; $\cos x\, dx=dz$.

So, $\int \frac{\cos x\, dx}{1+\sin^2 x}=\int \frac{dz}{1+z^2}=\tan^{-1}z=\tan^{-1}(\sin x)$.

4. $\int \frac{dx}{x\{1+(\log x)^2\}}$

Let, $\log x=z$. $\therefore$ $\frac{dx}{x}=dz$ and $1+(\log x)^2=1+z^2$.

$$\therefore \int \frac{dx}{x\{1+(\log x)^2\}}=\int \frac{dz}{1+z^2}=\tan^{-1}z=\tan^{-1}(\log x).$$

5. $\int \frac{e^x dx}{1-e^{2x}}$.

Let, $e^x=z$; $\therefore$ $e^x dx=dz$ and $e^{2x}=(e^x)^2=z^2$.

$$\therefore \int \frac{e^x dx}{1-e^{3x}}=\int \frac{dz}{1-z^2}=\frac{1}{2}\log\frac{1+z}{1-z}=\frac{1}{2}\log\frac{1+e^x}{1-e^x}$$

6. $\int \sec\theta\, d\theta \left[\theta<\frac{\pi}{2}\right]=\int\frac{1}{\cos\theta}d\theta=\int\frac{\cos\theta\, d\theta}{\cos^2\theta}=\int\frac{\cos\theta\, d\theta}{1-\sin^2\theta}$

Let, $\sin\theta=z$, $\cos\theta\, d\theta=dz$

$$\therefore\quad \int \sec\theta\, d\theta=\int\frac{dz}{1-z^2}=\tfrac{1}{2}\log\frac{1+z}{1-z}=\tfrac{1}{2}\log\frac{1+\sin\theta}{1-\sin\theta}$$

$$=\log\sqrt{\frac{1+\sin\theta}{1-\sin\theta}}=\log\frac{1+\sin\theta}{\sqrt{1-\sin^2\theta}}=\log\frac{1+\sin\theta}{\cos\theta}$$

$$=\log(\sec\theta+\tan\theta).$$

7. $\int\frac{3x^2\,dx}{x^6-1}(x>1).$

Let, $x^3=u$; $3x^2\,dx=du$

$$\therefore\quad \int\frac{3x^2\,dx}{x^6-1}=\int\frac{du}{u^2-1}=\tfrac{1}{2}\log\frac{u-1}{u+1}=\tfrac{1}{2}\log\frac{x^3-1}{x^3+1}.$$

8. $\int\frac{dx}{\sqrt{1-a^2x^2}}.$

Let, $ax=z$. $\therefore$ $adx=dz$, or, $dx=\frac{dz}{a}$.

$$\therefore\quad \int\frac{dx}{\sqrt{1-a^2x^2}}=\int\frac{dx}{a\sqrt{1-z^2}}=\frac{1}{a}\sin^{-1}z+c=\frac{1}{a}\sin^{-1}(ax)+c.$$

9. $\int\frac{dx}{6x^2+17x+12}=\int\frac{dx}{(2x+3)(3x+4)}=\int\left\{\frac{3}{3x+4}-\frac{3}{2x+3}\right\}dx.$

$$=3\int\frac{dx}{3x+4}-2\int\frac{dx}{2x+3}=3\frac{\log(3x+4)}{3}-2\left\{\frac{\log(2x+3)}{2}\right\}+c$$

$$=\log\frac{3x+4}{2x+3}+c.$$

10. $\int\frac{dx}{x^2+x+1}=\int\frac{dx}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}=\int\frac{dx}{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$

Now, let $x+\frac{1}{2}=z$, $\therefore$ $dx=dz$

$$\therefore\quad \text{Given integral}=\int\frac{dz}{z^2+\left(\frac{\sqrt{3}}{2}\right)^2}=\frac{1}{\frac{\sqrt{3}}{2}}\tan^{-1}\frac{z}{\frac{\sqrt{3}}{2}}+c$$

$$=\frac{2}{\sqrt{3}}\tan^{-1}\frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}}+c=\frac{2}{\sqrt{3}}\tan^{-1}\frac{2x+1}{\sqrt{3}}+c.$$

[In the above example, $\int\frac{dx}{x^2+x+1}$ is not a standard form. Putting $x+\frac{1}{2}=z$ the integral reduces to the standard form $\int\frac{dz}{z^2+\left(\frac{\sqrt{3}}{2}\right)^2}$]

11. $\int\frac{dx}{9x^2-12x+8}$ [Rajasthan, 1959]

$$\int\frac{dx}{9x^2-12x+8}=\int\frac{dx}{(3x-2)^2+2^2}$$

Let, $3x-2=z$. $\therefore$ $3dx=dz$. $dx=\frac{dz}{3}$.

$\therefore$ Given integral$=\frac{1}{3}\int\frac{dz}{z^2+(2)^2}=\frac{1}{6}\tan^{-1}\frac{z}{2}=c.$

$$=\frac{1}{6}\tan^{-1}\frac{3x-2}{2}+c.$$

12. Integrate $I=\int\frac{\sec^2 x\,dx}{\tan^2 x+2\tan x+3}$

Let, $\tan x=z$, $\therefore$ $\sec^2 x\,dx=dz$

$\therefore$ Given integral$=\int\frac{dz}{z^2+2z+3}=\int\frac{dz}{(z+1)^2+(\sqrt{2})^2}$

Now let, $z+1=u$ $\therefore$ $dz=du$.

and $I=\int\frac{du}{u^2+(\sqrt{2})^2}=\frac{1}{\sqrt{2}}\tan^{-1}\frac{u}{\sqrt{2}}+c$

$$=\frac{1}{\sqrt{2}}\tan^{-1}\frac{z+1}{\sqrt{2}}+c=\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\tan x+1}{\sqrt{2}}\right)+c.$$

13. $\int\frac{dx}{2x^2-6x+4}=\int\frac{dx}{2(x^2-3x+2)}=\frac{1}{2}\int\frac{dx}{(x-\frac{3}{2})-(\frac{1}{2})^2}$

$=\frac{1}{2}\int\frac{dz}{z^2-(\frac{1}{2})^2}$ [$x-\frac{3}{2}=z$ (say)]

$$=\frac{1}{2}\,\frac{1}{2.\frac{1}{2}}\log\frac{z-\frac{1}{2}}{z+\frac{1}{2}}+c=\frac{1}{2}\log\frac{x-\frac{3}{2}-\frac{1}{2}}{x-\frac{3}{2}+\frac{1}{2}}+c=\frac{1}{2}\log\frac{x-2}{x-1}+c.$$

14. $\int\frac{dx}{1+x-x^2}=\int\frac{dx}{1-(x^2-x)}=\int\frac{dx}{\frac{5}{4}-(x-\frac{1}{2})^2}$

$=\int\frac{du}{\left(\frac{\sqrt{5}}{2}\right)^2-u^2}$ [$u=x-\frac{1}{2}$ (say)]

$$=\frac{1}{2.\frac{\sqrt{5}}{2}}\log\frac{u+\frac{\sqrt{5}}{2}}{u-\frac{\sqrt{5}}{2}}+c=\frac{1}{\sqrt{5}}\log\frac{x-\frac{1}{2}+\frac{\sqrt{5}}{2}}{x-\frac{1}{2}-\frac{\sqrt{5}}{2}}+c$$

$$=\frac{1}{\sqrt{5}}\log\frac{2x-1+\sqrt{5}}{2x-1-\sqrt{5}}+c.$$

15. $\int\frac{7x-9}{x^2-2x+35}dx=\int\frac{\frac{7}{2}(2x-2)-2}{x^2-2x+35}dx$ [C. U. '33]

$$=\frac{7}{2}\int\frac{2x-2}{x^2-2x+35}dx-2\int\frac{dx}{x^2-2x+35}$$

$$=\frac{7}{2}\int\frac{du}{u}-2\int\frac{dx}{(x-1)^2+(\sqrt{34})^2}\quad[\,x^2-2x+35=u\text{ (say)}\,]$$

$$=\tfrac{7}{2}\log u-2\int\frac{dt}{t^2+(\sqrt{34})^2}\qquad[\,x-1=t\text{ say}\,]$$

$$=\tfrac{7}{2}\log(x^2-2x+35)-\frac{2}{\sqrt{34}}\tan^{-1}\frac{t}{\sqrt{34}}+c$$

$$=\tfrac{7}{2}\log(x^2-2x+35)-\frac{2}{\sqrt{34}}\tan^{-1}\frac{x-1}{\sqrt{34}}+c.$$

16. Integrate $\int\frac{2x+3}{2x^2+x-1}dx$

$$\int\frac{2x+3}{2x^2+x-1}dx=\int\frac{\frac{1}{2}(4x+1)+\frac{5}{2}}{2x^2+x-1}dx$$

$$=\frac{1}{2}\int\frac{4x+1}{2x^2+x-1}dx+\frac{5}{2}\int\frac{dx}{2x^2+x-1}$$

Now, $\int\frac{4x+1}{2x^2+x-1}dx=\log(2x^2+x-1)+c_1$

$$\left[\because\ \frac{d}{dx}(2x^2+x-1)=4x+1\right]$$

$$\int\frac{dx}{2x^2+x-1}=\tfrac{1}{2}\int\frac{dx}{\left(x+\frac{1}{4}\right)^2-\left(\frac{3}{4}\right)^2}$$

$$=\frac{1}{2.2\frac{3}{4}}\log\frac{x+\frac{1}{4}-\frac{3}{4}}{x+\frac{1}{4}+\frac{3}{4}}+c_2=\frac{1}{3}\log\frac{2x-1}{2(x+1)}+c_2$$

$\therefore$ Given integral

$$=\tfrac{1}{2}\log(2x^2+x-1)+\tfrac{5}{6}\log\frac{2x-1}{2(x+1)}+c.$$

17. $\int \frac{dx}{\sqrt{x^2-ax}}=\int \frac{dx}{(\sqrt{x(x-a)}}=2 \log (\sqrt{x}+\sqrt{x-a})$.

18. $\int \frac{dx}{\sqrt{x^2+7x+12}}=\int \frac{dx}{\sqrt{(x+3)(x+4)}}$

$=2 \log (\sqrt{x+3}+\sqrt{x+4})$.

19. $\int \frac{dx}{\sqrt{ax-x^2}}=\int \frac{dx}{\sqrt{x(a-x)}}$

$=2 \sin^{-1}\sqrt{\frac{x}{a-0}}=2 \sin^{-1}\sqrt{\frac{x}{a}}$.

20. $\int \frac{dx}{\sqrt{3x-x^2-2}}$ [C. U., '41]

Given integral$=\int \frac{dx}{\sqrt{3x-x^2-\frac{9}{4}+\frac{1}{4}}}=\int \frac{dx}{\sqrt{\frac{1}{4}-(x^2-3x+\frac{9}{4})}}$

$=\int \frac{dx}{\sqrt{(\frac{1}{2})^2-(x-\frac{3}{2})^2}}=\int \frac{dt}{\sqrt{(\frac{1}{2})-t^2}}$ [$x-\frac{3}{2}=t$ (say)]

$=\sin^{-1} \frac{t}{\frac{1}{2}}=\sin^{-1} 2t=\sin^{-1} 2(x-\frac{3}{2})=\sin^{-1}(2x-3)$.

21. $\int \frac{dx}{\sqrt{3x^2+4x+2}}$ [C. U., 1942]

Given integral$=\int \frac{dx}{\sqrt{3(x^2+\frac{4}{3}x+\frac{2}{3})}}$

$=\frac{1}{3}\int \frac{dx}{\sqrt{x^2+\frac{4}{3}x+\frac{4}{9}+\frac{2}{9}}}=\frac{1}{\sqrt{3}}\int \frac{dx}{\sqrt{\left(x+\frac{2}{3}\right)^2+\left(\frac{\sqrt{2}}{3}\right)^2}}$

$=\frac{1}{\sqrt{3}}\int \frac{dt}{\sqrt{t^2+\left(\frac{\sqrt{2}}{3}\right)^2}}$ [$x+\frac{2}{3}=t$ say]

$=\frac{1}{\sqrt{3}} \log \left\{t+\sqrt{t^2+\left(\frac{\sqrt{2}}{3}\right)^2}\right\}$

$=\frac{1}{\sqrt{3}} \log \left(x+\frac{2}{3}+\sqrt{x^2+\frac{4}{3}x+\frac{2}{3}}\right)$.

$=\frac{1}{\sqrt{3}} \log \left\{3x+2+\sqrt{3\left(3x^2+4x+2\right)}\right\}+c$

22. $\int \frac{dx}{\sqrt{1+x+x^2}}=\int \frac{dx}{\sqrt{\frac{3}{4}+(x+\frac{1}{2})^2}}$

$$=\int\frac{dx}{\sqrt{\left(\frac{\sqrt{3}}{2}\right)^2+z^2}}\quad [x+\tfrac{1}{2}=z \text{ (say)}]$$

$$=\log\left\{z+\sqrt{z^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right\}=\log x+\tfrac{1}{2}+\sqrt{(x+\tfrac{1}{2})^2+\tfrac{3}{4}}$$

$$=\log\left(\frac{2x+1}{2}+\sqrt{x^2+x+1}\right)=\log\left(\frac{2x+1+2\sqrt{x^2+x+1}}{2}\right)+c'$$

$$=\log(2x+1+2\sqrt{x^2+x+1})-\log 2+c'$$

Now as log 2 is a constant,

$$\therefore\ \int\frac{dx}{\sqrt{1+x+x^2}}=\log(2x+1+2\sqrt{x^2+x+1})+c.$$

23. Integrate $\int\frac{2x+9}{\sqrt{x^2+x+1}}dx$

$\frac{d}{dx}(x^2+x+1)=2x+1$ and $2x+9=(2x+1)+8$,

Now, $\int\frac{2x+9}{x^2+x+1}dx=\int\frac{2x+1}{\sqrt{x^2+x+1}}dx+\int\frac{8dx}{\sqrt{x^2+x+1}}$

$$=2\sqrt{x^2+x+1}+8\int\frac{dx}{\sqrt{(x+\frac{1}{2})^2+\frac{3}{4}}}+c_1$$

$$=2\sqrt{x^2+x+1}+8\log\left\{\left(x+\tfrac{1}{2}\right)+\sqrt{(x+\tfrac{1}{2})^2+\tfrac{3}{4}}\right\}+c_1+c_2$$

$$=2\sqrt{x^2+x+1}+8\log\left\{\frac{2x+1}{2}+\sqrt{x^2+x+1}\right\}+c.$$

24. Integrate $\int\frac{x+1}{\sqrt{x^2+1}}dx$

$$\int\frac{x+1}{\sqrt{x^2+1}}dx=\tfrac{1}{2}\int\frac{2x}{\sqrt{x^2+1}}dx+\int\frac{dx}{\sqrt{x^2+1}}$$

$$=\sqrt{x^2+1}+\log(x+\sqrt{x^2+1})+c.$$

25. Integrate : $\int\sqrt{\frac{1+x}{1-x}}dx$ [C. U. 1925, '28, '59]

$$\int\sqrt{\frac{1+x}{1-x}}dx=\int\frac{1+x}{\sqrt{1-x^2}}dx=\int\frac{dx}{\sqrt{1-x^2}}+\int\frac{x\,dx}{\sqrt{1-x^2}}=I_1+I_2 \text{ (say)}$$

Now, $I_1=\int\frac{dx}{\sqrt{1-x^2}}=\sin^{-1}x+c_1$

For, $I_2=\int\frac{x\,dx}{\sqrt{1-x^2}}$

Let, $1-x^2=z$, $\therefore$ $-2x\,dx=dz$, or, $x\,dx=-\frac{dz}{2}$

$\therefore$ $I_2=-\int\frac{dz}{2\sqrt{z}}=-\sqrt{z}+c_2=-\sqrt{1-x^2}+c_2$

$\therefore$ Given Integral$=I_1+I_2=\sin^{-1}x+c_1-\sqrt{1-x^2}+c_2$

$=\sin^{-1}x-\sqrt{1-x^2}+c.$ $[c_1+c_2=c]$

26. Integrate $\int\frac{3x+1}{\sqrt{(2-3x-2x^2)}}dx$

$$\int\frac{3x+1}{\sqrt{(2-3x-2x^2)}}dx=-\tfrac{3}{4}\int\frac{-4x-3}{\sqrt{2-3x-2x^2}}dx-\frac{5}{4}\int\frac{dx}{\sqrt{2-3x-2x^2}}$$

$$=-\tfrac{3}{2}\sqrt{(2-3x-2x^2)}-\frac{5}{4\sqrt{2}}\int\frac{dx}{\sqrt{\{(\frac{5}{4})^2-(x+\frac{3}{4})^2\}}}+c.$$

$$=-\tfrac{3}{2}\sqrt{(2-3x-2x^2)}-\frac{5}{4\sqrt{2}}\sin^{-1}\left(\frac{4x+3}{5}\right)+c.$$

27. Integrate $\int\frac{dx}{(x+1)\sqrt{x+2}}$.

Let, $x+2=u^2$. $\therefore$ $dx=2u\,du$ and $x+1=u^2-1$.

$$\therefore\quad\int\frac{dx}{(x+1)\sqrt{x+2}}=\int\frac{2u\,du}{(u^2-1).u}=2\int\frac{du}{u^2-1}$$

$$=\log\frac{u-1}{u+1}+c=\log\frac{\sqrt{x+2}-1}{\sqrt{x+2}+1}+c.$$

28. Integrate $\int\frac{dx}{x\sqrt{x+1}}$.

Let, $x+1=t^2$. $\therefore$ $dx=2t\,dt$ and $x=t^2-1$

$$\therefore\quad\int\frac{dx}{x\sqrt{x+1}}=\int\frac{2t\,dt}{(t^2-1).t}=2\int\frac{dt}{t^2-1}.$$

$$=2.\tfrac{1}{2}\log\frac{t-1}{t+1}+c=\log\frac{\sqrt{x+1}-1}{\sqrt{x+1}+1}+c.$$

29. Integrate $\int\frac{dx}{(2-x)\sqrt{1-2x+3x^2}}$ [Punjab, '60]

Let, $2-x=\frac{1}{t}$. $\therefore$ $-dx=-\frac{1}{t^2}dt$.

or, $dx=\dfrac{dt}{t^2}$; $1-2x+3x^2=1-2\left(2-\dfrac{1}{t}\right)+3\left(2-\dfrac{1}{t}\right)^2$

$$=1-4+\frac{2}{t}+12-\frac{12}{t}+\frac{3}{t^2}=\frac{9t^2-10t+3}{t^2}$$

$$\therefore\quad \int\frac{dx}{(2-x)\sqrt{1-2x+3x^2}}=\int\frac{dt}{t^2\cdot\frac{1}{t}\quad\dfrac{\sqrt{9t^2-10t+3}}{t}}$$

$$=\int\frac{dt}{\sqrt{9t^2-10t+3}}=\int\frac{dt}{3\sqrt{t^2-\frac{10}{9}t+\frac{1}{3}}}$$

$$=\tfrac{1}{3}\int\frac{dt}{\sqrt{(t-\frac{5}{9})^2+\left(\frac{\sqrt{2}}{9}\right)^2}}$$

$$=\tfrac{1}{3}\log\left\{(t-\tfrac{5}{9})+\sqrt{(t-\tfrac{5}{9})^2+\left(\frac{\sqrt{2}}{9}\right)^2}\right\}+c.$$

$$=\tfrac{1}{3}\log\left\{\left(\frac{1}{2-x}-\frac{5}{9}\right)+\sqrt{t^2-\tfrac{10}{9}t+\tfrac{1}{3}}\right\}+c$$

$$=\tfrac{1}{3}\log\left\{\frac{5x-1}{9(2-x)}+\sqrt{\frac{1}{(2-x)^2}-\frac{10}{9(2-x)}+\tfrac{1}{3}}\right\}+c$$

$$=\tfrac{1}{3}\log\left\{\frac{5x-1}{9(2-x)}+\frac{\sqrt{1-2x+3x^2}}{3(2-x)}\right\}+c.$$

Exercise 2B

Integrate :

1. (i) $\int\dfrac{dx}{x^2+9}$ (ii) $\int\dfrac{dx}{a^2+b^2x^2}$ (iii) $\int\dfrac{2x\,dx}{x^4+16}$

(iv) $\int\dfrac{\sec^2 x\,dx}{\sec^2 x+3}$ (v) $\int\dfrac{e^{2x}dx}{e^{4x}+4}$ (vi) $\int\dfrac{\cos x\,dx}{\sin^2 x+4}$

(vii) $\int\dfrac{dx}{(1+x^2)\{1+(\tan^{-1}x)^2\}}$ (viii) $\int\dfrac{dx}{x\{3+(\log x)^2\}}$

(ix) $\int\dfrac{dx}{e^x+e^{-x}}$ [C. U, '58]

2. (i) $\int\dfrac{dx}{x^2-2}$ $(x>\sqrt{2})$ (ii) $\dfrac{dx}{1-x^2}$ $(x<1)$

(iii) $\int\dfrac{dx}{x^2-2ax}$ $(x>2a)$ (iv) $\int\dfrac{dx}{2ax-x^2}$ $(x<2a)$

(v) $\int\dfrac{dx}{x\{1-(\log x)^2\}}$ (vi) $\int\dfrac{e^{2x}dx}{1-e^{4x}}$ (vii) $\int\dfrac{\sec^2\theta}{\sec^2\theta-2}\,d\theta$

(viii) $\int\operatorname{cosec}\theta\,d\theta$

3. (i) $\int \frac{dx}{\sqrt{x^2+9}}$ (ii) $\int \frac{dx}{\sqrt{a^2+b^2x^2}}$ (iii) $\int \frac{2x}{\sqrt{1+x^4}}\,dx.$

(iv) $\int \frac{dx}{\sqrt{a^2-b^2x^2}}$ (v) $\int \frac{dx}{(1+x^2)\sqrt{1-\tan^{-1}x)^2}}$

(vi) $\int \frac{\sec^2 x\,dx}{\sqrt{1-\tan^2 x}}.$

4. (i) $\int \frac{dx}{x^2+x-12}$ (ii) $\int \frac{dx}{3x^2+13x+14}$ (iii) $\int \frac{dx}{x^2+4x+5}$

(iv) $\int \frac{dx}{1-x-x^2}$ (v) $\int \frac{\cos x\,dx}{\sin^2 x+2\sin x+5}$

(vi) $\int \frac{dx}{x\{(\log x)^2+\log x+1\}}$ (vii) $\int \frac{x\,dx}{x^4+4x^2+3}$

(viii) $\int \frac{e^x dx}{2+3e^x-2e^{2x}}$ (ix) $\int \frac{\cos x\,dx}{5\sin^2 x-12\sin x+4}$ [C. U., '67]

5. (i) $\int \frac{x+1}{x^2+4x+5}$ [C. U., 1926. '28]

(ii) $\int \frac{x^2}{x^2-4}\,dx$ [C. U. 1935] (iii) $\int \frac{2x-1}{x^2+2x+3}\,dx$

(iv) $\int \frac{2x-3}{1-x-x^2}dx$ (v) $\int \frac{(1-x)\,dx}{4x^2-4x-3}$

6. (i) $\int \frac{dx}{\sqrt{x^2+x-2}}$ [C. U. 1931] $\int \frac{dx}{\sqrt{x^2-5x+6}}$

(iii) $\int \frac{dx}{\sqrt{5x-x^2-6}}$ (iv) $\int \frac{dx}{\sqrt{2+8x-3x^2}}$

7. (i) $\int \frac{dx}{\sqrt{x^2+2x+6}}$ (ii) $\frac{dx}{\sqrt{(5x-x^2-6)}}$

(iii) $\int \frac{dx}{\sqrt{2x^2+3x+4}}$ [P. P., 1932] (iv) $\int \frac{dx}{\sqrt{1-x-x^2}}$

(v) $\int \frac{dx}{\sqrt{2+x-3x^2}}$ [Gorakhpur, '63]

(vi) $\int \frac{dx}{\sqrt{3x^2-x-3}}$ [Agra, '61] (vii) $\int \frac{dx}{\sqrt{x^2+2x+5}}$ [Agra, '49]

(viii) $\int \frac{\sec^2 x\,dx}{\sqrt{5\tan^2 x-12\tan x+4}}$ (ix) $\int \frac{dx}{x\sqrt{(\log x)^2+2\log x+5}}$

8. (i) $\int \frac{2x+3}{\sqrt{x^2+x+1}}$ [C. U., '28, B. U., '45]

(ii) $\int \frac{x-2}{\sqrt{2x^2-8x+5}}dx$ [C. U., '26]

(iii) $\int \frac{(2x+5)\,dx}{\sqrt{(x^2+3x+1)}}$ (iv) $\int \frac{xdx}{\sqrt{x^2+x+1}}$ [Nagpur, '52]

(v) $\int \frac{x+3}{\sqrt{x^2+2x+2}}dx$ [Poona, '63]

(vi) $\int \frac{5-6x}{\sqrt{1+2x-3x^2}}dx$ (vii) $\int \frac{(2x+5)dx}{\sqrt{3x-x^2-2}}$

9\. (i) $\int \frac{dx}{(2+x)\sqrt{1+x}}$ (ii) $\int \frac{dx}{\sqrt{(2x+1)}\sqrt{3x+4}}$

(iii) $\int \frac{dx}{(x+2)\sqrt{x+3}}$ [C. U., '41]

10\. (i) $\int \frac{dx}{(1+x)\sqrt{1+x-x^2}}$ [Andhra, '63]

(ii) $\int \frac{dx}{(1+2x)\sqrt{1+x^2}}$ (iii) $\frac{dx}{x\sqrt{x^2+x+1}}$ [Poona, '63]

(iv) $\int \frac{dx}{(x-a)\sqrt{x^2-a^2}}$ (v) $\frac{dx}{(1+x)\sqrt{1-x^2}}$

(vi) $\int \frac{dx}{\sqrt{\frac{2}{3}x^3-x^2+\frac{1}{3}}}$.

§ **2·12.** Integration of integrals of the form

$$\int \frac{1}{a\cos x+b\sin x+c}dx.$$

In integration of integrals of this form use the formulas

$$\sin x=\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}} \quad \text{and} \quad \cos x=\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}$$

Put $\tan\frac{x}{2}=z$ and the integral will reduce to the form of § 2·8 (i). The method is illustrated in the following examples.

Ex. 1. Integrate : $\int \frac{dx}{5+4\cos x}$ [C. U. '74]

$$\int \frac{dx}{5+4\cos x}=\int \frac{dx}{5+4.\dfrac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}}$$

$$=\int\frac{\left(1+\tan^2\frac{x}{2}\right)dx}{5\left(1+\tan^2\frac{x}{2}\right)+4\left(1-\tan^2\frac{x}{2}\right)}=\int\frac{\sec^2\frac{x}{2}dx}{9+\tan^2\frac{x}{2}},$$

$$=\int\frac{2dz}{9+z^2}\ \left(\text{Let, }\tan\frac{x}{2}=z\quad\therefore\quad\frac{1}{2}\sec^2\frac{x}{2}dx=dz\right)$$

$$=2\int\frac{dz}{3^2+z^2}=2.\frac{1}{3}\tan^{-1}\frac{z}{3}+c,$$

$$=\frac{2}{3}\tan^{-1}\left(\frac{1}{3}\tan\frac{x}{2}\right)+c.$$

Ex. 2. Integrate : $\int\frac{dx}{3+2\sin x+\cos x}$. [C. U. '67]

$$\int\frac{dx}{3+2\sin x+\cos x}=\int\frac{dx}{3+2.\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}+\frac{\left(1-\tan^2\frac{x}{2}\right)}{1+\tan^2\frac{x}{2}}}$$

$$=\int\frac{\left(1+\tan^2\frac{x}{2}\right)dx}{3\left(1+\tan^2\frac{x}{2}\right)+4\tan\frac{x}{2}+\left(1-\tan^2\frac{x}{2}\right)}$$

$$=\int\frac{\sec^2\frac{x}{2}dx}{2\left(\tan^2\frac{x}{2}+2\tan\frac{x}{2}+2\right)}=\int\frac{2dz}{2(z^2+2z+2)}.$$

$$\left[\text{Let, }z=\tan\frac{x}{2}\quad\therefore\quad dz=\frac{1}{2}\sec^2\frac{x}{2}dx\right]$$

$$=\int\frac{dz}{(z+1)^2+1^2}=\tan^{-1}(z+1)=\tan^{-1}\left(\tan\frac{x}{2}+1\right).$$

Ex. 3. Integrate : $\int\frac{dx}{3\sin x-4\cos x}$. [C. U. '66]

$$\text{Given integral}=\int\frac{dx\left(1+\tan^2\frac{x}{2}\right)}{3.2\tan\frac{x}{2}-4\left(1-\tan^2\frac{x}{2}\right)}$$

$$=\int\frac{2dz}{4z^2+6z-4},\ \left[\text{Let, }z=\tan\frac{x}{2}\quad\therefore\quad\sec^2\frac{x}{2}dx=2dz\right]$$

$$=\int\frac{dz}{2\{(z+\frac{3}{4})^2-(\frac{5}{4})^2\}}=\frac{1}{2}\cdot\frac{1}{2.\frac{5}{4}}\log\frac{z+\frac{3}{4}-\frac{5}{4}}{z+\frac{3}{4}+\frac{5}{4}}+c$$

$$=\frac{1}{5}\log\frac{2\tan x-1}{2\tan\frac{x}{2}+4}+c.$$

Alternative method : $\int \frac{dx}{a \cos x + b \sin x}$ can also be integrated by the following method. Let $a = r \sin \theta$, $b = r \cos \theta$.

$\therefore \quad r = \sqrt{a^2+b^2}$ and $\tan \theta = \frac{a}{b}$ or, $\theta = \tan^{-1} \frac{a}{b}$

$\therefore$ Given integral

$$= \int \frac{dx}{a \cos x + b \sin x} = \int \frac{dx}{r(\cos x \sin \theta + \sin x \cos \theta)}$$

$$= \frac{1}{r}\int \frac{dx}{\sin(x+\theta)} = \frac{1}{r}\int \operatorname{cosec}(x+\theta) d\theta = \frac{1}{r} \log \tan \frac{x+\theta}{2}$$

$$= \frac{1}{\sqrt{a^2+b^2}} \log \tan \left(\frac{1}{2}x + \frac{1}{2} \tan^{-1} \frac{a}{b}\right)$$

Now, putting $a = -4$ and $b = 3$.

$$\int \frac{dx}{3 \sin x - 4 \cos x} = \frac{1}{\sqrt{3^2+4^2}} \log \tan \left(\tfrac{1}{2}x + \tfrac{1}{2} \tan^{-1} \frac{-4}{3}\right) + c$$

$$= \frac{1}{5} \log \tan \left(\frac{x}{2} - \tfrac{1}{2} \tan^{-1} \frac{4}{3}\right) + c.$$

Exercise 2C

Integrate :

1. $\int \frac{dx}{4+5 \cos x}$.
2. $\int \frac{dx}{5+4 \sin x}$.
3. $\int \frac{dx}{4-5 \sin x}$.
4. $\int \frac{dx}{3 \sin x + 4 \cos 4}$.
5. $\int \frac{dx}{2+\sin x + \cos x}$.

Miscellaneous Examples 2

1. Integrate : $\int \sqrt[5]{1+x}\, dx$, (ii) $\int \sqrt[4]{1+\tan x} \sec^2 x\, dx$,

(iii) $\int \frac{dx}{x\sqrt{1+\log x}}$ (iv) $\int \frac{(\sin^{-1} x+3)^2}{\sqrt{1-x^2}} dx$.

(i) Let $1+x = z$, $\therefore \ dx = dz$

$\therefore \quad \int \sqrt[5]{1+x}\, dx = \int \sqrt[5]{z}\, dz = \int z^{\frac{1}{5}} dz = \frac{5}{6} z^{\frac{6}{5}} + c = \frac{5}{6}(1+x)^{\frac{6}{5}} + c.$

(ii) Let $1+\tan x=z \quad \therefore \quad \sec^2 x\, dx=dz$

and $\int \sqrt[4]{1+\tan x}\, \sec^2 x\, dx=\int \sqrt[4]{z}\, dz=\int z^{\frac{1}{4}}\, dz$

$$=\tfrac{4}{5} z^{\frac{5}{4}}+c=\tfrac{4}{5}(1+\tan x)^{\frac{5}{4}}+c.$$

(iii) Let $1+\log x=u \quad \therefore \quad \frac{1}{x} dx=du$

and $\int \frac{dx}{x\sqrt{1+\log x}}=\int \frac{du}{\sqrt{u}}=\int u^{-\frac{1}{2}} du=2u^{\frac{1}{2}}+c$

$$=2\sqrt{u}+c=2\sqrt{1+\log x}+c.$$

(iv) Let $\sin^{-1} x+3=z \quad \therefore \quad \frac{1}{\sqrt{1-x^2}} dx=dz$

and $\int \frac{(\sin^{-1} x+3)^2}{\sqrt{1-x^2}} dx=\int z^2 dz=\frac{z^3}{3}+c=\frac{(\sin^{-1} x+3)^3}{3}+c.$

2. Integrate : (i) $\int \frac{x^2 dx}{\sqrt{x+2}}$. [Ranchi '63 ; P. U. '46]

Let $x+2=t^2 \quad \therefore \quad dx=2t\, dt$ and $x=t^2-2$

$\therefore \quad \int \frac{x^2 dx}{\sqrt{x+2}}=\int \frac{(t^2-2)^2\, 2t\, dt}{t}=2\int (t^4-4t^2+4) dt$

$$=2\left(\frac{t^5}{5}-\frac{4}{3} t^3+4t\right)+c$$

$$=\tfrac{2}{5}(x+2)^{\frac{5}{2}}-\tfrac{8}{3}(x+2)^{\frac{3}{2}}+8(x+2)^{\frac{1}{2}}+c.$$

(ii) $\int \frac{\sqrt{x+4}}{x} dx=\int \frac{x+4}{x\sqrt{x+4}} dx.$

$$=\int \frac{x}{x\sqrt{x+4}} dx+\int \frac{4dx}{x\sqrt{x+4}}$$

$$=\int \frac{dx}{\sqrt{x+4}}+4\int \frac{dx}{x\sqrt{x+4}}=I_1+4I_2$$

Now, let $x+4=z^2 \quad \therefore \quad dx=2z dz$ and $\sqrt{x+4}=z$

$\therefore \quad I_1=\int \frac{2z dz}{z}=2\int dz=2z=2\sqrt{x+4}$

and $I_2=\int \frac{2z\, dz}{(z^2-4)z}=2\frac{dz}{z^2-4}=\frac{2}{2.2} \log \frac{z-2}{z+2}$

$$=\frac{1}{2} \log \frac{\sqrt{x+4}-2}{\sqrt{x+4}+2}$$

So, the given integral$=2\sqrt{x+4}+2\log\dfrac{\sqrt{x+4}-2}{\sqrt{x+4}+2}$

3. Integrate : $\displaystyle\int\frac{dx}{\sqrt{1-(ax+b)^2}}$.

Let $ax+b=z \quad \therefore \quad a dx=dz, \quad$ or, $dx=\dfrac{dz}{a}$

$$\therefore \int\frac{dx}{\sqrt{1-(ax+b)^2}}=\int\frac{\frac{dz}{a}}{\sqrt{1-z^2}}=\frac{1}{a}\sin^{-1}z+c.$$

$$=\frac{1}{a}\sin^{-1}(ax+b)+c.$$

4. Integrate : (i) $\displaystyle\int\frac{cx+d}{\sqrt{ax+b}}dx$

(ii) $\int(cx+d)\sqrt{ax+b}\,dx$.

Let $ax+b=z^2$, $\therefore$ $a dx=2z dz$ and $x=\dfrac{z^2-b}{a}$.

(i) $$\int\frac{cx+d}{\sqrt{ax+b}}dx=\int\frac{c\left(\frac{z^2-b}{a}\right)+d}{z}\,2\frac{z}{a}dz$$

$$=\frac{2c}{a^2}\int z^2dz+\frac{2(ad-bc)}{a^2}\int dz=\frac{2c}{a^2}\frac{z^3}{3}+\frac{2(ad-bc)}{a^2}z+k$$

$$=\frac{2c}{3a^2}\left(ax+b\right)^{\frac{3}{2}}+\frac{2(ad-bc)}{a^2}\left(ax+b\right)^{\frac{1}{2}}+k.$$

(ii) $$\int(cx+d)\sqrt{ax+b}\,dx=\int\left(c.\frac{z^2-b}{a}+d\right).z.\frac{2z}{a}dz$$

$$=\frac{2c}{a^2}\int z^4dz+2\frac{ad-bc}{a^2}\int z^2dz=\frac{2c}{a^2}\frac{z^5}{5}+2\frac{(ad-bc)}{a^2}\frac{z^3}{3}+k$$

$$=\frac{2c}{5a^2}\left(ax+b\right)^{\frac{5}{2}}+\frac{2(ad-bc)}{3a^2}\left(ax+b\right)^{\frac{3}{2}}+k.$$

5. Integrate : $\displaystyle\int\frac{x^3+3x^2+3x+4}{x^2+2x+1}dx$ [C. U. '63]

$$\int\frac{x^3+3x^2+3x+4}{x^2+2x+1}dx=\int\frac{x(x^2+2x+1)+(x^2+2x+1)+3}{x^2+2x+1}dx$$

$$=\int\left\{x+1+\frac{3}{(x+1)^2}\right\}dx=\int x\,dx+\int dx+\int\frac{3}{(x+1)^2}dx$$

$$=\frac{x^2}{2}+x-\frac{3}{x+1}+c.$$

6. Integrate : $\int \frac{(x^2+1)x}{x^4+1}dx$ [C. U. '68]

$$\int \frac{(x^2+1)x}{x^4+1}dx = \int \frac{x^3 dx}{x^4+1} + \int \frac{x}{x^4+1}dx = I_1 + I_2$$

Now, $I_1 = \frac{x^3 dx}{x^4+1} = \int \frac{\frac{1}{4}dz}{z}$, [Let $x^4+1=z$ $\therefore$ $4x^3 dx = dz$]

$$= \tfrac{1}{4} \log (z) = \tfrac{1}{4} \log (x^4+1)$$

and $I_2 = \int \frac{x}{x^4+1}dx = \int \frac{\frac{1}{2}dt}{t^2+1}$ [Let $x^2=t$ $\therefore$ $2xdx=dt$]

$$= \tfrac{1}{2} \tan^{-1} t = \tfrac{1}{2} \tan^{-1}(x^2)$$

$\therefore$ $\int \frac{(x^2+1)x}{x^4+1}dx = \frac{1}{4} \log (x^4+1) + \frac{1}{2} \tan^{-1}(x^2) + c.$

7. $\int \tan^3 x \, dx = \int \tan x . \tan^2 x \, dx = \int \tan x (\sec^2 x - 1) dx$

$= \int \tan x \sec^2 x \, dx - \int \tan x \, dx$

$= \int z.dz - \log \sec x$, [In the first integral put $\tan x = z$, $\therefore$ $\sec^2 x \, dx = dz$]

$= \frac{z^2}{2} - \log \sec x + c = \frac{1}{2} \tan^2 x - \log \sec x + c.$

8. Integrate : $\int \frac{\sin x}{\sin x + \cos x}dx.$ [C. U. '58]

$$\int \frac{\sin x}{\sin x + \cos x}dx = \frac{1}{2}\int \frac{(\sin x + \cos x) - (\cos x - \sin x)}{\sin x + \cos x}dx$$

$= \frac{1}{2}\int dx - \frac{1}{2}\int \frac{\cos x - \sin x}{\sin x + \cos x}dx$

$= \frac{1}{2}x - \frac{1}{2} \log (\sin x + \cos x) + c$ [$\because$ the second integral is of the form $\int \frac{f'(x)}{f(x)}dx$]

Note : If an integral be of the form $\int \frac{a \sin x + b \cos x}{c \sin x + d \cos x}dx$ then express the integrand in the form $l \times$(denominator)$+m$ (denominator). See the illustrations.]

9. Integrate : $\int \frac{2 \sin x + 3 \cos x}{3 \sin x + 4 \cos x}dx$

Let $2 \sin x + 3 \cos x = l(3 \sin x + 4 \cos x) + m(3 \cos x - 4 \sin x)$

$= (3l - 4m) \sin x + (4l + 3m) \cos x$

Equating the coefficients of sin x and cos x from both sides we get, $\left.\begin{array}{l}2=3l-4m\\3=4l+3m\end{array}\right\}$. Solving, $l=\frac{18}{25}$, $m=\frac{1}{25}$

$\therefore$ Required integral

$$=\int\frac{\frac{18}{25}(3\sin x+4\cos x)+\frac{1}{25}(3\cos x-4\sin x)}{3\sin x+4\cos x}dx$$

$$=\int\tfrac{18}{25}dx+\tfrac{1}{25}\int\frac{3\cos x-4\sin x}{3\sin x+4\cos x}dx$$

$$=\tfrac{18}{25}x+\tfrac{1}{25}\log(3\sin x+4\cos x)+c.$$

10. Integrate : (i) $\int\frac{\cos 2x}{\cos x}dx$ (ii) $\int\frac{\cos x}{\cos 2x}dx$

(i) $\int\frac{\cos 2x}{\cos x}dx=\int\frac{2\cos^2x-1}{\cos x}dx=2\int\cos x\,dx-\int\sec x\,dx$

$=2\sin x-\log(\sec x+\tan x)+c.$

(ii) $\int\frac{\cos x}{\cos 2x}dx=\int\frac{\cos x}{1-2\sin^2x}dx=\frac{1}{\sqrt{2}}\int\frac{\sqrt{2}\cos x\,dx}{1-(\sqrt{2}\sin x)^2}$

$=\frac{1}{\sqrt{2}}\int\frac{dz}{1-z^2}$ [Taking $\sqrt{2}\sin x=z$, $\sqrt{2}\cos x\,dx=dz$]

$=\frac{1}{\sqrt{2}}\cdot\frac{1}{2}\cdot\log\frac{1+z}{1-z}=\frac{1}{2\sqrt{2}}\log\frac{1+\sqrt{2}\sin x}{1-\sqrt{2}\sin x}.$

11. Integrate : $\int\frac{dx}{(a\cos x+b\sin x)^2}$

$$\int\frac{dx}{(a\cos x+b\sin x)^2}=\int\frac{dx}{\cos^2x(a+b\tan x)^2}$$

$=\int\frac{\sec^2x\,dx}{(a+b\tan x)^2}=\int\frac{\frac{1}{b}dz}{z^2}$ $\qquad a+b\tan x=z$ (say) $\quad\therefore\ b\sec^2x\,dx=dz$

$=-\frac{1}{b}\cdot\frac{1}{z}+c=-\frac{1}{b}\cdot\frac{1}{a+b\tan x}=c.$

Alternative method : Let $a=r\cos\theta$, $b=r\sin\theta$

$\therefore\ a\cos x+b\sin x=r(\cos x.\cos\theta+\sin x.\sin\theta)$

$=r\cos(x-\theta)$

where $r=\sqrt{a^2+b^2}$ and $\theta=\tan^{-1}\frac{b}{a}$

$\therefore$ Required integral$=\int\frac{dx}{r^2\cos^2(x-\theta)}=\frac{1}{r^2}\int\sec^2(x-\theta)dx$

$=\frac{1}{a^2+b^2}\tan\left(x-\tan^{-1}\frac{b}{a}\right)+c'.$

12. Integrate : $\int \frac{\sin 2x\, dx}{(a+b\cos x)^2}$

Let $a+b\cos x=z \quad \therefore \quad -b\sin x\, dx=dz$

and $\cos x=\frac{z-a}{b}$.

Now, $\int \frac{\sin 2x\, dx}{(a+b\cos x)^2}=\int \frac{2\sin x\cos x\, dx}{(a+b\cos x)^2}$

$$=-\frac{2}{b}\int \frac{\frac{z-a}{b}dz}{z^2}=-\frac{2}{b^2}\int\frac{dz}{z}+\frac{2a}{b^2}\int\frac{dz}{z^2}$$

$$=-\frac{2}{b^2}\log z-\frac{2a}{b^2}.\frac{1}{z}+c$$

$$=-\frac{2}{b^2}\log(a+b\cos x)-\frac{2a}{b^2}.\frac{1}{a+b\cos x}+c$$

13. Integrate : (i) $\int \sin^6 x\cos^3 x\, dx$ [C. U. '68]

(ii) $\int \sin^2 x\cos^2 x\, dx$ (iii) $\int \frac{\sin^2 x}{\cos^6 x}dx$.

(i) $\int \sin^6 x\cos^3 x\, dx=\int \sin^6 x.(1-\sin^2 x)\cos x\, dx$,

[Let $\sin x=z \quad \therefore \quad \cos x\, dx=dz$]

$$=\int z^6(1-z^2)dz=\int z^6\, dz-\int z^8 dz$$

$$=\frac{z^7}{7}-\frac{z^9}{9}+c=\tfrac{1}{7}\sin^7 x-\tfrac{1}{9}\sin^9 x+c.$$

(ii) $\int \sin^2 x\cos^2 x\, dx=\int \tfrac{1}{4}.2\sin^2 x.2\cos^2 x\, dx$

$$=\tfrac{1}{4}\int(1-\cos 2x)(1+\cos 2x)dx=\tfrac{1}{4}\int(1-\cos^2 2x)dx$$

$$=\tfrac{1}{4}\int\left(1-\frac{1+\cos 4x}{2}\right)dx=\tfrac{1}{4}\int\left(\tfrac{1}{2}-\tfrac{1}{2}\cos 4x\right)dx$$

$$=\tfrac{1}{8}\left(x-\frac{\sin 4x}{4}\right)+c.$$

(iii) $\int \frac{\sin^2 x}{\cos^6 x}dx=\int \tan^2 x.(1+\tan^2 x).\sec^2 x\, dx$,

[Let $\tan x=z \quad \therefore \quad \sec^2 x\, dx=dz$]

$$=\int z^2(1+z^2)dz=\frac{z^3}{3}+\frac{z^5}{5}+c=\tfrac{1}{3}\tan^3 x+\tfrac{1}{5}\tan^5 x+c.$$

Note. If in integrals of the form $\int \sin^p x\cos^q x\, dx$ the power of one of $\sin x$ or $\cos x$ be odd, then the other is to be replaced by z. If both p and q be even positive integer, then express the integrand as sines or co-sines of multiple angles.

If both p and q be even integer and one of them be negative, then put $\tan x=z$ or $\cot x=z$.

14. Integrate : $\int \sqrt{1+\sec x}\, dx.$ [C. U. '62]

$$\int \sqrt{1+\sec x}\; dx=\int \frac{\sqrt{1+\cos x}}{\sqrt{\cos x}}dx=\int \frac{\sqrt{2}\cos \frac{x}{2}}{\sqrt{1-2\sin^2 \frac{x}{2}}}dx$$

$$=\int \frac{\sqrt{2}.\sqrt{2}\, dz}{\sqrt{1-z^2}} \quad \left[\sqrt{2}\sin \frac{x}{2}=z \text{ (say)} \right.$$

$$\left. \therefore\quad \frac{1}{\sqrt{2}}\cos \frac{x}{2}dx=dz \right]$$

$$=2\sin^{-1} z+c$$

$$=2\sin^{-1}\left(\sqrt{2}\sin \frac{x}{2}\right)+c.$$

15. **Integrate :** $\int \frac{\sin x}{\sqrt{1+\sin x}}dx.$ [C. U.]

$$\int \frac{\sin x}{\sqrt{1+\sin x}}dx=\int \frac{1+\text{sia } x-1}{\sqrt{1+\sin x}}dx$$

$$=\int \sqrt{1+\sin x}\; dx-\int \frac{1}{\sqrt{1+\sin x}}dx$$

$$=\int\left(\cos \frac{x}{2}+\sin \frac{x}{2}\right)dx-\int \frac{dx}{\cos \frac{x}{2}+\sin \frac{x}{2}},$$

$$\text{for, } \left(\cos \frac{x}{2}+\sin \frac{x}{2}\right)^2=1+\sin x$$

$$=2\sin \frac{x}{2}-2\cos \frac{x}{2}-\int \frac{dx}{\sqrt{2}\sin\left(\frac{x}{2}+\frac{\pi}{4}\right)}$$

$$=2\sin \frac{x}{2}-2\cos \frac{x}{2}-\frac{1}{\sqrt{2}}\int \text{cosec}\left(\frac{x}{2}+\frac{\pi}{4}\right)$$

$$=2\sin \frac{x}{2}-2\cos \frac{x}{2}-\sqrt{2}\log \tan\left(\frac{x}{4}+\frac{\pi}{8}\right)+c.$$

Miscellaneous Exercise 2

Integrate :

1. $\int(1+x)^5 dx$ 2. $\int \frac{\tan^3 x}{\cos^2 x}dx.$

3. $\int \frac{\cos 2x}{3+4\sin 2x}dx$. 4. $\int x a^{x^2} dx$. 5. $\int x^2 e^{x^3} dx$.

6. (i) $\int \frac{e^x dx}{3+4e^x}$. (ii) $\int e^{x^2+6x+9}.(x+3)dx$.

7. $\int \frac{\log(\log x)}{x.\log x}dx$. 8. $\int \frac{\log \sqrt{x}}{3x}dx$. [C. U. '64]

9. $\int \frac{dx}{x\sqrt{x^4-1}}dx$. [C. U. '62] 10. $\int \frac{x dx}{x^2+3x+2}$. [C. U. '74]

11. $\int \frac{\sin^4 x}{\cos^4 x}dx$. [C. U. '74] 12. $\int \frac{x-\alpha}{x^2-3x+4}dx$. [C. U.]

13. (i) $\int \frac{x^2+2x+3}{x^2+x+1}dx$. [C.U. '67] (ii) $\int \frac{x^3-2x+3}{x^2+x-2}dx$. [C.U. '63]

14. $\int \frac{dx}{x\sqrt{1+x^3}}$. [C. U. '66] [Hints : Put. $1+x^3=z^2$]

15. $\int \frac{dx}{\sqrt{x^2-7x+12}}$. 16. $\int \frac{dx}{\sqrt{x^2-3x+2}}$ [C. U. '62]

17. (i) $\int \sqrt{\frac{x}{a-x}}dx$. [C. U. '68] (ii) $\int \sqrt{\frac{a-x}{x}}dx$.

18. (i) $\int \frac{\sqrt{1+\sqrt{x}}}{\sqrt{x}}dx$. (ii) $\int \frac{dx}{\sqrt{x}\sqrt{1+\sqrt{x}}}$.

[Hints : Put $1+\sqrt{x}=z$]

19. (i) $\int \frac{dx}{a^2x^2-b^2}$ (ii) $\frac{dx}{\sqrt{b^2+a^2x^2}}$.

20. $\int \frac{e^x}{\sqrt{e^{2x}-2e^x+2}}dx$.

21. (i) $\int \sin^6 x \cos^3 x dx$ [C. U. '68]

(ii) $\int \sin^4 x. \cos^2 x\, dx$ (iii) $\int \frac{\sin^6 x}{\cos^4 x}dx$.

22. $\int \frac{\sin x\, dx}{\sin(x+a)}$. 23. $\int \sqrt{2x-x^2}\, dx$.

24. $\int \frac{\sqrt{x^2+2x}}{x}dx$. 25. $\int \frac{x+1}{(2x+x^2)\sqrt{2x+x^2}}dx$.

26. $\int \frac{\sqrt{x^2+4x}}{x^2}dx$. 27. $\int \frac{1}{1+\sin 2x}dx$.

28. $\int \frac{\sec x}{a+b\tan x}dx.$ [C. U. '43]

29. (i) $\int \frac{dx}{3+2\sin x}$. [C. U. '65] (ii) $\int \frac{dx}{3+2\cos x}$.

30. (i) $\int \frac{dx}{4-5\sin^2 x}$. (ii) $\int \frac{dx}{4-5\cos^2 x}$.

31. (i) $\int \frac{dx}{4\cos^2 x+3\sin^2 x}$ (ii) $\int \frac{dx}{(4\cos x+3\sin x)^2}$.

32. (i) $\int \frac{\cos x}{\cos x+\sin x}dx.$ (ii) $\int \frac{\cos x+2\sin x}{3\cos x+4\sin x}dx.$

33. $\int \frac{6+3\sin x+14\cos x}{3+4\sin x+5\cos x}dx.$

[Hints : Express the numerator as $l\times$(denominator) $+m\times$(denominator)$+n$.]

34. (i) $\int \frac{dx}{4\cos^3 x-3\cos x}$ (ii) $\int \frac{dx}{\cos 3x-\cos x}$.

35. (i) $\int \frac{\cos 3x}{\cos x}dx$ (ii) $\int \frac{\cos x}{\cos 3x}dx$

36. (i) $\int \frac{\sin 3x}{\sin x}dx.$ (ii) $\int \frac{\sin x}{\sin 3x}dx.$

37. Show that :

(i) $\int \frac{1}{1-\sin^4 x}dx=\frac{1}{2}\tan x+\frac{1}{2\sqrt{2}}\tan^{-1}(\sqrt{2}\tan x)$

(ii) $\int \frac{\sqrt{\tan x}}{\sin x\cos x}dx=2\sqrt{\tan x}.$

(iii) $\int \tan^6 x\,dx=-\frac{1}{5}\tan^5 x-\frac{1}{3}\tan^3 x+\tan x-x.$

(iv) $\int \cot^5 x\,dx=-\frac{1}{4}\cot^4 x+\frac{1}{2}\cot^2 x+\log\sin x.$

38. Show that, $\int \sin\{\phi(x)\}.\phi'(x).dx=-\cos\{\phi(x)\}$ and evaluate the following :

(i) $\int \sin(\log x).\frac{1}{x}dx.$ (ii) $\int \sin(e^x).e^x\,dx.$

(iii) $\int \sin(xe^x).e^x(x+1)\,dx$ (iv) $\int \sin(\tan x)\sec^2 x\,dx.$

39. If $\int f(x)dx = g(x)$ then show that

$\int f\{\sin x\}.\cos x\,dx = g\{\sin x\}$ and evaluate the following integrals :

(i) $\int e^{\sin x}.\cos x\,dx.$ (ii) $\int \sin^5 x.\cos x\,dx.$

(iii) $\int \frac{1}{\sin^3 x}.\cos x\,dx.$ (iv) $\int \log(\sin x)\cos x\,dx.$

(v) $\int \frac{1}{1+\sin^2 x}.\cos x\,dx.$ (vi) $\int \sin(\sin x)\cos x\,dx.$

40. Integrate : (i) $\int \frac{\cos x\,dx}{(1+\sin x)\sqrt{2+\sin x+\sin^2 x}}.$

(ii) $\int \frac{e^x}{(2e^x+1)\sqrt{e^x+2}}dx.$

CHAPTER THREE

Integration by Parts

§ 3·1. In this chapter we shall discuss integration of the product of two functions. Generally, these integrations are performed by the method of integration by parts.

Theorem. If u and v be two differentiable functions of the same variable x for all values of x, then

$$\int u\frac{dv}{dx}.dx = uv - \int v\frac{du}{dx}dx.$$

Proof : $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$

Hence from the definition of integral,

$$uv = \int\left(u\frac{dv}{dx} + v\frac{du}{dx}\right)dx.$$

$$= \int u\frac{dv}{dx}dx + \int v\frac{du}{dx}dx.$$

or, $\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx.$

In the above theorem both u and $\frac{dv}{dx}$ are functions of x.

Let $u = f(x)$ and $\frac{dv}{dx} = \phi(x)$.

as $\frac{dv}{dx} = \phi(x)$, $\therefore$ $dv = \phi(x)dx$.

or, $\int dv = \int \phi(x)dx$, or, $v = \int \phi(x)dx$

and $\frac{du}{dx} = \frac{d}{dx}\{f(x)\}$

Hence from the above theorem one can write,

$$\int f(x)\phi(x)dx = f(x)\int\phi(x)dx - \int\left[\left\{\frac{d}{dx}f(x)\right\}\int\phi(x)dx\right]dx.$$

i.e., the integral of the product of two functions
$=$(first function) $\times$ (integral of the second) $-$ integral
of {the differential coefficient of the first function
$\times$ the integral of the second function}.

This formula is called the formula for integration by parts.

Ex. 1. $\int x \sin x\, dx$

$$=\{x\int \sin x\, dx\}-\int\left[\left\{\frac{d}{dx}(x)\right\}\int \sin x\, dx\right]dx$$

$$=-x\cos x-\int\{(1)(-\cos x)\}dx$$

$$=-x\cos x+\int \cos x\, dx=-x\cos x+\sin x+c.$$

Ex. 2. $\int x^2 \cos x\, dx=x^2\int \cos x\, dx$

$$-\int\left[\left\{\frac{d}{dx}(x^2)\right\}\int \cos x\, dx\right]dx$$

$$=x^2\sin x-\int(2x).\sin x\, dx$$

$$=x^2\sin x-2\int x\sin x\, dx$$

$$=x^2\sin x-2(-x\cos x+\sin x+c) \quad [\text{ from Ex. 1 }]$$

$$=x^2\sin x+2x\cos x-2\sin x+c'.$$

Note that : One can take any of $f(x)$ and $\phi(x)$ as the first function ; but one should first determine the function which is to be taken as the first function, so that integration can be easily performed. In example 1, x has been taken as the first function and $\sin x$ as the second. As $\int x\, dx$ and $\int \sin x\, dx$ are both of the standard forms, so each of them can be easily determined. Let us now examine the situation if $\sin x$ is taken as the first function. In that case,

$$\int x\sin x\, dx=\sin x.\int x\, dx-\int\left\{\left(\frac{d}{dx}(\sin x)\int x\, dx\right)\right\}dx$$

$$=\frac{x^2}{2}\sin x-\int\frac{x^2}{2}\cos x\, dx\ldots\ldots\ldots(\alpha)$$

In the given integral, the integrand is the product of x and a trigonometric function. By taking $\sin x$ as the first function, we get in (α), $\int\frac{x^2}{2}\cos x\, dx$ and here also the integrand is the product of a trigonometric function and a power of x. In $\int\frac{x^2}{2}\cos x\, dx$, the power of x has increased and the integration will be lengthened. If in $\int\frac{x^2}{2}\cos x\, dx$, $\cos x$ is taken as the first function, you will find that integration of $\int\frac{x^3}{6}\sin x\, dx$ will be necessary and integration cannot be completed.

Hence if in integrals of the type $\int x \sin x\, dx$, trigonometric functions are taken as first functions, integration will never be

completed. So, you find that the success of the integration process depends upon the choice of the first function. There is no hard and fast rule for the choice of the first function. But a general rule is that the integral which cannot be evaluted easily is to be taken as the first function. In case of $\int x \sin x \, dx$, evaluations of both $\int x \, dx$ and $\int \sin x \, dx$ are easy. But if $\sin x$ is taken as the first function, integration cannot be completed. So, this rule is to be followed, when integration of one function is difficult. We give below a list of rules for the choice of the first function. There are exceptions to these rules. But in the primary stage they will be useful.

Rules for choice of the first function :

If the integrand is a product of

(1) an algebraic and a trigonometric function, take the algebraic function as the first function.

(2) an algebraic and an exponential function select the algebraic function as the first function.

(3) an algebraic and a logarithmic function, select the logarithmic function as the first function.

(4) an algebraic and an inverse circular function, take the inverse circular function as the first function.

(5) a trigonometric and an exponential function, take any of the functions as the first function.

(6) Sometimes in determination of integrals of the form $\int f(x) \, dx$, the integrand is expressed as $1.f(x)$ and in those cases $f(x)$ is to be selected as the first function.

The rules for choice of the first function can be remembered by the following artifice :

Remember the word "LIATE"

L stands for Logarithmic function.

I stands for Inverse circular function.

A stands for Algebraic function.

T stands for Trigonometric function.

E stands for Exponential function.

In the product of two functions, the letter which comes earlier in the word LIATE should be taken as the first function.

Examples. In $\int x^2 \sin x\, dx$

x^2 Is A and $\sin x$ is T. In 'LIATE' A comes earlier than T. So A i.e., x^2 is the first function.

In $\int \log x\, dx = \int \log x.\, 1\, dx$, L i.e., $\log x$ comes earlier than A i.e., 1 which is algebraic in 'LIATE'. So, L i.e., $\log x$ is to be chosen as the first function.

Ex. 3. $\int x \sec^2 x\, dx = x \int \sec^2 x\, dx - \int \left\{ \frac{d}{dx}(x) \int \sec^2 x\, dx \right\} dx$

$= x \tan x - \int \tan x\, dx = x \tan x - \log(\sec x) + c.$

Ex. 4. $\int x e^{ax} dx = x. \int e^{ax} dx - \int \left\{ \frac{d}{dx}(x) \int e^{ax} dx \right\} dx$

$= x. \frac{e^{ax}}{a} - \int \frac{e^{ax}}{a} dx = \frac{x e^{ax}}{a} - \frac{e^{ax}}{a^2} = \frac{e^{ax}}{a}\left(x - \frac{1}{a}\right).$

§ 3·2. Integration of logarithmic functions.

Ex. 1. $\int \log x\, dx = \int 1.\log x\, dx$

$= \log x \int 1.dx - \int \left\{ \frac{d}{dx}(\log x) \int 1.dx \right\} dx = x \log x - \int \frac{1}{x}.x\, dx$

$= x \log x - \int dx = x \log x - x = x(\log x - 1)$

Cor. $\int \log x^n dx = \int n \log x\, dx = n \int \log x\, dx = nx(\log x - 1).$

Ex. 2. $\int (\log x)^2 dx = \int 1.(\log x)^2 dx$

$= \left(\log x\right)^2 . \int 1.dx - \int \left\{ \frac{d}{dx}\left(\log x\right)^2 \int 1.dx \right\} dx$

$= x\left(\log x\right)^2 - \int \frac{2 \log x}{x}.\, x\, dx = x\left(\log x\right)^2 - 2\int \log x\, dx$

$= x(\log x)^2 - 2x(\log x - 1)$ [from Ex. 1]

$= x\{(\log x)^2 - 2 \log x + 2\}.$

§ 3·3. Integration of inverse circular functions.

Ex. 1. $\int \sin^{-1} x\, dx = \int 1.\sin^{-1} x\, dx$

$= \sin^{-1}.x. \int 1.dx - \int \left\{ \frac{d}{dx}(\sin^{-1} x) \int 1.dx \right\} dx$

$= x \sin^{-1} x - \int \frac{1}{\sqrt{1-x^2}} x\, dx$

Now, to determine $\int \frac{x\, dx}{\sqrt{1-x^2}}$, let $1 - x^2 = t^2$

$\therefore \quad -2x\,dx=2t\,dt$ or $x\,dx=-t\,dt$ and $\sqrt{1-x^2}=\sqrt{t^2}=t$

$\therefore \quad \int\frac{x\,dx}{\sqrt{1-x^2}}=-\int\frac{t\,dt}{t}=-\int dt=-t=-\sqrt{1-x^2}.$

$\therefore$ required integral$=x\sin^{-1}x+\sqrt{1-x^2}$.

Cor. $\int\cos^{-1}x\,dx=\int\left(\frac{\pi}{2}-\sin^{-1}x\right)dx$

$=\int\frac{\pi}{2}dx-\int\sin^{-1}x\,dx=\frac{\pi}{2}x-x\sin^{-1}x-\sqrt{1-x^2}$

$=\frac{\pi}{2}x-x\left(\frac{\pi}{2}-\cos^{-1}x\right)-\sqrt{1-x^2}=x\cos^{-1}x-\sqrt{1-x^2}.$

Ex. 2. $\int\tan^{-1}x\,dx=\int 1.\tan^{-1}x\,dx$

$=\tan^{-1}x.\int 1.dx-\int\left\{\frac{d}{dx}\left(\tan^{-1}x\right)\int 1.\,dx\right\}dx$

$=x\tan^{-1}x-\int\frac{1}{1+x^2}x\,dx=x\tan^{-1}x-\frac{1}{2}\log(1+x^2)$

Cor. $\int\cot^{-1}x\,dx$

$=\int\left(\frac{\pi}{2}-\tan^{-1}x\right)dx=\int\frac{\pi}{2}dx-\int\tan^{-1}x\,dx$

$=\frac{\pi}{2}x-\{x\tan^{-1}x-\frac{1}{2}\log(1+x^2)\}$

$=\frac{\pi}{2}x-x\left(\frac{\pi}{2}-\cot^{-1}x\right)+\frac{1}{2}\log(1+x^2)$

$=x\cot^{-1}x+\frac{1}{2}\log(1+x^2).$

Ex. 3. $\int\sec^{-1}x.dx=\int 1.\sec^{-1}x.dx$

$=\sec^{-1}x\int 1.\,dx-\int\left\{\frac{d}{dx}(\sec^{-1}x).\int 1.dx\right\}dx$

$=x\sec^{-1}x-\int\left\{\frac{d}{dx}(\sec^{-1}x)\int 1.dx\right\}dx$

$=x\sec^{-1}x-\int\frac{1}{x\sqrt{x^2-1}}.x\,dx=x\sec^{-1}x-\log(x+\sqrt{x^2-1}).$

Cor. $\int\operatorname{cosec}^{-1}x\,dx=x\operatorname{cosec}^{-1}x+\log(x+\sqrt{x^2-1}).$

Examples 3A

Ex. 1. $\int x^2\cos^2x\,dx=\int x^2\frac{1}{2}(1+\cos 2x)dx$

$=\frac{1}{2}[\int x^2dx+\int x^2\cos 2x\,dx]$

$=\frac{1}{2}\left[\frac{x^3}{3}+x^2\int\cos 2x\,dx-\int\left\{\frac{d}{dx}\left(x^2\right)\int\cos 2x\,dx\right\}dx\right]$

$$=\frac{x^3}{6}+\frac{1}{2}x^2\ \frac{\sin 2x}{2}-\frac{1}{2}\int 2x.\ \frac{\sin 2x}{2}dx$$

$$=\frac{x^3}{6}+\frac{x^2\ \sin 2x}{4}-\frac{1}{2}\int x\sin 2x\ dx$$

$$=\frac{x^3}{6}+\frac{x^2\ \sin 2x}{4}-\frac{1}{2}\left[x\int\sin 2x\ dx-\int\left\{\frac{d}{dx}(x)\int\sin 2x\ dx\right\}dx\right]$$

$$=\frac{x^3}{6}+\frac{x^2\ \sin 2x}{4}-\tfrac{1}{2}x\left(-\frac{\cos 2x}{2}\right)+\frac{1}{2}\int\left(-\frac{\cos 2x}{2}\right)dx$$

$$=\frac{x^3}{6}+\frac{x^2\ \sin 2x}{4}+\frac{x\cos 2x}{4}-\frac{1}{2}.\frac{\sin 2x}{4}+c$$

$$=\frac{x^3}{6}+\frac{x^2\ \sin 2x}{4}+\frac{x\cos 2x}{4}-\frac{\sin 2x}{8}+c.$$

Ex. 2. $\int x^2e^xdx=x^2\int e^xdx-\int\left\{\frac{d}{dx}(x^2).\int e^xdx\right\}dx$

$$=x^2e^x-\int 2x.e^xdx=x^2e^x-2\int xe^x\ dx$$

$$=x^2e^x-2x.e^x+2\int\left\{\frac{d}{dx}(x)\int e^xdx\right\}dx$$

$$=x^2e^x-2xe^x+2\int e^xdx=x^2e^x-2xe^x+2e^x=e^x(x^2-2x+2).$$

Ex. 3. $\int xe^{ax}dx=x.\int e^{ax}dx-\int\left\{\frac{d}{dx}(x)\int e^{ax}dx\right\}dx$

$$=x.\frac{e^{ax}}{a}-\int\frac{e^{ax}}{a}dx=\frac{xe^{ax}}{a}-\frac{e^{ax}}{a^2}=\frac{e^{ax}}{a}\left(x-\frac{1}{a}\right).$$

Ex. 4. $\int x^2\log x\ dx$

$$=\log x\int x^2dx-\int\left\{\frac{d}{dx}(\log x).\int x^2dx\right\}dx$$

$$=\frac{x^3}{3}\log x-\int\frac{1}{x}.\frac{x^3}{3}dx=\frac{x^3}{3}\log x-\int\frac{x^2}{3}dx$$

$$=\frac{x^3}{3}\log x-\frac{x^3}{9}=\frac{x^3}{9}\ (3\log x-1).$$

Ex. 5. $\int\log(x^2+7x+12)dx=\int\log\{(x+3)(x+4)\}dx$

$$=\int\log(x+3)dx+\int\log(x+4)dx$$

$$=\int\log u\ du+\int\log v\ dv\quad [x+3=u \text{ and } x+4=v \text{ (say)}]$$

$$=u(\log u-1)+v(\log v-1)$$

$$=(x+3)\{\log(x+3)-1\}+(x+4)\{\log(x+4)-1\}$$

$$=(x+3)\log(x+3)+(x+4)\log(x+4)-2x-7$$

$\therefore$ Given integral

$=(x+3)\log(x+3)+(x+4)\log(x+4)-2x+c$

Ex. 6. $\int(\sin^{-1}x)^2\,dx=\int 1.(\sin^{-1}x)^2\,dx$

$=(\sin^{-1}x)^2\int 1.dx-\int\left(\frac{d}{dx}(\sin^{-1}x)^2\int 1.dx\right)dx$

$=x(\sin^{-1}x)^2-\int\frac{2\sin^{-1}x}{\sqrt{1-x^2}}x\,dx$

$=x(\sin^{-1}x)^2-2\left[\sin^{-1}x\int\frac{x\,dx}{\sqrt{1-x^2}}-\int\left(\frac{d}{dx}(\sin^{-1}x)\int\frac{x}{\sqrt{1-x^2}}dx\right)dx\right]$

$=x(\sin^{-1}x)^2+2\left[\sin^{-1}x.\sqrt{1-x^2}+\int\frac{1}{\sqrt{1-x^2}}(-\sqrt{1-x^2})dx\right]$

$=x(\sin^{-1}x)^2+2\sqrt{1-x^2}\sin^{-1}x-\int 2.dx$

$=x(\sin^{-1}x)^2+2\sqrt{1-x^2}\sin^{-1}x-2x.$

7. $\int\cos^{-1}\sqrt{x}\,dx=\int 1.\cos^{-1}\sqrt{x}\,dx.$

$=\cos^{-1}\sqrt{x}\int 1\,dx-\int\left\{\frac{d}{dx}(\cos^{-1}\sqrt{x})\int 1.dx\right\}dx$

$=x\cos^{-1}\sqrt{x}-\int-\frac{1}{\sqrt{1-x}2\sqrt{x}}.x\,dx$

$=x\cos^{-1}\sqrt{x}+\frac{1}{2}\int\frac{\sqrt{x}\,dx}{\sqrt{1-x}}.$

Now, for $\int\frac{\sqrt{x}}{\sqrt{1-x}}dx$ let $x=\sin^2\theta$

$\therefore\quad\int\frac{\sqrt{x}\,dx}{\sqrt{1-x}}=\int\frac{\sin\theta\,2\sin\theta\cos\theta\,d\theta}{\cos\theta}=\int 2\sin^2\theta\,d\theta$

$=\int(1-\cos 2\theta)d\theta=\theta-\frac{\sin 2\theta}{2}=\sin^{-1}\sqrt{x}-\frac{2\sqrt{x}\sqrt{1-x}}{2}$

$=\sin^{-1}\sqrt{x}-\sqrt{x-x^2}$

$\therefore$ Given integral $=x\cos^{-1}\sqrt{x}+\frac{\sin^{-1}\sqrt{x}}{2}-\frac{\sqrt{x-x^2}}{2}$

$=x\cos^{-1}\sqrt{x}+\frac{1}{2}\left(\frac{\pi}{2}-\cos^{-1}\sqrt{x}\right)-\frac{\sqrt{x-x^2}}{2}$

$=(x-\frac{1}{2})\cos^{-1}\sqrt{x}-\frac{\sqrt{x-x^2}}{2}+\frac{\pi}{4}$

$\therefore \int \cos^{-1} \sqrt{x}dx = (x-\frac{1}{2})\cos^{-1} \sqrt{x} - \frac{\sqrt{x-x^2}}{2} + c.$

Ex. 8. $\int \sin^{-1}\left(\frac{2x}{1+x^2}\right)dx = \int \sin^{-1}\left(\frac{2 \tan \theta}{1+\tan^2\theta}\right). \sec^2\theta\, d\theta,$

[Let $x = \tan \theta$; $dx = \sec^2\theta\, d\theta$]

$= \int \sin^{-1}(\sin 2\theta). \sec^2\theta\, d\theta = 2\int \theta \sec^2\theta\, d\theta$

$= 2[\theta \tan \theta - \int 1. \tan \theta\, d\theta] = 2(\theta \tan \theta - \log \sec. \theta) + c$

$= 2(x \tan^{-1}x - \log \sqrt{1+x^2}) + c.$

Exercise 3A

Integrate :—

1. $\int x^3 \sin x\, dx.$ 2. $\int x^2 \sin 2x dx.$ 3. $\int (x+5)\sec^2 x\, dx.$

4. $\int (x^2+3x) \cos^3 x dx.$ 5. $\int x e^x dx.$ 6. $\int (x^2-2)e^{2x}dx.$

7. $\log ax dx.$ 8. $\int (\log x)^3 dx.$ 9. $\int x \log x\, dx.$

10. $\int \frac{\log x}{x^2} dx.$ 11. $\int (1+x^3) \log x\, dx.$

12. $\int (2x^2-5x+2) \log x\, dx.$ 13. $\int \log (\sin x) \cos x\, dx.$

14. $\int \tan^{-1} x\, dx.$ 15. $\int \sin^{-1}(2x\sqrt{1-x^2})\, dx.$

16. $\int \tan^{-1} \frac{2x}{1-x^2}\, dx.$ 17. $\int \cos^{-1} \frac{1-x^2}{1+x^2}\, dx.$

18. $\int \sin^{-1} \sqrt{x} dx.$ 19. $\int (\sin^{-1} x)^3 dx.$

20. $\int \cos^{-1} x dx$ [$\int \sin x^{-1}\, dx$ without determining]

21. $\int \cot^{-1} dx.$ [$\int \tan^{-1} x dx$ without determining]

22. $\int \operatorname{cosec}^{-1} x dx$ [$\int \sec^{-1} x\, dx$ without determining]

23. $\int \cos^{-1}(\frac{1}{x})dx$ [$\int \sec^{-1} x\, dx$ without determining]

§ 3·4. Standard forms :

$\int e^{ax} \cos bx\, dx$ and $\int e^{ax} \sin bx\, dx$.

Let $\int e^{ax} \cos bx\, dx = I_1$ and $\int e^{ax} \sin bx\, dx = I_2$

$\therefore\ I_1 = \int e^{ax} \cos bx\, dx.$

$= e^{ax}. \int \cos bx\, dx - \int \left\{\frac{d}{dx}(e^{ax} \int \cos bx\, dx\right\} dx.$

$$=e^{ax}\frac{\sin bx}{b}-\int ae^{ax}\frac{\sin bx}{b}dx$$

$$=\frac{e^{ax}\sin bx}{b}-\frac{a}{b}\int e^{ax}\sin bx\,dx \quad \cdots(1)$$

$$=\frac{e^{ax}}{b}\sin bx-\frac{a}{b}\,I_2 \quad \cdots(2)$$

Similarly, $I_2=-\dfrac{e^{ax}\cos bx}{b}+\displaystyle\int ae^{ax}\,\frac{\cos bx}{b}dx$

$$=-\frac{e^{ax}\cos bx}{b}+\frac{a}{b}\int e^{ax}\cos bx\,dx \quad \cdots(3)$$

$$=\frac{a}{b}\,I_1-\frac{e^{ax}\cos bx}{b} \quad \cdots(4)$$

From (2) by transposition we get

$$b\,I_1+a\,I_2=e^{ax}\sin bx \quad \cdots(5)$$

and from (4) by transposition we get

$$a\,I_1-b\,I_2=e^{ax}\cos bx \quad \cdots(6)$$

From $(5)\times b+(6)\times a$, we get

$$(a^2+b^2)\,I_1=e^{ax}(a\cos bx+b\sin bx)$$

or, $$I_1=\frac{e^{ax}(a\cos bx+b\sin bx)}{a^2+b^2} \quad \cdots(7)$$

Again, from $(5)\times a-(6)\times b$ we get,

$$(a^2+b^2)\,I_2=e^{ax}\,(a\sin bx-b\cos bx)$$

or, $$I_2=\frac{e^{ax}(a\sin bx-b\cos bx)}{a^2+b^2} \quad \cdots(8)$$

Alternative method.

In the first method I_1 and I_2 have been determined simultaneously. They can also be determined separately.

$$I_1=\int e^{ax}\cos bx\,dx=\frac{e^{ax}\sin bx}{b}-\frac{a}{b}\int e^{ax}\sin bx\,dx$$

$$=\frac{e^{ax}\sin bx}{b}+\frac{a}{b^2}\,e^{ax}\cos bx-\frac{a^2}{b^2}\int e^{ax}\cos bx\,dx$$

$$=\frac{e^{ax}(b\sin bx+a\cos bx)}{b^2}-\frac{a^2}{b^2}\,I_1$$

or, $\left(1+\frac{a^2}{b^2}\right)I_1=\frac{e^{ax}(a\cos bx+b\sin bx)}{b^2}$

or, $I_1=\frac{e^{ax}(a\cos bx+b\sin bx)}{a^2+b^2}$

Similarly, $I_2\ \frac{e^{ax}(a\sin bx-b\cos bx)}{a^2+b^2}$

Now, let $a=r\cos\theta$ and $b=r\sin\theta$.

$\therefore\ a^2+b^2=r^2$ or, $r=\sqrt{a^2+b^2}$ and $\theta=\tan^{-1}\frac{b}{a}$

$$I_1=e^{ax}(a\cos bx+b\sin bx)/(a^2+b^2)$$

$$=\frac{e^{ax}(r\cos\theta\cos bx+r\sin\theta\sin bx)}{r^2}$$

$$=\frac{e^{ax}}{r}\cos(bx-\theta)=\frac{e^{ax}}{\sqrt{a^2+b^2}}\cos\left(bx-\tan^{-1}\frac{b}{a}\right)$$

Similarly, $I_2=\frac{e^{ax}}{\sqrt{a^2+b^2}}\sin\left(bx-\tan^{-1}\frac{b}{a}\right)$.

$$\therefore\ \int e^{ax}\cos bx\,dx=\frac{e^{ax}(a\cos bx+b\sin bx)}{a^2+b^2}$$

$$=\frac{e^{ax}}{\sqrt{a^2+b^2}}\cos\left(bx-\tan^{-1}\frac{b}{a}\right)$$

and $\int e^{ax}\sin bx\,dx=\frac{e^{ax}(a\sin bx-b\cos bx)}{a^2+b^2}$

$$=\frac{e^{ax}}{\sqrt{a^2+b^2}}\sin\left(bx-\tan^{-1}\frac{b}{a}\right).$$

Remember these two forms as standard ones.

§ 3·5. $\int e^x\{f(x)+f'(x)\}dx$.

$\int e^x\{f(x)+f'(x)\}dx=\int e^x f(x)dx+\int e^x f'(x)dx$.

$=f(x)e^x-\int f'(x)e^x dx+\int e^x f(x)dx$ [Integrating $\int e^x f(x)dx$ by parts.]

$=f(x)e^x$.

§ 3·6. Standard forms.

(i) $\int\sqrt{x^2+a^2}\,dx$ (ii) $\int\sqrt{x^2-a^2}\,dx$ (iii) $\int\sqrt{a^2-x^2}\,dx$.

(i) $\int\sqrt{x^2+a^2}\,dx=\int 1.\sqrt{x^2+a^2}\,dx$

$=\sqrt{x^2+a^2}.x-\int\frac{2x}{2\sqrt{x^2+a^2}}.x\,dx$

$$=x\sqrt{x^2+a^2}-\int\frac{x^2}{\sqrt{x^2+a^2}}dx$$

$$=x\sqrt{x^2+a^2}-\int\frac{x^2+a^2-a^2}{\sqrt{x^2+a^2}}dx=x\sqrt{x^2+a^2}-\int\sqrt{x^2+a^2}\,dx$$

$$+a^2\int\frac{dx}{\sqrt{x^2+a^2}}$$

or, $2\int\sqrt{x^2+a^2}\,dx=x\sqrt{x^2+a^2}+a^2\log(x+\sqrt{x^2+a^2})$

(by transposition)

$$\therefore\quad\int\sqrt{x^2+a^2}\,dx=\frac{x\sqrt{x^2+a^2}}{2}+\frac{a^2}{2}\log(x+\sqrt{x^2+a^2}).$$

(ii) $\int\sqrt{x^2-a^2}\,dx=\int 1.\sqrt{x^2-a^2}\,dx$

$$=x\sqrt{x^2-a^2}-\int\frac{2x}{2\sqrt{x^2-a^2}}\,x\,dx$$

$$=x\sqrt{x^2-a^2}-\int\frac{x^2}{\sqrt{x^2-a^2}}dx\quad\cdots(1)$$

Again, $\int\sqrt{x^2-a^2}\,dx=\int\frac{x^2-a^2}{\sqrt{x^2-a^2}}dx$

$$=\int\frac{x^2}{\sqrt{x^2-a^2}}dx-\int\frac{a^2}{\sqrt{x^2-a^2}}dx\quad\cdots\cdots(2)$$

Adding, (1) and (2) we obtain,

$$2\int\sqrt{x^2-a^2}\,dx=x\sqrt{x^2-a^2}-a^2\int\frac{dx}{\sqrt{x^2-a^2}}$$

$$=x\sqrt{x^2-a^2}-a^2\log(x+\sqrt{x^2-a^2})$$

$$\therefore\quad\int\sqrt{x^2-a^2}\,dx=\frac{x\sqrt{x^2-a^2}}{2}-\frac{a^2}{2}\log(x+\sqrt{x^2-a^2})$$

(iii) $\int\sqrt{a^2-x^2}\,dx=\int 1.\sqrt{a^2-x^2}\,dx$

$$=x\sqrt{a^2-x^2}-\int\frac{-2x}{2\sqrt{a^2-x^2}}\,x\,dx$$

$$=x\sqrt{a^2-x^2}+\int\frac{x^2}{\sqrt{a^2-x^2}}dx$$

$$=x\sqrt{a^2-x^2}-\int\frac{a^2-x^2-a^2}{\sqrt{a^2-x^2}}dx$$

$$=x\sqrt{a^2-x^2}-\int\sqrt{a^2-x^2}\,dx+a^2\int\frac{dx}{\sqrt{a^2-x^2}}$$

on transposition,

$$2\int\sqrt{a^2-x^2}\,dx=x\sqrt{a^2-x^2}+a^2\sin^{-1}\frac{x}{a}$$

$$\therefore\quad\int\sqrt{a^2-x^2}\,dx=\frac{x\sqrt{a^2-x^2}}{2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}$$

Note. Putting $x=a\sin\theta$ you can easily determine this integral.

Example 1. $\int\sqrt{x^2+3}\,dx=\frac{x\sqrt{x^2+3}}{2.}+\frac{3}{2}\log(x+\sqrt{x^2+3}).$

Ex. 2. $\int\sqrt{x^2-16}\,dx=\frac{x\sqrt{x^2-16}}{2}-8\log(x+\sqrt{x^2-16}).$

Ex. 3. $\int\sqrt{a^2-b^2x^2}\,dx=\frac{1}{b}\int\sqrt{a^2-t^2}\,dt$

[Let $bx=t\quad\therefore\quad b\,dx=dt$]

$$=\frac{1}{b}\left\{\frac{t\sqrt{a^2-t^2}}{2}+\frac{a^2}{2}\sin^{-1}\frac{t}{a}\right\}=\frac{1}{b}\left\{\frac{bx\sqrt{a^2-b^2x^2}}{2}+\frac{a^2}{2}\sin^{-1}\frac{bx}{a}\right\}.$$

Ex. 4. $\int\sec^3 x\,dx.$ Let $\tan x=t.\quad\therefore\quad\sec^2 x\,dx=dt$

and $\sec x=\sqrt{1+\tan^2 x}=\sqrt{1+t^2}$

$$\therefore\quad\int\sec^3 x\,dx=\int\sec x\sec^2 x\,dx=\int\sqrt{1+t^2}\,dt$$

$$=\frac{t\sqrt{1+t^2}}{2}+\tfrac{1}{2}\log(t+\sqrt{1+t^2})$$

$$=\frac{\tan x\sec x}{2}+\tfrac{1}{2}\log(\sec x+\tan x).$$

Ex. 5. $\int(x-1)\sqrt{x^2+x+1}\,dx$

Here, $\frac{d}{dx}(x^2+x+1)=2x+1$

Now, $(x-1)\sqrt{x^2+x+1}$

$$=\tfrac{1}{2}(2x+1)\sqrt{x^2+x+1}-\tfrac{3}{2}\sqrt{x^2+x+1}$$

$\therefore$ Given integral $=\tfrac{1}{2}\int(2x+1)\sqrt{x^2+x+1}\,dx$

$$-\tfrac{3}{2}\int\sqrt{x^2+x+1}\,dx$$

$$=\tfrac{1}{2}\cdot\tfrac{2}{3}(x^2+x+1)^{\frac{3}{2}}-\tfrac{3}{2}\int\sqrt{(x+\tfrac{1}{2})^2+\left(\frac{\sqrt{3}}{2}\right)^2}\,dx$$

$$=\tfrac{1}{3}(x^2+x+1)^{\frac{3}{2}}-\frac{3(x+\frac{1}{2})\sqrt{x^2+x+1}}{4}$$

$$-\tfrac{3}{2}\cdot\tfrac{3}{8}\log(x+\tfrac{1}{2}+\sqrt{x^2+x+1})+c.$$

$$=\tfrac{1}{3}(x^2+x+1)^{\frac{3}{2}}-\tfrac{3}{8}(2x+1)\sqrt{x^2+x+1}$$

$$-\tfrac{9}{16}\log(x+\tfrac{1}{2}+\sqrt{x^2+x+1})+c.$$

Note. To reduce $(x-1)\sqrt{x^2+x+1}$ in the form
$\frac{1}{2}(2x+1)\sqrt{x^2+x+1}-\frac{3}{2}\sqrt{x^2+x+1}$
let $x-1=\alpha(2x+1)+\beta$, $\therefore$ $\alpha=\frac{1}{2}$ and $\alpha+\beta=-1$ or, $\beta=-\frac{3}{2}$.

Ex. 6. $\int\sqrt{(x-\alpha)(\beta-x)}\,dx$. $(\alpha<x<\beta)$

$$=\int\sqrt{-x^2+(\alpha+\beta)x-\alpha\beta}\,dx$$

$$=\int\sqrt{\left(\frac{\beta-\alpha}{2}\right)^2-\left(x-\frac{\alpha+\beta}{2}\right)^2}\,dx$$

$$=\frac{\left(x-\frac{\alpha+\beta}{2}\right)\sqrt{(x-\alpha)(\beta-x)}}{2}+\frac{\left(\frac{\beta-\alpha}{2}\right)^2}{2}\sin^{-1}\frac{x-\frac{\alpha+\beta}{2}}{\frac{\beta-\alpha}{2}}$$

$$=\tfrac{1}{4}\left\{(2x-\alpha-\beta)\sqrt{(x-\alpha)(\beta-x)}+\frac{(\beta-\alpha)^2}{8}\sin^{-1}\frac{2x-\alpha-\beta}{\beta-\alpha}\right\}.$$

Alternative method : Let $x=\alpha\cos^2\theta+\beta\sin^2\theta$

$\therefore$ $dx=(\beta-\alpha)\sin 2\theta\,d\theta$

$x-\alpha=\alpha\cos^2\theta+\beta\sin^2\theta-\alpha=(\beta-\alpha)\sin^2\theta.$

$\beta-x=\beta-\alpha\cos^2\theta-\beta\sin^2\theta=(\beta-\alpha)\cos^2\theta.$

$\therefore$ Given integral $=\int\sqrt{(\beta-\alpha)\sin^2\theta\,(\beta-\alpha)\cos^2\theta}$
$(\beta-\alpha)\sin 2\theta\,d\theta$

$$=\int\tfrac{1}{2}(\beta-\alpha)^2\sin^2 2\theta\,d\theta$$
$$=\tfrac{1}{4}(\beta-\alpha)^2\int(1-\cos 4\theta)\,d\theta$$
$$=\tfrac{1}{4}(\beta-\alpha)^2\theta-\tfrac{1}{16}(\beta-\alpha)^2\sin 4\theta.$$

Now, $x-\alpha=(\beta-\alpha)\sin^2\theta$ $\therefore$ $\sin\theta=\sqrt{\frac{x-\alpha}{\beta-\alpha}}$

Again, $\sin 4\theta=4\sin\theta\cos\theta.\cos 2\theta$

$$=4\sin\theta\cos\theta(\cos^2\theta-\sin^2\theta)$$
$$=4\sqrt{\frac{x-\alpha}{\beta-\alpha}}\sqrt{1-\frac{x-\alpha}{\beta-\alpha}}\left\{\frac{\beta-x}{\beta-\alpha}-\frac{x-\alpha}{\beta-\alpha}\right\}$$
$$=4\sqrt{\frac{x-\alpha}{\beta-\alpha}}\sqrt{\frac{\beta-x}{\beta-\alpha}}.\frac{\alpha+\beta-2x}{\beta-\alpha}$$
$$=\frac{4}{(\beta-\alpha)^2}\sqrt{(x-\alpha)(\beta-x)}(\alpha+\beta-2x)$$

$\therefore$ Given integral $=\frac{1}{4}(\beta-\alpha)^2\sin^{-1}\sqrt{\frac{x-\alpha}{\beta-\alpha}}$

$$-\tfrac{1}{4}\sqrt{(x-\alpha)(\beta-x)}(\alpha+\beta-2x).$$

Exercise IIIF

Integrate :—

1. $\int \sqrt{x^2+9}\, dx.$ 2. $\int \sqrt{16-9x^2}\, dx.$

3. $\int \sqrt{1-a^2x^2}\, dx.$ 4. $\int \frac{dx}{x-\sqrt{x^2-1}}.$

5. $\int \frac{x^2}{\sqrt{1-x^2}} dx.$ 6. $\int \frac{x^2}{\sqrt{x^2+1}} dx.$

7. $\int \sqrt{4-3x-2x^2}\, dx.$ 8. $\int \sqrt{5-2x+x^2}\, dx$

[C. U. '66]

Examples 3

Example 1. Integrate : $\int \frac{x+\sin x}{1+\cos x} dx.$ [C. U. 75]

$$\int \frac{x+\sin x}{1+\cos x} dx = \int \left(\frac{x}{1+\cos x} + \frac{\sin x}{1+\cos x}\right) dx$$

$$= \int \left(x.\, \frac{1}{2\cos^2 \frac{x}{2}} + \frac{2 \sin \frac{x}{2}. \cos \frac{x}{2}}{2 \cos^2 \frac{x}{2}} \right) dx$$

$$= \frac{1}{2}\int x \sec^2 \frac{x}{2}\, dx + \int \tan \frac{x}{2}\, dx$$

$$= \frac{1}{2} x. \int \sec^2 \frac{x}{2}\, dx - \int \left(\frac{1}{2}.1. \int \sec^2 \frac{x}{2}\, dx\right) dx + \int \tan \frac{x}{2}\, dx$$

(Integrating the first integral by parts)

$$= \frac{1}{2} x.\, 2 \tan \frac{x}{2} - \int \tan \frac{x}{2}\, dx + \int \tan \frac{x}{2}\, dx$$

$$= x \tan \frac{x}{2} + c.$$

Ex. 2. Evaluate : $\int e^x \log (e^{2x}+3e^x+2)\, dx.$

Let $e^x = z \quad \therefore \quad e^x dx = dz.$

Hence given integral $= \int \log (z^2+3z+2)\, dz$

$$= \int \log (z+1)(z+2)\, dz$$

$$= \int \log (z+1)\, dz + \int \log (z+2)\, dz$$

Now, $\int \log (z+1)\, dz = z. \log (z+1) - \int \frac{z}{z+1}\, dz$

$$= z \log (z+1) - \int dz + \int \frac{dz}{z+1}$$

$$= z \log (z+1) - z + \log (z+1)$$

Similarly, $\int \log (z+2)\, dz = z \log (z+2) - z + 2 \log (z+2)$.

Hence the given integral,

$= z \log (z+1) + z \log (z+2) - 2z + \log (z+1) + \log (z+2)^2$

$= z \log (z^2+3z+2) - 2z + \log (z+1)(z+2)^2$

$= e^x \log (e^{2x}+3e^x+2) - 2e^x + \log (e^x+1)(e^{2x}+4e^x+4)$.

Ex. 3. Integrate : $\int \cos x \sqrt{\sin^2 x - 4 \sin x + 5}\, dx$

Let $\sin x = z \quad \therefore \quad \cos x\, dx = dz$

$\therefore \quad \int \cos x \sqrt{\sin^2 x - 4 \sin x + 5}\, dx,$

$= \int \sqrt{z^2 - 4z + 5}\, dz = \int \sqrt{(z-2)^2+1}\, dz$

$= \frac{1}{2}(z-2)\sqrt{(z-2)^2+1} + \frac{1}{2} \log\{(z-2) + \sqrt{(z-2)^2+1}\} + c$

$= \frac{1}{2}(\sin x - 2) \sqrt{\sin^2 x - 4 \sin x + 5}$

$+ \frac{1}{2} \log (\sin x - 2 + \sqrt{\sin^2 x - 4 \sin x + 5}) + c$.

Ex. 4. Integrate : $\int 2^x \sin 3x\, dx$

$\int 2^x \sin 3x\, dx = \int e^{x \log 2} \sin 3x\, dx$

$[2^x = e^{\log 2^x} = e^{x \log 2}]$

$$= \frac{e^{x \log 2}}{(\log 2)^2 + 3^2} \{\log 2 \sin 3x - 3 \cos 3x\} + c$$

[Here $a = \log 2$, $b = 3$]

Ex. 5. Integrate : $\int x (\sin^{-1} x)^2 dx$

$\int x(\sin^{-1} x)^2 dx = \int \theta^2 \sin \theta . \cos \theta\, d\theta,$

[Let $\sin^{-1} x = \theta \quad \therefore \quad x = \sin \theta, \cos \theta\, d\theta = dx$]

$= \frac{1}{2}[\int \theta^2 \sin 2\theta\, d\theta]$

$$= \frac{1}{2}\left[\theta^2 \frac{-\cos 2\theta}{2} - \int 2\theta \frac{-\cos 2\theta}{2} d\theta\right]$$

$$= -\frac{\theta^2}{4} \cos 2\theta + \frac{1}{2} \int \theta \cos 2\theta\, d\theta$$

$$= -\frac{1}{4}\theta^2 \cos 2\theta + \frac{1}{2}\left\{\theta . \frac{\sin 2\theta}{2} - \int 1 . \frac{\sin 2\theta}{2} d\theta\right\}$$

$$= -\frac{1}{4}\theta^2 \cos 2\theta + \frac{1}{4}\theta \sin 2\theta + \frac{\cos 2\theta}{8} + c$$

$$= \frac{1}{4}\{(1 - 2\sin^2\theta) . \theta^2 + 2 \sin\theta . \cos\theta . \theta + \frac{1 - 2\sin^2\theta}{2} + c'\}$$

$$= \frac{1}{4}\{(1-2x^2)(\sin^{-1} x)^2 + 2x\sqrt{1-x^2} \sin^{-1} x - \frac{1}{2}x^2 + c'\}$$

Ex. 6. Integrate : $\int \frac{e^{m \tan^{-1} x}}{(1+x^2)^2} dx$ [C. U.]

$$= \int \frac{e^{m \tan^{-1} x}}{(1+x^2)^2} dx, \quad \left[\text{Putting } z = \tan^{-1} x, \ dz = \frac{1}{1+x^2} dx\right]$$

$$= \int \frac{e^{m \tan^{-1} x}}{1+x^2} \cdot \frac{dx}{1+x^2} = \int \frac{e^{mz}}{1+\tan^2 z} dz$$

$$= \int \cos^2 z . e^{mz} dz = \int \frac{1+\cos 2z}{2} e^{mz} dz$$

$$= \tfrac{1}{2} \int e^{mz} dz + \tfrac{1}{2} \int e^{mz} . \cos 2z \ dz$$

$$= \tfrac{1}{2} \frac{e^{mz}}{m} + \tfrac{1}{2} \frac{e^{mz}}{m^2+2^2} (m \cos 2z + 2 \sin 2z) + c$$

$$= \frac{e^{m \tan^{-1} x}}{2m} \left\{1 + \frac{m}{m^2+4}\left(m . \frac{1-x^2}{1+x^2} + \frac{4x}{1+x^2}\right)\right\} + c.$$

Ex. 7. Integrate : $\int \left\{\frac{1}{\log x} - \frac{1}{(\log x)^2}\right\} dx.$

$$\int \left\{\frac{1}{\log x} - \frac{1}{(\log x)^2}\right\} dx = \int \frac{1}{\log x} dx - \int \frac{1}{(\log x)^2} dx$$

$$= x . \frac{1}{\log x} - \int x . \frac{d}{dx}\left(\frac{1}{\log x}\right) dx - \int \frac{1}{(\log x)^2} dx$$

[Integrating the first integral by parts]

$$= \frac{x}{\log x} - \int x . \frac{-1}{(\log x)^2} \frac{1}{x} dx - \int \frac{1}{(\log x)^2} dx$$

$$= \frac{x}{\log x} + \int \frac{dx}{(\log x)^2} - \int \frac{1}{(\log x)^2} dx$$

$$= \frac{x}{\log x} + c.$$

Ex. 8. Integrate : $\int \sin^{-1} \sqrt{\frac{x}{a+x}} dx.$

$$\int \sin^{-1} \sqrt{\frac{x}{a+x}} dx$$

Let $\frac{x}{x+a} = \sin^2 \theta \quad \therefore \quad x = a \tan^2 \theta$

$\therefore$ Given integral $= \int \sin^{-1}(\sin \theta) . d(a \tan^2 \theta)$

$$= a \int \theta . d(\tan^2 \theta) = a[\theta \tan^2 \theta - \int 1 . \tan^2 \theta \, d\theta]$$

$$= a[\theta \tan^2 \theta - \int (\sec^2 \theta - 1) d\theta].$$

$$= a[\theta \tan^2 \theta - \tan \theta + \theta] + c$$

$$=a\left[\frac{x}{a}\tan^{-1}\sqrt{\frac{x}{a}}-\sqrt{\frac{x}{a}}+\tan^{-1}\sqrt{\frac{x}{a}}\right]+c$$

$$=(x+a)\tan^{-1}\sqrt{\frac{x}{a}}-\sqrt{ax}+c.$$

Ex. 9. Integrate : $\int\frac{1}{x+\sqrt{x^2+a^2}}dx.$

$$\int\frac{1}{x+\sqrt{x^2+a^2}}dx=\int\frac{\sqrt{x^2+a^2}-x}{x^2+a^2-x^2}dx$$

$$=\frac{1}{a^2}\int\sqrt{x^2+a^2}\,dx-\frac{1}{a^2}\int x\,dx$$

$$=\frac{1}{a^2}\left[\tfrac{1}{2}x\sqrt{x^2+a^2}+\frac{a^2}{2}\log\left(x+\sqrt{x^2+a^2}\right)-\frac{x^2}{2}\right]+c.$$

Ex. 10. Integrate : $\int\frac{dx}{(x^2+2x+3)^2}.$

$$\int\frac{dx}{(x^2+2x+3)^2}=\int\frac{dx}{\{(x+1)^2+2\}^2},\qquad \begin{bmatrix}\text{Let } x+1=\sqrt{2}\tan\theta\\ \therefore\ dx=\sqrt{2}\sec^2\theta\,d\theta\end{bmatrix}$$

$$=\int\frac{\sqrt{2}\sec^2\theta\,d\theta}{(2\tan^2\theta+2)^2}=\frac{\sqrt{2}}{4}\int\frac{\sec^2\theta\,d\theta}{\sec^4\theta}$$

$$=\frac{1}{2\sqrt{2}}\int\frac{1+\cos 2\theta}{2}d\theta=\frac{1}{4\sqrt{2}}\left[\theta+\frac{\sin 2\theta}{2}\right]$$

$$=\frac{1}{4\sqrt{2}}\left[\tan^{-1}\left(\frac{x+1}{\sqrt{2}}\right)+\frac{\sqrt{2}(x+1)}{2+(x+1)^2}\right]+c.$$

Ex. 11. Integrate : $\int xe^x\sin x.dx$

$$\int xe^x\sin x\,dx=x\int e^x\sin x-\int\left\{\frac{d}{dx}(x)\int e^x\sin x\,dx\right\}dx$$

$$=x.\frac{e^x}{2}(\sin x-\cos x)dx-\int 1.\frac{e^x}{2}(\sin x-\cos x)\,dx$$

$$=\tfrac{1}{2}xe^x(\sin x-\cos x)-\tfrac{1}{2}(-e^x\cos x)+c$$

[If $f(x)=-\cos x$, then $f'(x)=\sin x$]

$$=\tfrac{1}{2}e^x.\{x(\sin x-\cos x)+\cos x\}+c.$$

Ex. 12. If $I_n=\int\sin^n x\,dx$, then show that

$$I_n=-\frac{1}{n}\sin^{n-1}x\cos x+\frac{n-1}{n}I_{n-2}.$$

Also with the help of the above formula evaluate $\int\sin^5 x\,dx$.

$$I_n=\int\sin^n x\,dx=\int\sin^{n-1}x.\sin x\,dx$$

$$=\sin^{n-1}x\int\sin x\,dx-\int\left\{\frac{d}{dx}(\sin^{n-1}x)\int\sin x\,dx\right\}dx$$

$=\sin^{n-1}x.(-\cos x)-\int(n-1)\sin^{n-2}x.\cos x(-\cos x)dx$
$=-\sin^{n-1}x.\cos x+(n-1)\int\sin^{n-2}x.(1-\sin^2 x)dx$
$=-\sin^{n-1}.\cos x+(n-1)\int\sin^{n-2}x\,dx-(n-1)\int\sin^n x\,dx$
$=-\sin^{n-1}x.\cos x+(n-1)I_{n-2}-(n-1)I_n.$

Transposing,

$$I_n+(n-1)I_n=-\sin^{n-1}x.\cos x+(n-1)I_{n-2}$$

$$\therefore\quad I_n=-\frac{1}{n}\sin^{n-1}x.\cos x+\frac{n-1}{n}I_{n-2}$$

Now, $\int\sin^5 x\,dx=I_5=-\frac{1}{5}\sin^4 x\cos x+\frac{4}{5}I_3,$

[Putting $n=5$ in the above formula]

$=-\frac{1}{5}\sin^4 x.\cos x+\frac{4}{5}\{-\frac{1}{3}\sin^2 x.\cos x+\frac{2}{3}I_1\}$
$=-\frac{1}{5}\sin^4 x\cos x-\frac{4}{15}\sin^2 x.\cos x+\frac{8}{15}\int\sin x\,dx$
$=-\frac{1}{5}\sin^4 x\cos x-\frac{4}{15}\sin^2 x.\cos x-\frac{8}{15}\cos x+c.$

Exercise 3

Integrate :

1. (i) $\int\frac{1-\sin x}{x+\cos x}dx$ (ii) $\int\frac{x+\cos x}{1-\sin x}dx$

2. (i) $\int\frac{1-\cos x}{x-\sin x}dx$ (ii) $\int\frac{x-\sin x}{1-\cos x}dx$

3. (i) $\int\frac{x}{1+\cos x}dx$ (ii) $\int\frac{x}{1+\sin x}dx$

(iii) $\int\frac{x}{1-\cos x}dx$ (iv) $\int\frac{x}{1-\sin x}dx$

(v) $\int\frac{x(1+\sin x)}{\cos^2 x}dx.$

4. (i) $\int e^x\sqrt{(e^{2x}-3e^x+1)}\,dx$

(ii) $\int\frac{\sqrt{1-2e^x+2e^{2x}}}{e^{2x}}dx$ (iii) $\int x^2\sqrt{(x^6+x^3+1)}\,dx$

(iv) $\int(x+a)(x^2+b^2)\,dx$ [C. U. '66] (v) $\int\frac{\sqrt{1+x+x^2}}{x^3}dx$

5. (i) $\int 3^x\cos 4x\,dx$ (ii) $\int e^{2x}\sin x\cos x\,dx$ [C. U. '74]

(iii) $\int e^x\sin\left(\frac{\pi}{4}+x\right)dx$ [C. U. '64]

(iv) $\int e^{mx}\sin^3 x\,dx$ (v) $\int e^{-2x}\cos\frac{1}{2}x\,dx.$

6. (i) $\int x(\tan^{-1}x)^2\,dx$ (ii) $\int x^2\tan^{-1}x\,dx$

7. (i) $\int\frac{e^{m\tan^{-1}x}}{(1+x^2)^{3/2}}dx$ (ii) $\int\frac{e^{2\tan^{-1}x}}{(1+x^2)^{5/2}}$

(iii) $\int \frac{xe^{\sin^{-1} x}}{(1-x^2)^{\frac{1}{2}}} dx$ (iv) $\int \frac{x^3 e^{\sin^{-1} x}}{\sqrt{1-x^2}} dx$

8. (i) $\int \frac{\sin^{-1} x}{(1-x^2)^{3/2}} dx$ (ii) $\int \frac{x^2 \sin^{-1} x}{(1-x^2)^{5/2}} dx.$

9. (i) $\int \left\{ \frac{1}{(\log x)^2} - \frac{2}{(\log x)^3} \right\} dx$ (ii) $\int \left\{ \frac{1}{(\log x)^n} - \frac{n}{(\log x)^{n+1}} \right\} dx$

(iii) $\int \left\{ \log(\log x) + \frac{1}{(\log x)^2} \right\} dx$

10. (i) $\int \sin^{-1}(3x-4x^3)\, dx$ (ii) $\int \tan^{-1} \frac{3x-x^3}{1-3x^2} dx$

(iii) $\int \tan^{-1} \sqrt{\frac{x}{x+1}} dx$ (iv) $\int \tan^{-1} \sqrt{\frac{1-x}{1+x}} dx$

11. (i) $\int \sqrt{\frac{x}{a+x}} dx$ (ii) $\int \sqrt{\frac{x+a}{x}} dx$

(iii) $\int \sqrt{\frac{a-x}{x}} dx$ (iv) $\int \sqrt{\frac{x}{a-x}} dx$ [C. U. '62]

12. (i) $\int \frac{1}{x^2} (\tan^{-1} x)\, dx$ (ii) $\int x^6 \sin^{-1} x\, dx$

13. Show that $\int \cos^n x\, dx = \frac{\sin x \cos^{n-1} x}{n} + \frac{n-1}{n} \int \cos^{n-2} x\, dx$ and hence evaluate $\int \cos^6 x\, dx$.

14. Prove that : $\int \frac{x^2\, dx}{(x \sin x + \cos x)^2} = \frac{\sin x - x \cos x}{x \sin x + \cos x}.$

Integrate :

15. $\int e^x(\tan x - \log \cos x)\, dx.$ [C. U. '64]

16. $\int \frac{\log \sqrt{x}}{3x} dx.$ [C. U. '64]

17. $\int e^x(1+x) \log (xe^x)\, dx.$ [C. U. '63]

18. $\int e^x(\tan x + \sec^2 x)\, dx.$ [C. U. '63]

19. $\int \log (x + \sqrt{x^2+a^2}) dx.$ 20. $\int x \log (x + \sqrt{x^2+a^2}) dx.$

21. $\int \frac{1}{\sqrt{x+a} + \sqrt{x+b}} dx$

22. (i) $\int \log (1+x) dx.$ (ii) $\int x^2 \log x\, dx.$

23. $\int \tan^{-1} \sqrt{x}\, dx.$

CHAPTER FOUR

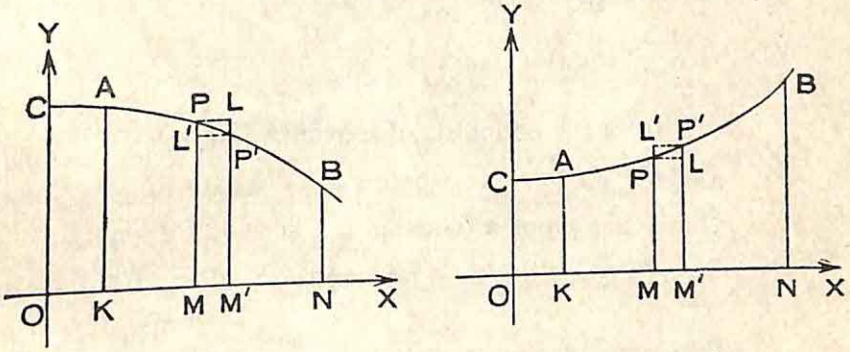
Definite Integral

§ 4·1. Area and Definite Integral.

You have previously learnt how to determine area of rectilineal figures with the help of geometry.

But area of figures enclosed by curved lines such as circles, ellipse or areas enclosed by curves and straight lines such as segment of a circle, portion of a parabola between the x-axis and an ordinate cannot be determined by those geometrical methods. These areas are determined by integration. Actually, the subject of Integral Calculus originated from the attempt of determination of areas of regular figures enclosed by curved lines.

Let the function $f(x)$ be continuous in the interval $0 \leqslant x \leqslant b$ and the curve CAB be the graph of the function in this interval. Since the function is continuous in the interval, so the portion



CAB of the graph of the function in this interval is continuous. Let the curve intersect the y-axis at the point C.

Let P(x, y) be a point on the curve such that $0 < x < b$. $\overline{PM}$ is perpendicular from the point P on the x-axis.

Let the measure of the area COMP be A, i.e., the area enclosed by the x-axis, y-axis, $\overline{PM}$, and the curve $y = f(x)$ be A.

Let P$'(x + \Delta x, y + \Delta y)$ be a point on the curve very close to P and $\overline{P'M'}$, is perpendicular on the x-axis.

So, $MM' = \triangle x$, $P'M' = y + \triangle y$.

Let the measure of the area $CO'M'P' = A + \triangle A$

$\therefore$ The measure of the area $PMM'P' = \triangle A$.

From P draw $\overline{PL}$ perpendicular on $\overline{P'M'}$ and from P' draw $\overline{P'L'}$, perpendicular on $\overline{PM}$.

Now, Area of the rectangle $PMM'L = y.\triangle x$

and area of the rectangle $L'MM'P' = (y + \triangle y).\triangle x$

Hence in fig. (i), $y.\triangle x > \triangle A > (y + \triangle y).\triangle x$

or, $y > \dfrac{\triangle A}{\triangle x} > y + \triangle y \quad \cdots(1)$

Now, as $f(x)$ is continuous in $0 \leqslant x \leqslant b$,

$\therefore$ if $\triangle x \to 0$, then $\triangle y$ will tend to 0.

Hence from (1) we get, $\lim\limits_{\triangle x \to 0} \dfrac{\triangle A}{\triangle x} = y$

or, $\dfrac{dA}{dx} = y = f(x)$.

Again, in fig. (ii), $y < \dfrac{\triangle A}{\triangle x} < y + \triangle y \quad \cdots(2)$

So, as before we get $\dfrac{dA}{dx} = y = f(x)$.

Hence in both cases $\dfrac{dA}{dx} = f(x) \quad \cdots(3)$

$\therefore$ from the definition of indefinite integrals,

$A = \int f(x)\, dx + c \cdots(4)$, where c is a constant of integration.

Hence if $F(x)$ be a function of x such that $F'(x) = f(x)$,

or, $F(x) = \int f(x)\, dx$. [we assume that there is no constant of integration in $F(x)$]

then $A = F(x) + c \quad \cdots(5)$

Now, when the point P coincides with the point C, then $\overline{PM}$ coincides with the y-axis i.e., $\overline{OC}$ and then $x = 0$, $A = 0$.

$\therefore$ From (5), $0 = F(0) + C \quad \cdots(6)$

Subtracting (6) from (5) we obtain,

$$A = F(x) - F(0) \quad \cdots(7)$$

Now, if $A\{(a, f(a)\}$ and $B\{(b, f(b)\}$ be two points on the curve and $a < b$, then putting in succession $x = a$ and $x = b$ in (7) we obtain

Area OKAC $=F(a)-F(0)$ $\cdots(8)$

and area ONBC $=F(b)-F(0)$ $\cdots(9)$

Subtracting (8) from (9) we get,

$$\text{area AKNB}=F(b)-F(a).$$

Hence the measure of the area between the curve $y=f(x)$, the x-axis and the ordinates $x=a$ and $x=b$, is $F(b)-F(a)$, where $F'(x)=f(x)$.

So, the measure of the area enclosed between the curve $y=f(x)$, the x axis and the ordinates $x=a$ and $x=b$ [where the function $f(x)$ is continuous in $a \leqslant x \leqslant b$] can be obtained by evaluating $\int f(x)\,dx$ in the form $F(x)$ and then subtracting $F(a)$ from $F(b)$.

$F(b)-F(a)$ is said to be the value of $\int_a^b f(x)\,dx$. $\int_a^b f(x)\,dx$ is the definite integral of $f(x)$ with respect to x from the limit a to the limit b. a and b are respectively said to be the lower and upper limits of x.

Note. 1, In the above discussion, the curve $y=f(x)$ is above the x-axis i.e., y has been assumed to be positive. If the curve is situated below the x-axis, then the value of the area will become negative. If the value of an area or a definite integral be zero, then numerical values of the areas of the portions above and below the x-axis are equal.

2. Though the indefinite integral of a function is not unique, the definite integral of a function cannot have more than one value.

Note that in the value of the definite integral of a function, there is no constant of integration.

3. In the above discussion it has been assumed that the value of the integral can be determined i.e., the function $f(x)$ is an integrable function. When $\int_a^b f(x)\,dx$ can be determined, then the function is said to be integrable in the interval $a \leqslant x \leqslant b$. A function may not be integrable in an interval. But in this book we shall discuss only integrable functions.

4. In $\int_a^b f(x)\,dx$, the upper limit b is greater than the lower limit a. If $b > a$, then $\int_b^a f(x)\,dx$ is defined as follows :

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx.$$

5. To determine $\int_a^b f(x)\,dx$ we write

$$\int_a^b f(x)\,dx = \Big[F(x)\Big]_a^b = F(b) - F(a).$$

$\Big[\quad\Big]_a^b$ means that put b and a successively for x in the function $F(x)$ within the third bracket and then subtract $F(a)$ from $F(b)$.

Example 1. $\int_a^b x^2\,dx = \left[\frac{x^3}{3}\right]_a^b = \frac{b^3}{3} - \frac{a^3}{3} = \frac{b^3 - a^3}{3}.$

Ex. 2. $\int_1^4 \sqrt{x}\,dx = \left[\frac{2}{3}x^{\frac{3}{2}}\right]_1^4 = \frac{2}{3}.8 - \frac{2}{3}.1 = \frac{16}{3} - \frac{2}{3} = \frac{14}{3}.$

Ex. 3. $\int_0^{\frac{\pi}{2}} \cos x\,dx = \Big[\sin x\Big]_0^{\frac{\pi}{2}} = \sin\frac{\pi}{2} - \sin 0 = 1 - 0 = 1.$

Ex. 4. $\int_1^2 x^n\,dx (n \neq -1) = \left[\frac{x^{n+1}}{n+1}\right]_1^2$

$$= \frac{2^{n+1}}{n+1} - \frac{1^{n+1}}{n+1} = \frac{1}{n+1}(2^{n+1} - 1).$$

Ex. 5. $\int_0^1 (x^2+3)\,dx = \left[\frac{x^3}{3} + 3x\right]_0^1 = (\frac{1}{3}+3) - (0+0) = 3\frac{1}{3}.$

Ex. 6. $\int_0^{\frac{\pi}{2}} \cos^2 x\,dx$ [C. U.]

$$\int \cos^2 x\,dx = \tfrac{1}{2}\int (1+\cos 2x)\,dx = \tfrac{1}{2}\left(x + \frac{\sin 2x}{2}\right)$$

$$\therefore \int_0^{\frac{\pi}{2}} \cos^2 x\,dx = \tfrac{1}{2}\left[x + \frac{\sin 2x}{2}\right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{2}\left[\left(\frac{\pi}{2}+0\right) - \left(0+0\right)\right] = \frac{\pi}{4}.$$

Ex. 7. $\int_0^{\frac{\pi}{4}} \tan^2 x\, dx$ [G. U.]

$\int \tan^2 x\, dx = \int(\sec^2 x - 1)\, dx = \int \sec^2 x\, dx - \int dx = \tan x - x$

$$\therefore \int_0^{\frac{\pi}{4}} \tan^2 x\, dx = \Big[\tan x - x\Big]_0^{\frac{\pi}{4}}$$

$$= \left(\tan\frac{\pi}{4} - \frac{\pi}{4}\right) - \left(\tan 0 - 0\right) = 1 - \frac{\pi}{4}$$

Ex. 8. $\int_0^{\frac{\pi}{2}} \sin^3 x\, dx$.

$\int \sin^3 x\, dx = \frac{1}{4}\int(3\sin x - \sin 3x)\, dx = -\frac{3}{4}\cos x + \frac{1}{12}\cos 3x.$

$$\therefore \int_0^{\frac{\pi}{2}} \sin^3 x\, dx = \Big[-\tfrac{3}{4}\cos x + \tfrac{1}{12}\cos 3x\Big]_0^{\frac{\pi}{2}}$$

$$= \left(-\frac{3}{4}\cos\frac{\pi}{2} + \frac{1}{12}\cos\frac{3\pi}{2}\right) - \left(-\frac{3}{4}\cos 0 + \frac{1}{12}\cos 0\right)$$

$$= 0 + \tfrac{3}{4} - \tfrac{1}{12} = \tfrac{2}{3}.$$

Ex. 9. $\int_{-1}^{1} \frac{dx}{1+x^2}$

$$\int \frac{dx}{1+x^2} = \tan^{-1} x. \quad \therefore \int_{-1}^{1} \frac{dx}{1+x^2} = \Big[\tan^{-1} x\Big]_{-1}^{1}$$

$$= \tan^{-1} 1 - \tan^{-1}(-1) = \frac{\pi}{4} - \left(-\frac{\pi}{4}\right) = \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}.$$

Ex. 10. $\int_0^{\frac{\pi}{2}} \cos 2x \cos x\, dx$.

$$\int \cos 2x \cos x\, dx = \tfrac{1}{2}\int(\cos 3x + \cos x)dx = \tfrac{1}{2}\Big[\sin\frac{3x}{3} + \sin x\Big]$$

$$\therefore \int_0^{\frac{\pi}{2}} \cos 2x \cos x\, dx = \frac{1}{2}\Big[\frac{\sin 3x}{3} + \sin x\Big]_0^{\pi}$$

$$= \frac{1}{2}\left(\frac{\sin\frac{3\pi}{2}}{3} + \sin\frac{\pi}{2}\right) - \frac{1}{2}\left(\frac{\sin 0}{3} + \sin 0\right) = \frac{1}{2}\left(-\frac{1}{3} + 1\right) - 0$$

$$= \tfrac{1}{2}\left(\tfrac{2}{3}\right) = \tfrac{1}{3}$$

Ex. 11. $\int_m^n \frac{mx}{m+n} dx$

$$\int \frac{mx}{m+n} dx = \frac{m}{m+n} \frac{x^2}{2}.$$

$$\therefore \quad \int_m^n \frac{mx}{m+n} dx = \frac{m}{m+n}\left[\frac{x^2}{2}\right]_m^n = \frac{m}{2(m+n)}(n^2-m^2)$$

$$= \frac{m(n-m)}{2}.$$

Ex. 12. $\int_{-1}^{1} \frac{2x+3}{4} dx.$

$$\int \frac{2x+3}{4} dx = \frac{1}{2}\int x\, dx + \tfrac{3}{4}\int dx = \tfrac{1}{4}x^2 + \tfrac{3}{4}x.$$

$$\therefore \quad \int_{-1}^{1} \frac{2x+3}{4} dx = \left[\frac{1}{4}x^2 + \frac{3}{4}x\right]_{-1}^{1}$$

$$= (\tfrac{1}{4}+\tfrac{3}{4}) - (\tfrac{1}{4}-\tfrac{3}{4}) = 1 - (-\tfrac{1}{2}) = \tfrac{3}{2}.$$

Ex. 13. $\int_0^1 e^x dx = \left[e^x\right]_0^1 = e-1.$

Ex. 14. $\int_0^{\frac{\pi}{2}} x \sin x\, dx$

$$\int x \sin x\, dx = x(-\cos x) - \int -(\cos x)\, dx$$

$$= -x \cos x + \int \cos x\, dx = -x \cos x + \sin x$$

$$\therefore \quad \int_0^{\frac{\pi}{2}} x \sin x\, dx = \Big[-x \cos x + \sin x\Big]_0^{\frac{\pi}{2}}$$

$$= \left(-\frac{\pi}{2}\cos\frac{\pi}{2} + \sin\frac{\pi}{2}\right) - (0 \cos 0 + \sin 0) = 1 - 0 = 1.$$

Ex. 15. $\int_1^e (x \log x)\, dx$

$$\int x \log x\, dx = (\log x)\frac{x^2}{2} - \int \frac{1}{x}\cdot\frac{x^2}{2} dx$$

$$= \frac{x^2}{2}\log x - \tfrac{1}{2}\int x\, dx = \frac{x^2}{2}\log x - \frac{x^2}{4}.$$

$$\therefore \int_1^e (x \log x)\,dx = \left[\frac{x^2}{2}\log x - \frac{x^2}{4}\right]_1^e$$

$$= \left(\frac{e^2}{2}\log e - \frac{e^2}{4}\right) - \left(\frac{1}{2}\log 1 - \frac{1}{4}\right)$$

$$= \frac{e^2}{2} - \frac{e^2}{4} + \frac{1}{4} = \frac{1}{4} + \frac{e^2}{4} = \frac{e^2+1}{4} \quad [\because \log e = 1]$$

Ex. 16. $\int_0^1 \sin^{-1}x\,dx$

$$\int \sin^{-1}x\,dx = x\sin^{-1}x + \sqrt{1-x^2}$$

$$\therefore \int_0^1 \sin^{-1}x\,dx = \left[x\sin^{-1}x + \sqrt{1-x^2}\right]_0^1$$

$$= \sin^{-1}1 - 1 = \frac{\pi}{2} - 1.$$

Ex. 17. Show that $\int_2^e \left\{\frac{1}{\log x} - \frac{1}{(\log x)^2}\right\}dx = e - \frac{2}{\log 2}$

$$\int_2^e \left\{\frac{1}{\log x} - \frac{1}{(\log x)^2}\right\} = \int_2^e \frac{1}{\log x}\cdot 1dx - \int_2^e \frac{1}{(\log x)^2}dx$$

$$= \left[\frac{1}{\log x}\cdot x\right]_2^e - \int_2^e \left\{-\frac{1}{(\log x)^2}\cdot\frac{1}{x}\cdot x\,dx\right\} - \int_2^e \frac{1}{(\log x)^2}dx$$

$$= \frac{e}{\log e} - \frac{2}{\log 2} + \int_2^e \frac{1}{(\log x)^2}dx - \int_2^e \frac{1}{(\log x)^2}dx$$

$$= e - \frac{2}{\log 2}\left[\because \log e = 1\right]$$

Ex. 18. Show that $\int_{\frac{1}{2}}^2 (4x^2+1)dx = \int_{\frac{1}{2}}^2 \frac{8}{x^2}dx$

$$\int_{\frac{1}{2}}^2 (4x^2+1)dx = \left[\frac{4}{3}x^3 + x\right]_{\frac{1}{2}}^2 = \left(\frac{4}{3}\cdot 8 + 2\right) - \left(\frac{4}{3}\cdot\frac{1}{8} + \frac{1}{2}\right)$$

$$= \frac{32}{3} + 2 - \frac{1}{6} - \frac{1}{2} = \frac{64+12-1-3}{6} = 12$$

$$\int_{\frac{1}{2}}^2 \frac{8}{x^2}dx = \left[-\frac{8}{x}\right]_{\frac{1}{2}}^2 = -4 + 16 = 12.$$

So, $\int_{\frac{1}{2}}^2 (4x^2+1)dx = \int_{\frac{1}{2}}^2 \frac{8}{x^2}dx.$

Exercise IVA

Evaluate :

1. $\int_1^{10} x^8 . \, dx$ 2. $\int_0^9 \left(x^{\frac{1}{2}}+x^{-\frac{1}{2}}\right) dx$ 3. $\int_0^{\frac{\pi}{6}} \cos 3x \, dx$

4. $\int_0^{\frac{\pi}{2}} (\cos \theta - \sin \theta) \, d\theta$ 5. $\int_0^{\frac{\pi}{2}} \sin^2 x \, dx$

6. $\int_0^1 \tan^{-1} x \, dx$ 7. $\int_1^{\sqrt{e}} x \log x \, dx$ 8. $\int_0^4 \sqrt{1+2x} \, dx$

9. $\int_{-2}^{-1} \frac{dx}{(x-2)^3}$ 10. $\int_0^1 x e^x \, dx.$ [C. U. 1936]

11. $\int_0^{\frac{\pi}{4}} \sec \theta (\sec \theta - \tan \theta) d\theta.$

If m and n are both positive integers (Ex. 12—14).

12. $\int_0^{\pi} \sin mx \sin nx \, dx.$

13. $\int_0^{\pi} \sin mx \cos nx \, dx.$ 14. $\int_0^{\pi} \cos mx \cos nx \, dx$

15. $\int_0^1 \frac{dx}{\sqrt{1-x^2}}.$ 16. $\int_a^b e^{mx} dx.$ 17. $\int_1^4 \log x \, dx.$

18. $\int_0^{\frac{1}{2}} \sin^{-1} x \, dx$ [C. U. '70]. 19. $\int_0^{\pi} x \sin x \, dx.$

20. $\int_0^{\frac{\pi}{2}} x \sin^2 x \, dx$ 21. $\int_0^1 x \log (x+3) \, dx$

22. $\int_0^1 x^2 \tan^{-1} x \, dx$ 23. $\int_0^{\frac{\pi}{2}} e^x (\sin x + \cos x) \, dx.$

24. $\int_{\frac{\pi}{2}}^{\pi} (x+\sin 2x) \, dx.$

§ 4·2. Substitution of variable in definite integral.

In the last article we have found that to evalute the definite integral of a function, one has to determine its indefinite integral first. You have frequently determined indefinite integrals by substitution of variables; the indefinite integral is finally expressed in terms of the origininal variable. But in case of evaluation of definite integrals, one may not express the result in terms of the original variable. After substitution of the original variable, one may determine the corresponding limits of the new variable and evaluate the definite integral of the new variable between these new limits. This process is also frequently found convenient. Hence in case of evaluation of definite integrals by substitution of variables, the integrand, the differential and the limits of integration all are to be substituted.

Example 1. Evaluate : $\int_0^1 \frac{dx}{\sqrt{3-2x}}$

Let $3-2x=u$ $\therefore$ $-2dx=du$ or, $dx=-\frac{du}{2}$

When $x=0$, then $u=3-2.0=3$

When $x=1$, then $u=3-2.1=1$

$\therefore$ Given Integral$=\int_3^1 \frac{-du}{2\sqrt{u}}=-\frac{1}{2}\int_3^1 \frac{du}{\sqrt{u}}$

$=\frac{1}{2}\int_1^3 \frac{du}{\sqrt{u}}$ [See Note § 4·1–4]

$=\frac{1}{2}\left[2\cdot u^{\frac{1}{2}}\right]_1^3=\left[u^{\frac{1}{2}}\right]_1^3=\sqrt{3}-1\cdot$

Ex. 2. Evaluate : $\int_0^a \sqrt{a^2-x^2}\,dx$

Let $x=a\sin\theta$ $\therefore$ $dx=a\cos\theta\, d\theta$

When $x=0$, then $a\sin\theta=0$ or, $\theta=0$

When $x=a$, then $a\sin\theta=a$ or, $\theta=\frac{\pi}{2}$

and $\sqrt{a^2-x^2}=\sqrt{a^2-a^2\sin^2\theta}=a\cos\theta$

$\therefore$ Given integral $=\int_0^{\frac{\pi}{2}} a^2\cos^2\theta\, d\theta$

Now, $\int a^2\cos^2\theta\, d\theta=a^2\int\cos^2\theta\, d\theta$

$$=\frac{a^2}{2}\int(1+\cos 2\theta)\, d\theta=\frac{a^2}{2}\left(\theta+\frac{\sin 2\theta}{2}\right)$$

$$\therefore \quad \int_0^{\frac{\pi}{2}} a^2\cos^2\theta\, d\theta=\left[\frac{a^2}{2}\left(\theta+\frac{\sin 2\theta}{2}\right)\right]_0^{\frac{\pi}{2}}=\frac{a^2\pi}{4}.$$

Ex. 3. Evalute : $\int_0^1 \frac{3x\, dx}{4-x^2}$

Let $4-x^2=u$ $\quad\therefore\quad -2x\, dx=du$

$\therefore\quad 3x\, dx=-\frac{3}{2}du$

When $x=0$, then $u=4$; when $x=1$, then $u=3$

$$\therefore \quad \text{Given integral}=-\frac{3}{2}\int_4^3 \frac{du}{u}=\frac{3}{2}\int_3^4 \frac{du}{u}=\frac{3}{2}\Big[\log u\Big]_3^4$$

$$=\tfrac{3}{2}(\log 4-\log 3).$$

Ex. 4. $\int_0^1 \frac{e^x dx}{1+e^{2x}}$.

Let $e^x=z$ $\therefore$ $e^x dx=dz$. and $1+e^{2x}=1+z^2$.

When $x=0$, $z=e^0=1$ and when $x=1$, $z=e^1=e$.

$$\therefore \quad \int_0^1 \frac{e^x dx}{1+e^{2x}}=\int_1^e \frac{dz}{1+z^2}=\Big[\tan^{-1} z\Big]_1^e=\tan^{-1} e-\tan^{-1} 1$$

$$=\tan^{-1} e-\frac{\pi}{4}.$$

Ex. 5. Show that $\int_2^6 \sqrt{(6-x)(x-2)}\, dx=2\pi$.

Let $x=6\cos^2\theta+2\sin^2\theta$

$\therefore\quad dx=\{12\cos\theta\,(-\sin\theta)+4\sin\theta\cos\theta\}\, d\theta.$

$\qquad =-8\sin\theta\cos\theta\, d\theta=-4\sin 2\theta\, d\theta.$

$6-x=6-6\cos^2\theta-2\sin^2\theta=6\sin^2\theta-2\sin^2\theta=4\sin^2\theta.$

$x-2=6\cos^2\theta+2\sin^2\theta-2=6\cos^2\theta-2\cos^2\theta=4\cos^2\theta.$

$\therefore\quad \sqrt{(6-x)(x-2)}=4\sin\theta\cos\theta=2\sin 2\theta.$

when $x=2$, then $6\cos^2\theta+2\sin^2\theta=2$

or, $6\cos^2\theta=2-2\sin^2\theta$

or, $6\cos^2\theta=2\cos^2\theta \quad \therefore \quad 4\cos^2\theta=0$

or, $\cos\theta=0 \quad \therefore \quad \theta=\frac{\pi}{2}$.

when $x=6$, then $6=6\cos^2\theta+2\sin^2\theta$

or, $6\sin^2\theta=2\sin^2\theta \quad \therefore \quad \sin^2\theta=0$

or, $\sin\theta=0 \quad \therefore \quad \theta=0$.

Hence given integral

$$=\int_{\frac{\pi}{2}}^{0} 2\sin 2\theta\,(-4\sin 2\theta)\,d\theta=-4\int_{\frac{\pi}{2}}^{0} 2\sin^2 2\theta\,d\theta$$

$$=4\int_{0}^{\frac{\pi}{2}}(1-\cos 4\theta)\,d\theta=4\left[\theta-\frac{\sin 4\theta}{4}\right]_0^{\frac{\pi}{2}}$$

$$=4.\frac{\pi}{2}=2\pi.$$

Ex. 6. Evaluate $\int_{\alpha}^{\beta}\frac{dx}{\sqrt{(x-\alpha)(\beta-x)}}\quad(\beta>\alpha)$.

Let $x-\alpha=z^2 \quad \therefore \quad dx=2z\,dz$

when $x=\alpha$, then $z=0$, when $x=\beta$, then $z=\sqrt{\beta-\alpha}$

$$\therefore \quad \int_{\alpha}^{\beta}\frac{dx}{\sqrt{(x-\alpha)(\beta-x)}}=\int_{0}^{\sqrt{\beta-\alpha}}\frac{2z\,dz}{z\sqrt{\beta-\alpha-z^2}}$$

$$=2\int_{0}^{\sqrt{\beta-\alpha}}\frac{dz}{\sqrt{\beta-\alpha-z^2}}=2\left[\sin^{-1}\frac{z}{\sqrt{\beta-\alpha}}\right]_0^{\sqrt{\beta-\alpha}}$$

$$=2(\sin^{-1}1-\sin^{-1}0)=2.\frac{\pi}{2}=\pi.$$

Ex. 7. Evaluate $\int_{a}^{b}\frac{\log x}{x}\,dx$

Let $\log x=z \quad \therefore \quad \frac{dx}{x}=dz$ and when $x=a$ and b then z $=\log a$ and $\log b$ respectively.

$$\therefore \int_a^b \frac{\log x}{x} dx = \int_{\log a}^{\log b} z\, dz = \left[\frac{z^2}{2}\right]_{\log a}^{\log b}$$

$$= \frac{(\log b)^2 - (\log a)^2}{2} = \tfrac{1}{2} \log (ab) \log \left(\frac{b}{a}\right).$$

Ex. 8. Evaluate $\int_0^{2a} \sqrt{2ax - x}\, dx.$

Let $x = a(1-\cos\theta)$ $\therefore$ $dx = a \sin\theta\, d\theta.$

$$\sqrt{2ax - x^2} = \sqrt{2a^2(1-\cos\theta) - a^2(1-\cos\theta)^2}$$

$= a\sqrt{2 - 2\cos\theta - 1 + 2\cos\theta - \cos^2\theta}$

$= a\sqrt{1-\cos^2\theta} = a \sin\theta.$

When $x=0$, then $\theta=0$; when $x=2a$, then $\theta=\pi$.

$$\therefore \int_0^{2a} \sqrt{2ax - x^2}\, dx = \int_0^{\pi} a^2 \sin^2\theta\, d\theta$$

$$= \frac{a^2}{2}\int_0^{\pi} (1-\cos 2\theta)\, dx = \frac{a^2}{2}\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\pi} = \frac{a^2\pi}{2}.$$

Exercise IVB

Evalute :

1. $\int_0^1 \sqrt{4-3x}\, dx$

2. $\int_0^{\frac{\pi}{2}} \frac{\cos x\, dx}{1+\sin^2 x}$ [G. U. 1970]

3. $\int_0^1 \frac{dx}{\sqrt{4-x^2}}$

4. $\int_0^1 x^3 \sqrt{1+3x^4}\, dx$

5. $\int_0^1 \frac{\sin^{-1} x}{\sqrt{1-x^2}}\, dx$ [C. U.]

6. $\int_{\alpha}^{\beta} \sqrt{(x-\alpha)(\beta-x)}\, dx$ [C. U.]

[Hints : Put $x = \alpha \cos^2\theta + \beta \sin^2\theta$]

7. $\int_1^2 \left(\frac{x^2-1}{x^2}\right) e^{x+\frac{1}{x}}\, dx.$

8. $\int_2^5 \frac{dx}{\sqrt{(x-2)(5-x)}}$

9. $\int_2^3 \frac{dx}{\sqrt{(x-1)(3-x)}}$

10. $\int_0^{\frac{\pi}{2}} \sin^4 x \cos^3 x\, dx$

11. $\int_0^{\frac{1}{2}} \frac{dx}{(1-2x^2)\sqrt{1-x^2}}$ 12. $\int_0^{\frac{\pi}{2}} \frac{\cos x \, dx}{(1+\sin x)(2+\sin x)}$

13. $\int_0^1 \frac{5x \, dx}{(x+2)(x^2+1)}$ 14. $\int_0^3 \frac{dx}{(x+2)\sqrt{x+1}}$

15. $\int_0^1 2e^{-x^2} x \, dx$ 16. $\int_0^1 \frac{\tan^{-1}x}{1+x^2} dx$

17. $\int_e^{e^2} \frac{dx}{x \log x}$ 18. $\int_0^{\frac{\pi}{2}} \frac{\sin 2x \, dx}{a^2\sin^2 x+b^2\cos^2 x}$ $[a^2 \neq b^2]$

19. $\int_1^{e^2} \frac{dx}{x(1+\log x)^2}$ 20. $\int_0^3 \frac{x \, dx}{\sqrt{x+1}+\sqrt{5x+1}}$

§ 4·3. Definition of definite integral as the limit of the sum of a special class of series :

In § 4·1 definite integral of a function has been defined as an area. In this article we shall give a more generalised definition of definite integral. In the next article it will be shown that these definitions are consistent with each other.

Bounded Function : If in an interval $a \leqslant x \leqslant b$, a function $f(x)$ is defined and if there exist two finite numbers M and m such that for all values of $f(x)$ in the interval, $m \leqslant f(x) \leqslant M$ then the function $f(x)$ is said to be bounded in the interval $a \leqslant x \leqslant b$.

If corresponding to every value of x in an interval $a \leqslant x \leqslant b$, one can get one and only one value of $f(x)$, then the function $f(x)$ is said to be single valued in the interval.

Let a and b be two finite quantities and $b > a$. Let $f(x)$ be a bounded, single valued and continuous function of x defined in the interval $a \leqslant x \leqslant b$. Divide the interval $a \leqslant x \leqslant b$ into n equal subintervals each of length h,

$$a \leqslant x \leqslant a+h, \; a+h \leqslant x \leqslant a+2h, \; \dots, \; a+(n-1)h \leqslant x \leqslant a+nh=b$$

$$\therefore \; nh = b-a.$$

$$\underset{h \to 0}{Lt} \; h\{f(a)+f(a+h)+f(a+2h)+\dots+f(a+\overline{n-1}h)\} \qquad (1)$$

or, $\underset{h \to 0}{Lt}\ h \sum_{r=0}^{n-1} f(a+rh)$ is defined as the definite intergal of the function $f(x)$ with respect to x and is written as $\int_a^b f(x)\,dx$.

Note 1. It can be easily proved that

$$\underset{h \to 0}{Lt}\ h\{f(a)+f(a+h)+f(a+2h)+ \quad +f(a+\overline{n-1}h)\} \qquad \cdots(1)$$

$$= \underset{h \to 0}{Lt}\ h\{f(a+h)+f(a+2h)+\cdots+f(a+\overline{n-1}h)+f(a+nh)\}$$

$$\therefore \underset{h \to 0}{Lt}\ h\{f(a+h)+f(a+2h)+\cdots+f(a+\overline{n-1}h)$$

$$+f(a+nh)\} \qquad \cdots(2)$$

$$=\int_a^b f(x)\,dx.$$

Any of the limits (1) and (2) can be taken as the definition of $\int_a^b f(x)\,dx$,

Note 2. As $nh=b-a \quad \therefore \quad h=\dfrac{b-a}{n}$;

$\therefore$ when $h \to 0$, then $n \to \infty$,

§ 4·4. Geometrical Interpretation of the definition of definite Integral as the limit of a sum.

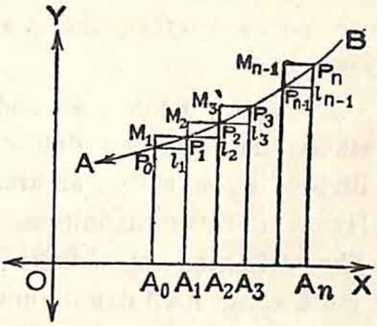
Let a function $f(x)$ be finite and continuous everywhere within the interval $a \leqslant x \leqslant b$ and the curve AB be the graph of the function $y=f(x)$. The ordinates A_0P_0 and A_nP_n at the points $A_0(a, 0)$ and $A_n(b, 0)$ intersect the curve at P_0 and P_n respectively.

Now, $A_0A_n = OA_n - OA_0 = b-a$.

Divide the line segment $\overline{A_0A_n}$ into n equal parts each of length h.

$\therefore \quad nh=b-a, \quad$ or, $\quad a+nh=b.$

Draw perpendiculars to the x-axis at each of the points $(a+h, 0)$, $(a+2h, 0)\cdots$ $\{a+(n-1)h, 0\}$ and complete the rectangles below and above the curve as shown in the figure.



Let A be the measure of the area enclosed by the x-axis, the curve $y=f(x)$ and the ordinates $x=a$ and $x=b$.

Let A_1 and A_2 be respectively the sum of the areas of the lower rectangles and the upper rectangles.

From figure it is evident that $A_1 < A < A_2$ $\cdots(1)$

Now, $A_1=hf(a)+hf(a+h)+\cdots+hf\{a+(n-1)h\}$

$$=h\sum_{r=0}^{n-1} f(a+rh)$$

and $A_2=hf(a+h)+hf(a+2h)+\cdots+hf(a+nh)$

$$=h\sum_{r=0}^{n-1} f(a+rh)-hf(a)+hf(b)$$

Now, if the value of h be very small i.e., $h\to 0$, then n will be very large i.e., n will tend to infinity.

Hence when $n\to\infty$, then as $f(a)$ and $f(b)$ are finite $hf(a)$ and $hf(b)$ will both tend to zero.

$\therefore$ A_1 and A_2 will respectively approach

$$\underset{h\to 0}{Lt}\ h\sum_{r=0}^{n-1} f(a+rh)=\int_a^b f(x)\,dx$$

and $$\underset{h\to 0}{Lt}\ h\sum_{r=0}^{n-1} f(a+rh)=\int_a^b f(x)\,dx$$

Hence from (1) we get,

$$A=\int_a^b f(x)\,dx$$

Hence $\int_a^b f(x)\,dx$ is the measure of the area enclosed by the curve $y=f(x)$, the x-axis and the ordinates $x=a$ and $x=b$.

Note. 1. From § 4·1 and this article, you can now understand that the two definitions of definite integral as the limit of a sum and as an area have same geometical meaning. Hence the two definitions are consistent with each other. The definition of definite integral as the limit of a sum is a more generalised definition than that given as an area.

Example 1. Evaluate from the first principle $\int_0^1 3dx$.

$$\int_0^1 3dx = \underset{n\to\infty}{Lt}\ h\sum_{r=1}^{n} f(rh)\ ,\ \text{where } nh=1-0=1$$

$$= \underset{h\to 0}{Lt}\ h(3+3+3+\cdots\text{to } n \text{ terms})$$

$$= \underset{h\to 0}{Lt}\ (h.\ 3n)=3\ \underset{h\to 0}{Lt}\ (nh)=3\ \underset{h\to 0}{Lt}\ 1.=3.1=3.$$

Ex. 2. Evaluate : $\int_0^1 (ax^2+b)\,dx$.

$$\int_0^1 (ax^2+b)dx = \underset{h\to 0}{Lt}\ \sum_{r=1}^{n}\{a(rh)^2+b\},\ nh=1-0=1$$

$$= \underset{h\to 0}{Lt}\ h\{a(h^2)+a(2h)^2+a(3h)^2+\cdots a(nh)^2\}$$

$$+ \underset{h\to 0}{Lt}\ h\ (b+b+b+\cdots\cdots\text{to } n \text{ terms})\}$$

$$= \underset{h\to 0}{Lt}\ h.ah^2(1^2+2^2+3^2+\cdots+n^2)+\underset{h\to 0}{Lt}\ h(nb)$$

$$= \underset{h\to 0}{Lt}\left\{ah^3\ \frac{n(n+1)(2n+1)}{6}\right\}+b\ \underset{h\to 0}{Lt}\ (nh)$$

$$=\frac{2a}{6}\ \underset{h\to 0}{Lt}\left\{n^3h^3\left(1+\frac{1}{n}\right)\left(1+\frac{1}{2n}\right)\right\}+b\ Lt\ 1. \quad \cdots(1)$$

$$=^{a}\ \underset{h\to 0}{Lt}\ \left\{\left(1+h\right)\left(1+\frac{1}{2}h\right)\right\}+b.1$$

$$=\frac{a}{3}+b.$$

Ex. 3. Find from the first principle or from the definition the value of $\int_0^1 x^3 dx$.

Here $a=0,\ b=1$; $f(x)=x^3$, $nh=b-a=1-0=1$

$\therefore$ By definition

$$\int_0^1 x^3 dx = \underset{h\to 0}{Lt}\ h\{f(a+h)+f(a+2h)+\cdots+f(a+nh)\}$$

$$= Lt\ h\{h^3+2^3h^3+3^3h^3+\cdots+n^3h^3\}\quad [\because\ a=0]$$

$$= \underset{h\to 0}{Lt}\ h^4(1^3+2^3+\cdots+n^3)$$

$$= \underset{h\to 0}{Lt}\ h^4.\frac{n^2(n+1)^2}{4} = \underset{h\to 0}{Lt}\ \frac{n^4h^4\left(1+\frac{1}{n}\right)^2}{4}$$

$$= \underset{n\to\infty}{Lt}\ \frac{\left(1+\frac{1}{n}\right)^2}{4} = \frac{1}{4}.\qquad \left[\because\ nh=1 \text{ and as } h\to 0 \text{ then } n\to\infty\right]$$

Ex. 4. Evaluate from the first principle $\int_a^b e^x dx$. [C.U.]

Here $f(x)=e^x$; $nh=b-a$

Now, $$\int_b^b e^x dx = \underset{h\to 0}{Lt}\ h\left\{e^a+e^{2+h}+e^{a+2h}+\cdots e^{a+(n-1)h}\right\}$$

$$= \underset{h\to 0}{Lt}\ he^x\{1+e^h+e^{2h}+\cdots+e^{(n-1)h}\}$$

$$= \underset{h\to 0}{Lt}\ he^a\ \frac{e^{nh}-1}{e^h-1} = Lt\ he^x\ \frac{e^{b-a}-1}{e^h-1}$$

$$= e^a\left(e^{b-a}-1\right)\underset{h\to 0}{Lt}\ \frac{h}{e^h-1} = \left(e^b-e^a\right)\frac{1}{\underset{h\to 0}{Lt}\ \frac{e^h-1}{h}}$$

$$= (e^b-e^a).1 = e^b-e^a.$$

Ex. 5. Evaluate :

$$\underset{n\to\infty}{Lt}\ \frac{1+2^{10}+3^{10}+\cdots+n^{10}}{n^{11}}$$ [C. U. '58]

$$\underset{n\to\infty}{Lt}\ \frac{1+2^{10}+3^{10}+\cdots+n^{10}}{n^{11}}$$

$$= \underset{n\to\infty}{Lt} \frac{1}{n}\left[\left(\frac{1}{n}\right)^{10} + \left(\frac{2}{n}\right)^{10} + \cdots + \left(\frac{n}{n}\right)^{10}\right]$$

$$= \underset{h\to 0}{Lt} \left[h^{10} + (2h)^{10} + \cdots + (nh)^{10}\right],$$

[where $nh=1$ and so as $n\to\infty$, then $h\to 0$.]

$$= \underset{h\to 0}{Lt}\ h \sum_{r=1}^{n} (rh)^{10} = \int_0^1 x^{10}\,dx = \left[\frac{x^{11}}{11}\right]_0^1 = \frac{1}{11}.$$

Ex. 6. Evaluate :

$$\underset{n\to\infty}{Lt}\left[\frac{n}{n^2+1^2} + \frac{n}{n^2+2^2} + \frac{n}{n^2+3^2} + \cdots + \frac{n}{n^2+n^2}\right]$$

[C. U. '62 : '67]

Given limit

$$= \underset{n\to\infty}{Lt}\left[\frac{\frac{n}{n^2}}{\frac{n^2}{n^2}+\frac{1^2}{n^2}} + \frac{\frac{n}{n^2}}{\frac{n^2}{n^2}+\frac{2^2}{n^2}} + \frac{\frac{n}{n^2}}{\frac{n^2}{n^2}+\frac{3^2}{n^2}} + \quad + \frac{\frac{n}{n^2}}{\frac{n^2}{n^2}+\frac{n^2}{n^2}}\right]$$

$$= \underset{n\to\infty}{Lt} \frac{1}{n}\left\{\frac{1}{1+\frac{1^2}{n^2}} + \frac{1}{1+\frac{2^2}{n^2}} + \cdots + \frac{1}{1+\frac{n^2}{n^2}}\right\}$$

$$= \underset{h\to 0}{Lt}\ h.\left\{\frac{1}{1+h^2} + \frac{1}{1+2^2h^2} + \quad + \frac{1}{1+n^2h^2}\right\}$$

[Let $nh=1=1-0$; so as $n\to\infty$, then $h\to 0$]

$$= \int_0^1 \frac{1}{1+x^2}\,dx \text{ (By def.)}$$

$$= \left[\tan^{-1}x\right]_0^1 = \tan^{-1}1 - \tan^{-1}0 = \frac{\pi}{4}.$$

Ex. 7. Evaluate :

$$\lim_{n\to\infty}\left\{\frac{1^2}{n^3+1^3} + \frac{2^2}{n^3+2^3} + \cdots + \frac{n^2}{n^3+n^3}\right\}$$

[C. U. '63]

$$\text{Given limit} = \lim_{n\to\infty}\left\{\frac{\frac{1^2}{n^3}}{1+\left(\frac{1}{n}\right)^3} + \frac{\frac{2^2}{n^3}}{1+\left(\frac{2}{n}\right)^3} + \cdots + \frac{\frac{n^2}{n^3}}{1+\left(\frac{n}{n}\right)^3}\right\}$$

$$= \lim_{h\to 0}\left\{\frac{1^2h^3}{1+h^3} + \frac{2^2h^3}{1+2^3h^3} + \cdots + \frac{n^2h^3}{1+n^3h^3}\right\}$$

[Putting $nh=1=1-0$]

$$= \lim_{h\to 0} h \left\{ \sum \frac{r^2h^2}{1+r^3h^3} \right\}$$

$$= \int_0^1 \frac{x^2}{1+x^3}\, dx = \tfrac{1}{3}\Big[\log(1+x^3) \Big]_0^1$$

$$= \tfrac{1}{3}(\log 2 - \log 1) = \tfrac{1}{3}\log 2.$$

Exercise IV C

1. Evaluate from the first principle :

(i) $\int_0^1 x^2\, dx$ (ii) $\int_0^1 (ax^2+b)\, dx$ (iii) $\int_0^1 \sqrt{x}\, dx$

(iv) $\int_1^9 \frac{dx}{\sqrt{x}}$ (v) $\int_1^4 \sqrt{x}\, dx.$

2. Evaluate the following limits :

(i) $\underset{n\to\infty}{Lt} \left\{ \frac{1}{\sqrt{n^2-1}} + \frac{1}{\sqrt{n^2-2^2}} + \cdots + \frac{1}{\sqrt{n^2-(n-1)^2}} \right\}$ [C U.]

(ii) $\underset{n\to\infty}{Lt} \left\{ \frac{1}{n} + \frac{n^2}{(n+1)^3} + \frac{n^2}{(n+2)^3} + \cdots + \frac{1}{8n} \right\}$ [C. U.]

(iii) $\underset{n\to\infty}{Lt} \sum_{r=1}^{n} \left(\frac{n+r}{n^2+r^2} \right).$

(iv) $\underset{n\to\infty}{Lt} \frac{1+2^m+3^m+\cdots+n^m}{n^{m+1}} \left[m > -1 \right]$

§ 4·5. Fundamental Theorem of Integral Calculus.

At the very beginning we have indicated that the mutual relationship of indefinite and definite integrals are found in the Fundamental Theorem of Integral Calculus. We now state the theorem without proof. The proof is outside the scope of the syllabus.

Fundamental Theorem of Integral Calculus. If two functions $f(x)$ and $\phi(x)$ be such that the function $f(x)$ is integrable in the interval $a \leqslant x \leqslant b$ and $\phi'(x) = f(x)$ everywhere in the interval, then

$$\int_a^b f(x)\, dx = \phi(b) - \phi(a).$$

Properties of Definite Integrals :

(1) By definition, $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$

(2) $\int_a^b f(x)\,dx = \int_a^b f(z)\,dz$.

Proof : If $\int f(x)\,dx = \phi(x)$, then $\int f(z)\,dz = \phi(z)$.

Now, $\int_a^b f(x)\,dx = \phi(b) - \phi(a)$ and $\int_a^b f(z)\,dz = \phi(b) - \phi(a)$

$\therefore \int_a^b f(x)\,dx = \int_a^b f(z)\,dz.$

(3) If $a < c < b$

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

Proof : Let $\int f(x)\,dx = \phi(x)$.

$\therefore \int_a^b f(x)\,dx = \phi(b) - \phi(a)$;

$\int_a^c f(x)\,dx = \phi(c) - \phi(a)$ and $\int_c^b f(x)\,dx = \phi(b) - \phi(c)$.

$\therefore \int_a^b f(x)\,dx = \phi(b) - \phi(a)$

$= \{\phi(c) - \phi(a)\} + \{\phi(b) - \phi(c)\}$

$= \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$

Cor. : If $a < c_1 < c_2 < \cdots\cdots < c_n < b$, then

$$\int_a^b f(x)\,dx = \int_a^{c_1} f(x)\,dx + \int_{c_1}^{c_2} f(x)\,dx + \cdots\cdots + \int_{c_{n-1}}^{c_n} f(x)\,dx + \int_{c_n}^{b} f(x)\,dx$$

(4) $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx.$

Proof : Let $a-x=z \quad \therefore \quad -dx=dz$.

Again, when $x=0$, then $z=a$, when $x=a$, then $z=0$.

$$\therefore \quad \int_0^a f(a-x)\,dx=-\int_a^0 f(z)\,dz$$

$$=\int_0^a f(z)\,dz \text{ [By (1)]}=\int_0^a f(x)\,dx. \qquad \text{[By (2)]}$$

(5) If $f(x)=f(a+x)$ then

$$\int_0^{na} f(x)\,dx=n\int_0^a f(x)\,dx.$$

Proof : Let $x=a+z \quad \therefore \quad dx=dz$.

when $x=a$, then $z=0$, when $x=2a$ then $z=a$.

$$\therefore \quad \int_0^{2a} f(x)\,dx=\int_0^a f(a+z)\,dz=\int_0^a f(a+x)\,dx \text{ [By (2)]}$$

$$=\int_0^a f(x)\,dx \quad [\because \; f(a+x)=f(x)\,]$$

Similarly, $\displaystyle\int_{2a}^{3a} f(x)\,dx=\int_a^{2a} f(x)\,dx=\int_0^a f(x)\,dx.$

$$\int_{3a}^{4a} f(x)\,dx=\int_{2a}^{3a} f(x)\,dx=\int_0^a f(x)\,dx$$

$$\int_{(n-1)a}^{na} f(x)\,dx=\int_{(n-2)a}^{(n-1)a} f(x)\,dx=\int_0^a f(x)\,dx.$$

Now, $\displaystyle\int_0^{na} f(x)\,dx=\int_0^a f(x)\,dx+\int_a^{2a} f(x)\,dx+$

$$\int_{2a}^{3a} f(x)\,dx+\cdots+\int_{(n-1)a}^{na} f(x)\,dx$$

$$=\int_0^a f(x)\,dx+\int_0^a f(x)\,dx+\cdots+\int_0^a f(x)\,dx.$$

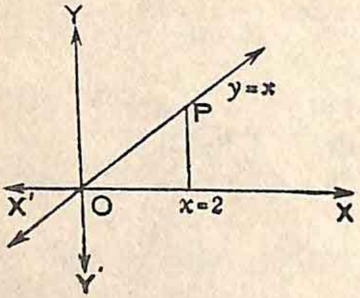
$$=n\int_0^a f(x)\,dx.$$

§ 4·6. Determination of Area.

In the first article of this chapter it has been shown that the area of the space enclosed by the x-axis, the two ordinates $x=a$ and $x=b$ $(a<b)$ and the curve $y=f(x)$ is

$$\int_a^b f(x)\,dx \text{ or } \int_a^b y\,dx.$$

Similarly, the area enclosed by the y-axis, the ordinates $y=c$ and $y=d$ and the curve $x=f(y)$ is $\int_c^d f(y)\,dy$ or $\int_c^d x\,dy$.



Ex. 1. Determine by integration the measure of the area enclosed by the x-axis, the y-axis, the ordinate $x=2$ and the straight line $y=x$.

Here, $f(x)=x$. The limits of x are from 0 to 2. Hence the required area

$$=\int_0^2 y\,dx=\int_0^2 f(x)\,dx=\int_0^2 x\,dx=\left[\frac{x^2}{2}\right]_0^2=2 \text{ sq. units.}$$

Ex. 2. Find the area enclosed by the straight line $\frac{x}{a}+\frac{y}{b}=1$ and the axes of co-ordinates and justify the formula, "the area of a triangle $=\frac{1}{2}\times$base$\times$height sq. units."

$$\frac{x}{a}+\frac{y}{b}=1, \quad \text{or,} \quad y=b\left(1-\frac{x}{a}\right)=\frac{b}{a}(a-x)$$

The limits of x in the area enclosed by the straight line and the axes of co-ordinates are from 0 to a

Hence the required area

$$=\int_0^a y\,dx=\int_0^a \frac{b}{a}(a-x)\,dx=\frac{b}{a}\left[ax-\frac{x^2}{2}\right]_0^a$$

$$=\frac{b}{a}\left[a^2-\frac{a^2}{2}\right]=\frac{b}{a}\cdot\frac{a^2}{2}=\frac{1}{2}ab \text{ sq. units.}$$

Now, if the straight line $\frac{x}{a}+\frac{y}{b}=1$ intersects the the axes of co-ordinates at the points A and B, then the area is $\triangle$ABC.

Now from the formula,

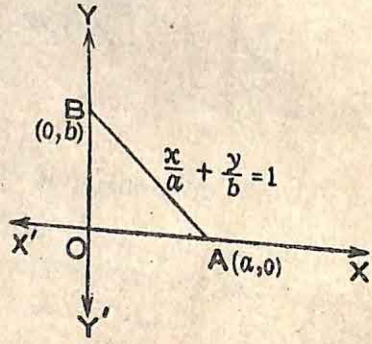
area of a triangle $=\frac{1}{2}\times$base$\times$height, we obtain

$$m\triangle \text{ABC}=\tfrac{1}{2}\text{OA}\cdot\text{OB}$$

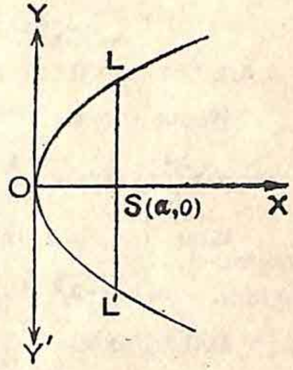
Now the co-ordinates of A are $(a, 0)$ and those of B are $(0, b)$. So, OA$=a$, OB$=b$.

$\therefore\ m\triangle$ABC$=\frac{1}{2}ab$ sq. units.

Hence the formula is verified.



Ex. 3. Determine the area enclosed by a parabola and its latus-rectum.



Let the equation of the parabola be $y^2=4ax$.

The equation of the latus-rectum of the parabola is $x=a$ and it intersects the parabola at the points $(a, 2a)$ and $(a, -2a)$ and the x-axis at the point $(a, 0)$.

Now we are to determine the area LOL'SL. You know that a parabola is symmetrical about its axis (which is the x-axis in this case). Hence the required area = twice the area LOSL *i.e.*. $2m$ a ea (LOSL).

Now, the area LOSL is the area enclosed by the x-axis, the parabola $y^2=4ax$ and the ordinates $x=0$ and $x=a$.

Hence m area(LOSL) $=\int_0^a y\,dx$.

Now, for the portion of the parabola above the x-axis, $y=2\sqrt{ax}$.

$$\therefore \quad \text{Required area}=2\int_0^a y\,dx=2\int_0^a 2\sqrt{ax}\,dx$$

$$=4\sqrt{a}\left[\frac{2}{3}x^{\frac{3}{2}}\right]_0^a=4\sqrt{a}\cdot\tfrac{2}{3}a^{\frac{3}{2}}=\tfrac{8}{3}a^2$$

Ex. 4. Prove that the area enclosed by the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is πab sq. units.

Ellipse is symmetrical about both its major and minor axes. Hence the area of an ellipse is four times the area of a quadrant of the ellipse.

So, let us first determine the area of the first quadrant of the ellipse.

$$\text{Now} \quad y^2=b^2\left(1-\frac{x^2}{a^2}\right)=\frac{b^2}{a^2}\left(a^2-x^2\right)$$

$$\therefore \quad y=\pm\frac{b}{a}\sqrt{a^2-x^2}.$$

For, the first quadrant y is positive.

$\therefore\ y=\frac{b}{a}\sqrt{a^2-x^2}$. Again, for this quadrant the limits of x are between 0 and a.

Hence the area of the first quadrant of the ellipse is

$$A=\int_0^a y\,dx=\int_0^a \frac{b}{a}\sqrt{a^2-x^2}\,dx=\frac{b}{a}\int_0^a \sqrt{a^2-x^2}\,dx.$$

Now, let $x=a\sin\theta$. $\therefore\ dx=a\cos\theta\,d\theta$.

$$\sqrt{a^2-x^2}=\sqrt{a^2-a^2\sin^2\theta}=a\cos\theta$$

and when $x=0$, then $\theta=0$; when $x=a$, than $\theta=\frac{\pi}{2}$.

$$\therefore\ A=\frac{b}{a}\int_0^{\frac{\pi}{2}} a^2\cos^2\theta\,d\theta=ab\int_0^{\frac{\pi}{2}}\cos^2\theta\,d\theta.$$

$$=ab\int_0^{\frac{\pi}{2}}\tfrac{1}{2}(1+\cos 2\theta)\,d\theta$$

$$=\frac{ab}{2}\left[\theta+\frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{2}}=\frac{ab}{2}\cdot\frac{\pi}{2}=\frac{\pi ab}{4}\text{ sq. units.}$$

$\therefore$ The measure of the whole area enclosed by the ellipse $=4A=4.\ \frac{\pi ab}{4}=\pi ab$ sq. units.

Ex. 5. The radius of a circle is r. Show that the area enclosed by the circle is πr^2.

Let the centre of the circle be the origin and its equation be $x^2+y^2=r^2$.

Now, for the first quadrant of the circle, $y=\sqrt{a^2-x^2}$ and the limits of x are between 0 and r.

$$\therefore\ \text{Required area}=4\int_0^r y\,dx=4\int_0^r \sqrt{a^2-x^2}\,dx$$

$$=4.\pi\frac{r^2}{4}=\pi r^2.$$

Note. If we put $b=a$ in the integral $4\int_0^a \frac{b}{a}\sqrt{a^2-x^2}\,dx$ for determination of the area of an ellipse, we get the area of the circle of radius a.

Ex. 6. Find the area enclosed between the hyperbola $xy=k^2$, the x-axis and the ordinates $x=a$ and $x=b$.

Here the required area $=\int_a^b y\,dx=\int_a^b \frac{k^2}{x}dx$

$$=k^2\Big[\log x\Big]_a^b=k^2\Big(\log b-\log a\Big)=k^2\log\frac{b}{a}.$$

Ex. 7. Find the area enclosed by the x-axis and an wave of the curve $y=\sin x$.

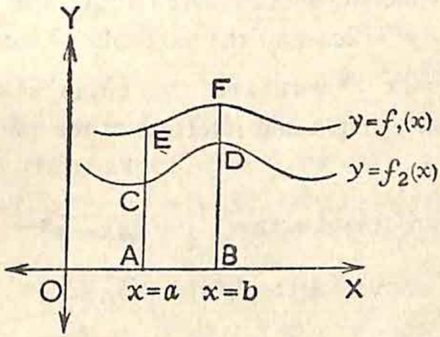
The curve $y=\sin x$ intersects the x-axis at the points $(n\pi, 0)$ $[n=0, \pm 1, \pm 2, \cdots]$, $(0, 0)$ and $(0, \pi)$ are two conseoutive points of intersection. Hence it is convenient to find the area of the portion between these points.

Here the required area $=\int_0^\pi y\,dx=\int_0^\pi \sin x\,dx$

$$=\Big[-\cos x\Big]_0^\pi=-\cos\pi+\cos 0=-(-1)+1=2 \text{ sq. units.}$$

Ex. 8. Find the measure of the area enclosed by the curves $y=f_1(x)$, $y=f_2(x)$ and the two ordinates $x=a$ and $x=b$.

Let A and B be the two points $(a, 0)$ and $(b, 0)$ and the



ordinates at these points intersect the curve $y=f_1(x)$ at the points E and F and the curve $y=f_2(x)$ at the points C and D.

Now, measure of the area CDFE

$=m$(Area ABFE)$-m$(Area ABDC)

$$=\int_a^b f_1(x)\,dx-\int_a^b f_2(x)\,dx$$

$$=\int_a^b \{f_1(x)-f_2(x)\}\,dx.$$

Ex. 9. Find the area enclosed by the parabola $y^2=4ax$ and the straight line $y=x$

Putting $y=x$ in the equation of the parabola we get, $x^2=4ax$, $\therefore$ $x=0$ or $4a$.

So, $y=0$ or $4a$.

So, the straight line $y=x$ intersects the parabola $y^2=4ax$ at the points $(0, 0)$ and $(4a, 4a)$.

Hence the required area

$=\int_0^{4a} (\sqrt{4ax}-x)dx$ [$\because$ from the equation of the

parabola $y=\sqrt{4ax}$.]

$=\int_0^{4a} 2\sqrt{a}\sqrt{x}\, dx-\int_0^{4a} x\, dx$

$=2\sqrt{a}\frac{2}{3}\left[x^{\frac{3}{2}}\right]_0^{4a}-\left[\frac{x^2}{2}\right]_0^{4a}=\frac{4\sqrt{a}}{3}\cdot(8a\sqrt{a})-8a^2$

$=\frac{32a^2}{3}-8a^2=\frac{8}{3}a^2$ square units.

Ex. 10. Find the area enclosed above the x-axis between the circle $x^2+y^2=2ax$ and the parabola $y^2=ax$.

The parabola $y^2=ax$ and the circle $x^2+y^2=2ax$ touch each other at the origin and intersect at the point (a, a) above the x-axis.

Hence the required area $=\int_0^a \{\sqrt{2ax-x^2}-\sqrt{ax}\}\, dx.$

[See Ex. 8 above ; Here $f_1(x)=\sqrt{2ax-x^2}$

and $f_2(x)=\sqrt{ax}$]

$=\int_0^a \sqrt{2ax-x^2}\, dx-\int_0^a \sqrt{a}\sqrt{x}\, dx.$

Now, to determine $\int_0^a \sqrt{2ax-x^2}\, dx$ put $x=a\,(1-\cos\theta)$

$\therefore$ $dx=a\sin\theta\, d\theta.$

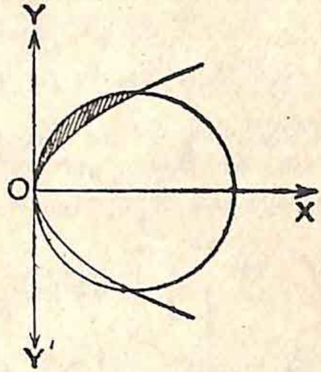
$\sqrt{2ax-x^2}=\sqrt{2a^2(1-\cos\theta)-a^2(1-\cos\theta)^2}$

$=a\sqrt{2-2\cos\theta-1+2\cos\theta-\cos^2\theta}$

$=a\sqrt{1-\cos^2\theta}=a\sin\theta.$

$\therefore$ when $x=0$, then $\theta=0$

and when $x=a$, then $\theta=\frac{\pi}{2}$.



$$\therefore \int_0^a \sqrt{2ax-x^2}\,dx$$

$$=\int_0^{\frac{\pi}{2}} a^2\sin^2\theta\,d\theta.$$

$$=a^2\int_0^{\frac{\pi}{2}} \tfrac{1}{2}(1-\cos 2\theta)\,d\theta.$$

$$=\frac{a^2}{2}\left[\theta-\frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{2}}=\frac{\pi a^2}{4}.$$

and $\int_0^a \sqrt{x}\,dx=\left[\frac{2}{3}x^{\frac{3}{2}}\right]_0^a=\frac{2}{3}a^{\frac{3}{2}}$

$\therefore \sqrt{a}\int_0^a \sqrt{x}\,dx=\sqrt{a}.\frac{2}{3}a^{\frac{3}{2}}=\frac{2}{3}a^2.$

Hence required area $=\frac{\pi a^2}{4}-\frac{2}{3}a^2=a^2\left(\frac{\pi}{4}-\frac{2}{3}\right)$.

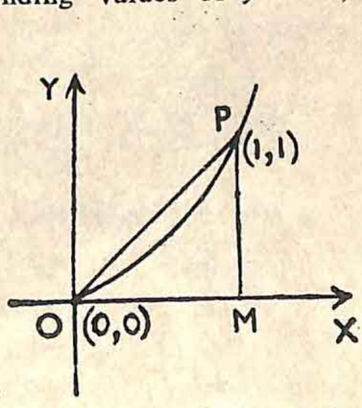
Ex. **11.** Find the measure of the area enclosed by the curve $y^2=x^3$ and the straight line $y=x$.

Putting $y=x$ in the equation $y^2=x^3$, we get

$$x^2=x^3 \quad \text{or,} \quad x^3-x^2=0 \quad \text{or,} \quad x^2(x-1)=0.$$

$\therefore$ $x=0, 0, 1$. The corresponding values of y are 0, 0 and ± 1.

But, $x=1$, $y=-1$ does not satisfy the equation $y=x$. Hence the straight line $y=x$ touches the curve $y^2=x^3$ at the point (0, 0) and intersects at the point (1, 1).



Let P be the point (1, 1) and $\overline{PM}$ be perpendicular on the axis of x.

Hence required area

=(Area of $\triangle$OPM)−(the area enclosed by the curve $y^2=x^3$, the x-axis and the ordinates $x=0$ and $x=1$)$=\triangle-$A (say).

Now $\triangle=\frac{1}{2}$OM.PM sq. units$=\frac{1}{2}.1.1=\frac{1}{2}$ sq. units

and A$=\int_0^1 x^{\frac{3}{2}}\,dx$ $\left[\text{for the portion of the curve above the } x\text{-axis } y=+x^{\frac{3}{2}}\right]$

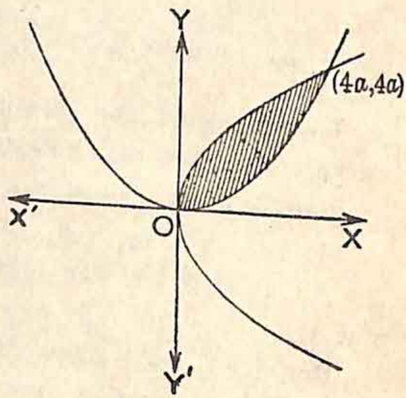
$$\left[\frac{x^{\frac{5}{2}}}{5}\right]_0^1=\tfrac{2}{5}.1 \text{ sq. units}=\tfrac{2}{5} \text{ sq. units.}$$

$\therefore$ required area$=\frac{1}{2}-\frac{2}{5}=\frac{1}{10}$ sq. units.

Ex. 12. Show that the area enclosed between the curves $y^2=4ax$ and $x^2=4ay$ is $\frac{16}{3}a^2$ [C. U. 1928]

The curve $y^2=4ax$ and $x^2=4ay$ intersect each other at the points (0, 0) and ($4a$, $4a$).

Hence the required area=(Area enclosed by the curve $y^2=4ax$, the x-axis and the ordinates $x=0$, $x=4a$)−(Area enclosed by the curve $x^2=4ay$, the x-axis and the ordinates $x=0$, $x=4a$)



$$=\int_0^{4a}\left(2\sqrt{ax}-\frac{x^2}{4a}\right)dx$$

$$=2\sqrt{a}\cdot\tfrac{2}{3}\left[x^{\frac{3}{2}}\right]_0^{4a}-\frac{1}{4a}\left[\frac{x^3}{3}\right]_0^{4a}$$

$$=2\sqrt{a}\cdot\tfrac{2}{3}8a^{\frac{3}{2}}-\frac{1}{4a}\,\frac{64a^3}{3}$$

$$=\frac{32}{3}a^2-\frac{16a^2}{3}=\frac{16a^2}{3} \text{ square units.}$$

§ 4·7. Sign of an area.

In examples 4 and 5 of the last section in determining areas of an ellipse and a circle we have first determined the areas lying in the first quadrant. As the areas are situated in the region above the x-axis, the area have been found positive.

As the ellipse or circle are symmetrical about both the axes of Co-ordinates, the areas enclosed by the ellipse or circle are obtained by multiplying the corresponding areas in the first quadrant by 4. If an area is completely below the x-axis, then as $y=f(x)$ is negative, the area will be negative. But actually area has no regard to sign. So, though from the algebraic point of view one can obtain negative area, actually we do not attach any sign to areas. Follow the following examples.

Example 1. Find the area enclosed by the parabola

$y=x^2-5x+6$ and the x-axis.

The parabola $y=x^2-5x+6$ intersects the x-axis at points $(2, 0)$ and $(3, 0)$. In the figure the shaded region is the area between the parabola and the x-axis.

The region is situated wholly below the x-axis.

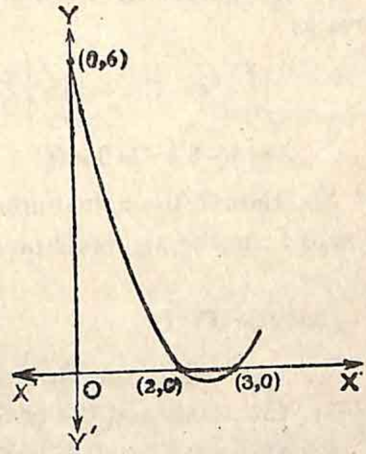
Now the required area

$$=\int_2^3 (x^2-5x+6)\, dx$$

$$=\left[\frac{x^3}{3}-5\frac{x^2}{2}+6x\right]_2^3$$

$$=\left(9-\frac{45}{2}+18\right)-\left(\frac{8}{3}-10+12\right)$$

$$=\tfrac{9}{2}-\tfrac{14}{3}=-\tfrac{1}{6}.$$



Here the area is found to be negative and this is correct according to the algebraic point of view. But actually the area is without sign and it is $\frac{1}{6}$ square units.

Ex. 2. Determine the area enclosed between the curve

$y=x(x-1)(x-2)$ and the x-axis.

The curve $y=x(x-1)(x-2)$ intersects the x-axis at the points $(0, 0)$, $(1, 0)$ and $(2, 0)$. The total area enclosed between the x-axis and the curve has two portions (i) the portion OAB situated wholly above the x-axis and (ii) the portion BCD situated wholly below the x-axis.

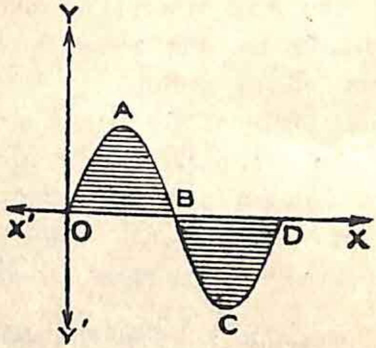
Now the area of the region OAB

$=\int_0^1 x(x-1)(x-2)\,dx$

$=\int_0^1 (x^3-3x^2+2x)\,dx$

$=\left[\frac{x^4}{4}-x^3+x^2\right]_0^1$

$=\frac{1}{4}-1+1=\frac{1}{4}$ square units



The area of the region BCD

$=\int_1^2 x(x-1)(x-2)\,dx=\left[\frac{x^4}{4}-x^3+x^2\right]_1^2=-\frac{1}{4}$ square units.

If we now neglect the sign of this portion BCD below the x-axis then its actual area is $\frac{1}{4}$ square units and the total required area is $(\frac{1}{4}+\frac{1}{4})=\frac{1}{2}$ square unit.

Now integrating from $x=0$ to $x=2$ we get the required area as

$$\int_0^2 x(x-1)(x-2)=\left[\frac{x^4}{4}-x^3+x^2\right]_0^2$$

$$=4-8+4=0=(\tfrac{1}{4})+(-\tfrac{1}{4}).$$

So, though the actual area of the region is $\frac{1}{2}$ square units it is zero form the algebraic point of view.

Exercise IV D

1. Determine the area enclosed between the straight line $y=3x$, the x-axis and the ordinates $x=1$ and $x=2$.

2. Determine the area enclosed between the straight line $y=-x$, the x-axis and the ordinates $x=1$ and $x=2$.

Determine the area enclosed by the following

3. The curve $y=2x+3x^2$, $y=0$, $x=0$ and $x=4$.

4. $y=\cos x$, $y=0$, $x=0$ and $x=\frac{\pi}{2}$.

5. The curve $y=\frac{1}{x^2}$, the x-axis and the ordinates $x=1$ and $x=2$.

6. The curve $y=\sin 2x$, $y=0$, $x=0$ and $x=\frac{\pi}{6}$

7. The curve $y=\sin x+\cos x$, the x-axis and the ordinates $x=0$ and $x=\frac{\pi}{4}$.

8. The x-axis and the curve $y=(x-1)(x-2)$.

9. The area enclosed by the parabola $y^2=4ax$ and the double ordinate $x=h$.

10. The portions of the curve $y=(x-2)(x-3)$ intercepted by the x-axis.

11. Prove that the area enclosed by $\sqrt{x}+\sqrt{y}=a$ and the x-axis is $\frac{1}{6}a^2$.

12. Find the area enclosed between the parabola $y^2+x-1=0$ and the circle $x^2+y^2=1$

13. Find the area common to the two ellipses $ax^2+by^2=1$ and $bx^2+ay^2=1$.

14. Find the area enclosed by the parabolas $y^2=4x$ and $y^2=x$ and the ordinates $x=1$ and $x=2$. [Burdwan 1970]

15. (*a*) Find the measure of the area enclosed by the parabolas $y^2=6x$ and $x^2=6y$. [C. U. 1972]

(*b*) Prove that the curves $y^2=4x$ and $x^2=4y$ divide the square formed by $y=0$, $y=4$, $x=0$ and $x=4$ into three equal parts.

16. Find the area enclosed by the curve $y=x^3$ and the straight line $y=2x$.

17. Find the areas enclosed between the curve $y=\cos x$, the x-axis (*i*) between the ordinates $x=0$ and $x=\frac{\pi}{2}$ (*ii*) between the ordinates $x=\frac{\pi}{2}$ and $x=\pi$ and (*iii*) between the ordinates $x=0$ and $x=\pi$.

18. Find the areas enclosed between the curve $y=\sin x$, the x-axis (*i*) between the ordinates $x=0$ and $x=\pi$ (*ii*) between the ordinates $x=\pi$ and $x=2\pi$ and (*iii*) between the ordinates $x=0$ and $x=2\pi$.

Examples 4

Example 1. Evalute : $\int_0^1 \frac{1-x}{1+x}dx$.

$$\int\frac{1-x}{1+x}dx=\int\left(\frac{2}{1+x}-1\right)dx=2\int\frac{dx}{1+x}-\int dx$$

$$=2\log(1+x)-x$$

$$\therefore \quad \int_0^1 \frac{1+x}{1-x}\,dx=\Big[2\log(1+x)-x\Big]_0^1=2\log 2-1$$

Ex. 2. Evaluate : $\int_0^{\frac{\pi}{2}} \sin^6\theta\cos^3\theta\,d\theta$.

Let, $\sin\theta=x$. $\therefore$ $\cos\theta\,d\theta=dx$.

when $\theta=0$, then $x=0$; when $\theta=\frac{\pi}{2}$ then $x=\sin\frac{\pi}{2}=1$.

$\sin^6\theta=x^6$, $\cos^2\theta=1-\sin^2\theta=1-x^2$.

$$\therefore \quad \int_0^{\frac{\pi}{2}} \sin^6\theta\cos^3\theta\,d\theta=\int_0^{\frac{\pi}{2}} \sin^6\theta\cos^2\theta\cos\theta\,d\theta$$

$$=\int_0^1 x^6(1-x^2)dx=\int_0^1 (x^6-x^8)\,dx.$$

$$=\left[\frac{x^7}{7}-\frac{x^9}{9}\right]_0^1=\tfrac{1}{7}-\tfrac{1}{9}=\tfrac{2}{63}.$$

Ex. 3. Evaluate from the first principle

$$\int_{-1}^1 \frac{2x+3}{4}dx \qquad \text{[C. U. 1964]}$$

$$\int_{-1}^1 \frac{2x+3}{4}dx=\int_{-1}^1 \tfrac{1}{2}x\,dx+\int_{-1}^1 \tfrac{3}{4}dx$$

$$= \lim_{n\to\infty} \frac{h}{2}\Big[(-1)+(-1+h)+(-1+2h)+\cdots\cdots+ \{-1+(n-1)h\}\Big]$$

$$+\lim_{n\to\infty} h\left[\tfrac{3}{4}+\tfrac{3}{4}+\cdots+\tfrac{3}{4}\right]$$

[Taking $nh=1-(-1)=2$]

$$= \lim_{n\to\infty} \tfrac{1}{2}\left[-nh+h\{h+2h+\cdots+(n-1)h\right]+\lim_{n\to\infty} \tfrac{3}{4}nh$$

$$= \underset{n\to\infty}{Lt} \left[-\frac{nh}{2}+\tfrac{1}{4}h^2n(n-1)\right]+\tfrac{3}{2}. \quad [\because \ nh=2]$$

$$= \underset{n\to\infty}{Lt} \ [-\tfrac{2}{2}+\tfrac{1}{4}nh\ nh(1-\tfrac{1}{n})]+\tfrac{3}{2}$$

$$= \underset{n\to\infty}{Lt}\ (-1)+ \underset{n\to\infty}{Lt}\ \tfrac{1}{4}.2.2(1-\tfrac{1}{n})+\tfrac{3}{2}$$

$$=-1+1+\tfrac{3}{2}=\tfrac{3}{2}.$$

Ex. 4. Evaluate from the definition : $\int_a^b x^2\, dx$

$$\int_a^b x^2dx= \underset{h\to 0}{Lt}\ h[a^2+(a+h)^2+(a+2h)^2 + \cdots+(a+\overline{n-1}h)^2]$$

$$= \underset{h\to 0}{Lt}\ h[na^2+2ah\{1+2+\cdots+(n-1)\} + h^2\{1^2+2^2+\cdots+(n-1)^2\}]$$

$$= \underset{h\to 0}{Lt}\ h\left[na^2+2ah\frac{n(n-1)}{2}+h^2\frac{n(n-1)(2n-1)}{6}\right]$$

$$= \underset{h\to 0}{Lt}\ \left\{nha^2+anh(nh-h) + \frac{nh(nh-1)(2n-1)h}{6}\right\}$$

$$= \underset{h\to 0}{Lt}\ \left\{(b-a)a^2+a(b-a)(b-a-h) + \frac{(b-a)(b-a-h)(2b-2a-h)}{6}\right. \quad [\because \ nh=b-a]$$

$$=(b-a)a^2+a(b-a)^2+\frac{(b-a)^2\ 2(b-a)}{6}$$

$$=(b-a)a^2+a(b-a)^2+\tfrac{1}{3}(b-a)^3$$

$$=a(b-a)(a+b-a)+\tfrac{1}{3}(b-a)^3$$

$$=\tfrac{1}{3}[(b-a)^3+3ab(b-a)]$$

$$=\tfrac{1}{3}(b^3-a^3).$$

Ex. 5. Evaluate : $\int_0^{\frac{\pi}{2}} \frac{\sin^2 x\, dx}{\sin x+\cos x}$

Let $I=\int_0^{\frac{\pi}{2}} \frac{\sin^2 x\, dx}{\sin x+\cos x}$

$$=\int_0^{\frac{\pi}{2}} \frac{\sin^2\left(\frac{\pi}{2}-x\right)dx}{\sin\left(\frac{\pi}{2}-x\right)+\cos\left(\frac{\pi}{2}-x\right)}=\int_0^{\frac{\pi}{2}} \frac{\cos^2 x\, dx}{\cos x+\sin x}$$

$\therefore\quad 2I=I+I$

$$=\int_0^{\frac{\pi}{2}} \frac{\sin^2 x\, dx}{\sin x+\cos x}+\int_0^{\frac{\pi}{2}} \frac{\cos^2 x\, dx}{\cos x+\sin x}$$

$$=\int_0^{\frac{\pi}{2}} \frac{\sin^2 x+\cos^2 x}{\sin x+\cos x}dx$$

$$=\int_0^{\frac{\pi}{2}} \frac{dx}{\sin x+\cos x}=\int_0^{\frac{\pi}{2}} \frac{\frac{1}{\sqrt{2}}dx}{\sin x\cos\frac{\pi}{4}+\cos x\sin\frac{\pi}{4}}$$

$$=\frac{1}{\sqrt{2}}\int_0^{\frac{\pi}{2}} \frac{dx}{\sin\left(\frac{\pi}{4}+x\right)}$$

$$=\frac{1}{\sqrt{2}}\int_0^{\frac{\pi}{2}} \operatorname{cosec}\left(\frac{\pi}{4}+x\right)dx$$

$$=\frac{1}{\sqrt{2}}\left[\log\tan\left(\frac{\pi}{8}+\frac{x}{2}\right)\right]_0^{\frac{\pi}{2}}$$

$$=\frac{1}{\sqrt{2}}\left[\log\tan\frac{3\pi}{8}-\log\tan\frac{\pi}{8}\right]$$

$$=\frac{1}{\sqrt{2}}\left[\log\frac{\cot\left(\frac{\pi}{2}-\frac{3\pi}{8}\right)}{\tan\frac{\pi}{8}}\right]$$

$$=\frac{1}{\sqrt{2}}\log\frac{\cot\left(\frac{\pi}{2}-\frac{\pi}{2}+\frac{\pi}{8}\right)}{\tan\frac{\pi}{8}}=\frac{1}{\sqrt{2}}\log\frac{\cot\frac{\pi}{8}}{\tan\frac{\pi}{8}}$$

$$=\frac{1}{\sqrt{2}}\log\frac{\cos^2\frac{\pi}{8}}{\sin^2\frac{\pi}{8}}=\frac{1}{\sqrt{2}}\log\frac{\frac{1}{2}\left(1+\cos\frac{\pi}{4}\right)}{\frac{1}{2}\left(1-\cos\frac{\pi}{4}\right)}$$

$$=\frac{1}{\sqrt{2}}\log\frac{(\sqrt{2}+1)^2}{2-1}=\frac{1}{\sqrt{2}}\,2\log(\sqrt{2}+1)$$

$$\therefore\quad I=\frac{1}{\sqrt{2}}\log(\sqrt{2}+1).$$

Ex. 6. Prove that $\int_0^{\frac{\pi}{4}}\log(1+\tan\theta)\,d\theta=\frac{\pi}{8}\log 2$

Let, $I=\int_0^{\frac{\pi}{4}}\log(1+\tan\theta)\,d\theta$

$$\therefore\quad I=\int_0^{\frac{\pi}{4}}\log\left\{1+\tan\left(\frac{\pi}{4}-\theta\right)\right\}d\theta$$

Now, $1+\tan\left(\frac{\pi}{4}-\theta\right)=1+\frac{1-\tan\theta}{1+\tan\theta}=\frac{2}{1+\tan\theta}$

$$\therefore\quad I=\int_2^{\frac{\pi}{4}}\log\frac{2}{1+\tan\theta}\,d\theta=\int_0^{\frac{\pi}{4}}\left\{\log 2-\log(1+\tan\theta)d\theta\right\}$$

$$=\int_0^{\frac{\pi}{4}}\log 2-\int_0^{\frac{\pi}{4}}\log(1+\tan\theta)\,d\theta=\frac{\pi}{4}\log 2-I$$

$\therefore\quad 2I=\frac{\pi}{4}\log 2$ (on transposition)

$$\therefore\quad I=\frac{\pi}{8}\log 2.$$

Ex 7. Prove that : $\int_0^{\frac{\pi}{2}}(a\cos^2x+b\sin^2x)\,dx=(a+b)\frac{\pi}{4}$

Let $I=\int_0^{\frac{\pi}{2}}(a\cos^2x+b\sin^2)\,dx$

$$=\int_0^{\frac{\pi}{2}} \left\{a \cos^2\left(\frac{\pi}{2}-x\right)+b \sin^2\left(\frac{\pi}{2}-x\right)\right\}dx$$

$$=\int_0^{\frac{\pi}{2}} (a \sin^2 x+b \cos^2 x)dx$$

$$\therefore \quad 2\text{I}=\text{I}+\text{I}=\int_0^{\frac{\pi}{2}} (a \cos^2 x+b \sin^2 x)dx$$

$$+\int_0^{\frac{\pi}{2}} (a \sin^2 x+b \cos^2 x)dx$$

$$=\int_0^{\frac{\pi}{2}} \{(a \cos^2 x+b \sin^2 x)+(a \sin^2 x+b \cos^2 x)\}dx$$

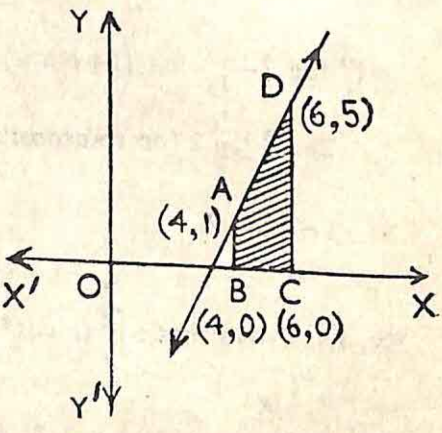
$$=\int_0^{\frac{\pi}{2}} \{a(\cos^2 x+\sin^2 x)+b(\sin^2 x+\cos^2 x)\}dx$$

$$=\int_0^{\frac{\pi}{2}} (a+b)\ dx=\Big[(a+b)\ x\Big]_0^{\frac{\pi}{2}}=(a+b)\frac{\pi}{2}.$$

$$\therefore \quad \text{I}=(a+b)\frac{\pi}{4}.$$

Ex. 8. Determine the measure of the area enclosed by the straight line $y=2x-7$, x-axis and the ordinates $x=4$ and $x=6$. Also verify the formula : the area of a trapezium $=\frac{1}{2}$(sum of the lengths of the parallel sides) $\times$ (distance between the parallel sides).

The two sides $x=4$ and $x=6$ of the quadrilateral enclosed by the four given straight lines are parallel and the sides $y=2x-7$ and $y=0$ are intersecting. Hence the quadrilateral is a trapezium.



Now, the area of the trapezium

$$=\int_4^6 y\ dx=\int_4^6 (2x-7)\ dx$$

$$=\Big[x^2-7x\Big]_4^6$$

$$=(36-42)-(16-28)=6 \text{ sq. units.}$$

Now the side $x=4$ intersects the sides $y=2x-7$ and $y=0$ at the points A (4, 1) and B (4, 0). Hence the length of the side $\overline{AB}$ of the trapezium is 1 unit.

Again, the side $x=6$ intersects the two sides $y=2x-7$ and $y=0$ at the points D(6, 5) and C (6, 0) respectively. Hence the length of the side $\overline{CD}$ of the trapezium is 5 units.

Now, from the given formula, the area of the trapezium

$=\frac{1}{2}$(sum of the lengths of the parallel sides) × (distance between the parallel sides)

$=\frac{1}{2}(1+5).2$ sq. units $=6$ sq. units.

Hence from both considerations we get the same measure of the area of the trapezium. Hence the given formula is verified.

Ex. 9. The slope of a curve at the point (x, y) is $4x-3$ and the curve passes through the point (2, 2). Find the area enclosed by the curve, the x-axis and the ordinates $x=0$ and $x=2$

Since the slope of the curve at the point (x, y) is $4x-3$,

$$\therefore \quad \frac{dy}{dx}=4x-3 \text{ or, } dy=(4x-3)\,dx$$

$$\text{or, } y=4\frac{x^2}{2}-3x+c \text{ [By integration]}$$

$$\text{or, } \quad y=2x^2-3x+c.$$

Now since the curve passes through the point (2, 2),

$$\therefore \quad 2=8-6+c \qquad \text{or,} \quad c=0.$$

Hence the equation of the curve is $y=2x^2-3x$.

Now, required area $=\int_0^2 y dx=\int_0^2 (2x^2-3x)\, dx$

$$=\left[2.\frac{x^3}{3}-\frac{3x^2}{2}\right]_0^2=\frac{16}{3}-6=-\frac{2}{3}$$

Hence the required area $=\frac{2}{3}$ sq. units.

Exercise 4

Integrate :

1. $\int_0^{\frac{1}{2}} \frac{dx}{(1-2x^2)\sqrt{1-x^2}}$ [C. U. 1933]

2. $\int_2^3 \sqrt{(x-2)(3-x)}\, dx$ [C. U. 1965]

3. $\int_0^{\frac{\pi}{2}} \sin^4 x \cos^3 x \, dx.$ [C. U. 1965]

4. $\int_2^3 \frac{dx}{\sqrt{(x-1)(5-x)}}$ [C. U. 1966]

5. $\int_0^\pi \frac{dx}{1-2a\cos x+a^2}$ $(0<a<1)$ [C. U. '47]

6. $\int_0^{\frac{\pi}{2}} \frac{\sin\theta \, d\theta}{\sin\theta+\cos\theta}$ [C. U.]

7. $\int_0^{\frac{\pi}{2}} \frac{\sin^3 x}{\cos^3 x+\sin^3 x} dx.$

8. $\int_0^\pi \frac{x\sin x}{1+\cos^2 x} dx.$ [North Bengal '69]

9. $\int_{-a}^{a} \frac{xe^{x^4}}{1+x^2} dx.$ [C. U. '66]

10. $\underset{n\to\infty}{Lt} \frac{1^4+2^4+\cdots+n^4}{n^5}$ [C. U. '64]

11. $\underset{n\to\infty}{Lt} \left[\frac{1}{\sqrt{2n-1^2}}+\frac{1}{\sqrt{4n-2^2}}+\frac{1}{\sqrt{6n-3^2}}+\cdots+\frac{1}{n}\right]$

[C. U. '66]

12. $\underset{n\to\infty}{Lt} \left\{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\cdots\cdots\cdots\left(1+\frac{n}{n}\right)\right\}^{\frac{1}{n}}$

[C. U. '67]

13. Find the measure of the area enclosed by the curve $y=x^3$ and the straight line $y=3x$.

14. Find the area enclosed by the curve $y=8+12x-x^3$, the axes of co-ordinates and the ordinates of the point of maxima of the curve.

CHAPTER FIVE

Differential Equations

§ 5·1- Equations involving derivatives or differentials of functions are called Differential equations.

Hence $x^2\frac{dy}{dx}+y-1=0$ is a differential equation due to the presence of the derivative $\frac{dy}{dx}$.

$y\,dx-xdy=xy\,dx$ is a differential equation, since the differentials dx, dy are present in the equation.

Let us now consider the three equations,

$$\frac{dy}{dx}=0,\ dy=\sin x\,dx \text{ and } y\,dy=dx.$$

In the first equation, x and y are both absent, though $\frac{dy}{dx}$ is present; y and x are respectively absent in the second and third equations. But all the three equations, due to the presence of derivatives or differentials are differential equations. So, in a differential equation both or any of the dependent and independent variables may not be directly present.

The differential equation, in which the derivatives present are obtained by differentiating the dependent variable with respect to a single independent variable only, is called an **ordinary differential equations.** In this book we shall discuss ordinary differential equations and by differential equations shall mean ordinary differential equations. So, the epithet ordinary will not be used any farther.

§ 5·2. **Order and degree of a differential equation.**

The order of a differential equation is the order of the highest order derivative or differential present in the equation.

Hence, $x\frac{dy}{dx}=2a$ $\quad\cdots(1)$

$$x\,dy+y\,dx=0 \qquad \cdots(2)$$

$$\left(\frac{dy}{dx}\right)^2=x^2+y^2 \qquad \cdots(3)$$

are equations of the first order. The two equations,

$$\frac{d^2y}{dx^2}-5\frac{dy}{dx}+6=0 \qquad \cdots(4)$$

$$x\left(\frac{dy}{dx}\right)^2-\left(\frac{d^2y}{dx^2}\right)=x^3 \qquad \cdots(5)$$

are examples of second order differential equations.

Similarly examples of third, fourth or nth-order differential equations can be obtained.

By the degree of a differential equation is meant the highest power of the highest order derivative present in the equation.

In the equation $\frac{dy}{dx}=\frac{y}{x}$, the highest order derivative present is $\frac{dy}{dx}$ and its degree is 1 and hence the degree of the equation is 1 and the equation is of the first order and degree.

In the equation $x\left(\frac{dy}{dx}\right)^2-\frac{d^2y}{dx^2}=x^2$, the highest order derivative present is $\frac{d^2y}{dx^2}$ and its degree is 1. Hence the equation is of degree 1.

The equation $\frac{d^2y}{dx^2}-5\frac{dy}{dx}+6y=0$ is an equation of the second order but of the first degree. In this equation the highest order derivative present is $\frac{d^2y}{dx^2}$ whose dergee is 1. Similarly one can find examples of differential equations of the n-th order and m-th degree. Now, let us consider the equation

$$\frac{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}=\rho, \quad \cdots(i)$$

This equation can be written as

$$\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}=\rho\frac{d^2y}{dx^2}$$

or, $$\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^3=\rho^2\left(\frac{d^2y}{dx^2}\right)^2$$

or; $\rho^2\left(\frac{d^2y}{dx^2}\right)^2-1-3\left(\frac{dy}{dx}\right)^2-3\left(\frac{dy}{dx}\right)^4-\left(\frac{dy}{dx}\right)^6=0\cdots$(ii)

The equation (ii) is obtained from the equation (i) by making the latter free of fractions and radicals or fractional indices. The equation (ii) is evidently an equation of the second order and second degree and so the equation (i) is also called an equation of the second order and second degree.

Similarly the equation $x+\dfrac{\frac{dy}{dx}}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}=a$ can be written

as $\dfrac{\frac{dy}{dx}}{\sqrt{\left(1+\frac{dy}{dx}\right)^2}}=a-x.$

or; $\left(\frac{dy}{dx}\right)^2=(a-x)^2\left\{1+\left(\frac{dy}{dx}\right)^2\right\}$

So, the equation is of the first order and second degree. Hence the degree of a differential equation free from fraction and radicals or fractional indices is the highest degree of the highest order derivative present in the equation. For determination of the degree of an equation, the equation should be first made free from fractions and radicals.

§ 5·3. Formation of Differential Equations.

You know that the equation $x^2+y^2-2ax=0\cdots$(i) is the equation of a circle passing through the origin and the centre of the circle lies on the x-axis. Now for different values of a, the equation will represent different circles passing through the origin and having centres on the x-axis. Hence the equation $x^2+y^2-2ax=0$ represents a family of circles. The constant a is called the parameter of the equation. Now, differentiating both sides of the equation (1) we get,

$$2x+2y\frac{dy}{dx}-2a=0$$

or, $a=x+y\dfrac{dy}{dx}$ ......(2)

Putting this value of a in equation-(1) we get,

$$x^2+y^2-2\left(x+y\frac{dy}{dx}\right)x=0$$

or, $$y^2-2xy\frac{dy}{dx}-x^2=0 \qquad \cdots(3)$$

Now the equation-(3) is free from a. The equation-(3) is said to be the differential equation of the family of circles represented by the equation (1). Note that in the equation-(1) there is only one arbitrary constant or parameter and equation (2) has been obtained by single differentiation of both sides of equation (1). The differential equation (3) has been obtained after eliminating a from the equations (1) and (2)

Let us now discuss equations with two parameters.

In the equation $y=mx+c\cdots(4)$, m and c are parameters. For different values of m and c equation-(5) represents different straight lines. Now, differentiating both sides of equation-(4) we obtain

$$\frac{dy}{dx}=m \qquad \cdots(5)$$

and the parameter c in equation-(4) is eliminated. To eliminate the parameter m, another differentiation is required; differentiating both sides of equation-(5) we obtain

$$\frac{d^2y}{dx^2}=0 \qquad \cdots(6)$$

In equation-(6) both the parameters m and c are absent. Equation-(6) is the differential equation of the family of straight lines in a plane.

In general if n independent parameters be present in an equations, then the n parameters are eliminated from the total number of $(n+1)$ equations obtained after successive n times differentiation of both sides of the given equation. The differential equation obtained after elimination of the n parameters is the differential equation originated from the given equation.

§ 5·4. Solution of a differential equation.

If a relation between the dependent and independent variables of a differential equation not containing any derivative or differential satisfy the differential equation, then the relation is said to be a solution of the differential equation. So the solution of a differential equation is the equation, free from differentials or derivatives, from which the differential equation can be obtained after one or more succcssive differentiations.

Let us now consider the equation $\frac{d^2y}{dx^2}-5\frac{dy}{dx}+6y=0.$

Let $y=e^{2x}$.

$\therefore \quad \frac{dy}{dx}=2e^{2x}$ and $\frac{d^2y}{dx^2}=4e^{2x}$.

Hence the left hand side of the given differential equation becomes $4e^{2x}-5.2e^{2x}+6\,e^{2x}=0$.

Hence the relation $y=e^{2x}$ is a solution of the equation.

Now, if $y=Ae^{2x}$, then $\frac{dy}{dx}=2Ae^{2x}$ and $\frac{d^2y}{dx^2}=4Ae^{2x}$.

Now putting these values of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in the left hand side of the given differential equation, we find that the equation is satisfied. Again $y=Ae^{3x}$ and $y=Be^{3x}$ both satisfy the differential equation and both of them are solutions of the equation.

Now, as $y=Ae^{2x}$ and $y=Be^{3x}$ both satisfy the given differential equation, so $y=Ae^{2x}+Be^{3x}$ [where A and B are arbitrary constants] will also satisfy the equation.

$y=Ae^{2x}+Be^{3x}$ is the general solution of the given differential equation

$$\frac{d^2y}{dx^2}-5\frac{dy}{dx}+6y=0.$$

We state below what is meant by the general solution of a differential equation.

If the number of independent arbitrary constants in a solution of differential equation be equal to the order of the

equation, then that solution of the equation is called the general solution of the equation.

Note 1. In the relation $y=\log \frac{x}{a}+c$, the two quantities a and c are constants. But $\log \frac{x}{a}+c$ can be written as $\log x-\log a+c=\log x+k$ (taking $c-\log a=k$). So the two constants can be absorbed in a single constant. Here the two constants a and c are not independent.

Similarly, the relation $y=Ae^{c+x}$ can be expressed as $y=A.e^{c}.e^{x}=Be^{x}$ which contains only one arbitrary constant.

If a relation containing n arbitrary constants cannot be expressed as a relation containing less than n arbitrary constants, then the n arbitrary constants are mutually independent in the relation.

2. In § 5·3 it has been found that to eliminate a relation containing only one parameter one has to differentiate both sides of the equation only once and the eliminant becomes a differential equation of the first order. To eliminate two independent arbitrary constants one has to differentiate twice and the eliminant is a differential equation of the second order. Hence it is quite natural to presume that the general solution of a differential equation of the n-th order, will contain n mutually independent arbitrary constants.

3. Determination of the solution of a differential equation means determination of the general solution.

4. A particular solution of a differential equation is obtained by assigning particular values to the arbitrary constants in the general solution of the equation.

Exercise V (A)

1. Determine the order and degree of the following equations.

(i) $\frac{dy}{dx}=\sin x$. (ii) $x^2\left(\frac{dy}{dx}\right)^2-2xy\frac{dy}{dx}+2y^2-x^2=0$;

(iii) $\frac{d^2y}{dx^2}-a\left(\frac{dy}{dx}\right)^2=0$ (iv) $\frac{d^2y}{dx^2}-7\frac{dy}{dx}+12y=0$.

(v) $y-\frac{dy}{dx}=\sqrt{1+\left(\frac{dy}{dx}\right)^2}(x^2+y^2)$.

2. Form differential equations from the following relations.

(i) $xy=c^2$. (ii) $y=Ae^{mx}+Be^{-mx}$.

(iii) $ax^2+by^2=1$. [C. U. 1945]

(iv) $y=ax+bx^2$. (v) $r=a+b\cos\theta$.

3. Eliminate c from the equation $ay^2=(x-c)^3$.

4. Prove that the differential equation of the family of straight lines passing through the origin is $\frac{dy}{dx}=\frac{y}{x}$.

§ 5·5. Solution of differential equations of the first order and of the first degree by the method of separation of variables.

You know that all equations are not solvable. So also all differential equations are not solvable. At the primary stage conditions of existence of solution of differential equations are not necessary. The methods of solution of solvable equations are also different. In this book we shall discuss only one type of solvable equations. They are Equations of the first order and first degree which can be solved by the method of separation of variables. If a differential equation of the first order and degree can be expressed in the form $f_1(x)dx+f_2(y)dy=0$, where $f_1(x)$ and $f_2(y)$ are functions of only x and only y respectively, then we can say that in the equation variables have been separated. In a differential equation variables are generally separated (i) by inspection and (ii) by substitution.

§ 5·6. The method of separation of variables.

Example 1. Solve : $xdx+ydy=0$.

In this equation the variables are already separated. So integrating we get, $\int xdx+\int ydy=0$

or, $\frac{x^2}{2}+c_1+\frac{y^2}{2}+c_2=0$

or, $\dfrac{x^2}{2}+\dfrac{y^2}{2}=-c_1-c_2$

or, $\dfrac{x^2}{2}+\dfrac{y^2}{2}=c.$ $[-c_1-c_2=c]$

or, $x^2+y^2=2c$

or, $x^2+y^2=a^2$ $[a^2=2c]$

Ex. 2. $(1-x)dy-(1+y)dx=0.$

or, $(1-x)dy=(1+y)dx.$

or, $\dfrac{dy}{1+y}=\dfrac{dx}{1-x}$

Integrating both sides,

$\log(1+y)=-\log(1-x)+\log c.$

or, $\log(1+y)+\log(1-x)=\log c$

or, $\log\{(1+y)(1-x)\}=\log c.$

or, $(1+y)(1-x)=c.$

Ex. 3. $\dfrac{dy}{dx}+\sqrt{\dfrac{1-y^2}{1-x^2}}=0.$

or, $\dfrac{dy}{\sqrt{1-y^2}}+\dfrac{dx}{\sqrt{1-x^2}}=0$

Integrating we get,

$\sin^{-1} y+\sin^{-1} x=\sin^{-1} c.$

or, $\sin^{-1}(x\sqrt{1-y^2}+y\sqrt{1-x^2})=\sin^{-1}c.$

or, $x\sqrt{1-y^2}+y\sqrt{1-x^2}=c.$

Ex. 4. Find the general solution of

$\sec^2 x \tan y\, dx+\sec^2 y \tan x\, dy=0.$

or, $\dfrac{\sec^2 x\, dx}{\tan x}+\dfrac{\sec^2 y\, dy}{\tan y}=0.$

or, $\displaystyle\int\frac{\sec^2 x\, dx}{\tan x}+\int\frac{\sec^2 y\, dy}{\tan y}=c'.$

Let, $\tan x=z$ $\therefore$ $\sec^2 x\, dx=dz$

$\therefore$ $\displaystyle\int\frac{\sec^2 x\, dx}{\tan x}=\int\frac{dz}{z}=\log z=\log(\tan x)$

Similarly $\int \frac{\sec^2 y\, dy}{\tan y} = \log(\tan y)$

Hence the required general solution is

$\log \tan x + \log \tan y = \log c \quad [c' = \log c]$

or, $\log(\tan x \tan y) = \log c.$

or, $\tan x \tan y = c.$

Ex. 5. Find the general solution of

$$e^{x-y}\, dx + e^{y-x}\, dy = 0$$

or, $\frac{e^x}{e^y} dx + \frac{e^y}{e^x}\, dy = 0$ or, $e^{2x}\, dx + e^{2y}\, dy = 0.$

or, $\int e^{2x}\, dx + \int e^{2y}\, dy = c'.$

or, $\frac{e^{2x}}{2} + \frac{e^{2y}}{2} = c'$ or, $e^{2x} + e^{2y} = 2c' = c.$

Ex. 6. Find the general solution of

$$\frac{dy}{dx} + \frac{y^2+y+1}{x^2+x+1} = 0.$$

From the given equation.

$$\frac{dy}{y^2+y+1} + \frac{dx}{x^2+x+1} = 0.$$

or, $\int \frac{dy}{y^2+y+1} + \int \frac{dx}{x^2+x+1} = c'$

Now, $\int \frac{dy}{y^2+y+1} = \int \frac{dy}{\left(y+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$

$$= \int \frac{dz}{z^2 + \left(\frac{\sqrt{3}}{2}\right)^2} \quad [\, z = y + \tfrac{1}{2} \text{ (say) }]$$

$$= \frac{2}{\sqrt{3}} \tan^{-1} \frac{2z}{\sqrt{3}} = \frac{2}{\sqrt{3}} \tan^{-1} \frac{2(y+\frac{1}{2})}{\sqrt{3}}$$

$$= \frac{2}{\sqrt{3}} \tan^{-1} \left(\frac{2y+1}{\sqrt{3}}\right)$$

Similarly, $\int \frac{dx}{x^2+x+1} = \frac{2}{\sqrt{3}} \tan^{-1} \left(\frac{2x+1}{\sqrt{3}}\right)$

Hence the required general solution

is $\frac{2}{\sqrt{3}} \tan^{-1} \frac{2y+1}{\sqrt{3}} + \frac{2}{\sqrt{3}} \tan^{-1} \frac{2x+1}{\sqrt{3}} = \frac{2}{\sqrt{3}} \tan^{-1} c.$

$$\left[c' = \frac{2}{\sqrt{3}} \tan^{-1} c \text{ (say)} \right]$$

or, $\tan^{-1} \frac{2y+1}{\sqrt{3}} + \tan^{-1} \frac{2x+1}{\sqrt{3}} = \tan^{-1} c.$

or, $\tan^{-1} \dfrac{\frac{2y+1}{\sqrt{3}} + \frac{2x+1}{\sqrt{3}}}{1 - \frac{2y+1}{\sqrt{3}} \cdot \frac{2x+1}{\sqrt{3}}} = \tan^{-1} c.$

or, $\dfrac{\frac{2x+2y+2}{\sqrt{3}}}{\frac{3-4xy-2x-2y-1}{3}} = c.$

or, $\frac{2(x+y+1)}{2(1-2xy-x-y)} = \frac{c}{\sqrt{3}} = \frac{1}{A}$

or, $A(x+y+1) = 1-2xy-x-y.$

or, $2xy+x+y+A(x+y+1) = 1$

Ex. 7. If $f'(x) = \log x$ and $f(1) = -5$ then find $f(x)$.

[C. U. 1964]

Let $y = f(x)$. $\therefore \frac{dy}{dx} = f'(x) = \log x$

or, $dy = \log x \, dx$. or, $\int dy = \int \log x \, dx$

or, $y = \int 1. \log x = \log x.x - \int dx = x(\log x - 1) + c.$

Now, when $x = 1$, then $f(x) = f(1) = -5$

$\therefore -5 = 1.(\log 1 - 1) + c = -1 + c \quad \therefore c = -4$

$\therefore y = f(x) = x(\log x - 1) - 4.$

Ex. 8. $\frac{ds}{dt} = -\frac{s^2}{30}$ and when $t = 0$, then $s = 15$. Find t when $s = 10$.

$\frac{ds}{dt} = -\frac{s^2}{30}$ or, $-\frac{ds}{s^2} = \frac{1}{30} dt$

or, $\int -\frac{ds}{s^2} = \int \frac{1}{30} dt$ or, $\frac{1}{s} = \frac{1}{30} t + c.$

Now, when $t=0$, then $s=15$

$\therefore \quad \frac{1}{15}=\frac{1}{30}.0+c \quad \therefore \quad c=\frac{1}{15} \quad \therefore \quad \frac{1}{s}=\frac{1}{30}t+\frac{1}{15}.$

Now, when $s=10$, then $\frac{1}{10}=\frac{1}{30}t+\frac{1}{15}$

or, $\frac{1}{30}t=\frac{1}{10}-\frac{1}{15}=\frac{1}{30} \quad \therefore \quad t=1.$

Ex. 9. $\frac{dt}{dQ}=-\frac{CR}{Q}$ and $C=0{\cdot}006, \quad R=75$;

If $Q=10$ when $t=0$, then find t in terms of Q

$$\frac{dt}{dQ}=-\frac{CR}{Q}$$

or, $dt=-CR\frac{dQ}{Q}$ or, $\int dt=\int -CR\frac{dQ}{Q}$

or, $t=-CR \log Q+A=CR \log \frac{1}{Q}+A$

Now, when $t=0$, then $Q=10$, $\therefore \quad 0=CR \log \frac{1}{10}+A$

$\therefore \quad A=-CR \log \frac{1}{10} \quad \therefore \quad t=CR \log \frac{1}{Q}-CR \log \frac{1}{10}$

$$=CR \log \frac{10}{Q}={\cdot}006\times 75 \log \frac{10}{Q}=0{\cdot}45 \log \frac{10}{Q}$$

[Putting the values of C and R]

Ex. 10. Prove that if the normal to a curve at every point passes through a fixed point, then the curve is a circle.

Let the fixed point be the origin and the equation of the curve with respect to a set of rectangular axes be $y=f(x)$. Now the gradient of the normal at every point (x, y) of the curve is $-\frac{dx}{dy}$. Also as the normal passes through the fixed point i.e., the origin, so the gradient of the normal is $\frac{y}{x}$.

$$\therefore \quad -\frac{dx}{dy}=\frac{y}{x}$$

or, $-x\,dx=y\,dy$ or, $x\,dx+y\,dy=0$

or, $\frac{x^2}{2}+\frac{y^2}{2}=c$ or, $x^2+y^2=a^2$ [Let $2c=a^2$]

which is the equation of a circle. Hence the curve is a circle.

Exercise V B

Determine the general solution of the following equations :—

1. $\frac{dy}{dx}=\frac{x+1}{y+1}$ 2. $\frac{dy}{dx}=\frac{x^2+x+1}{y^2+y+1}$

3. $x\,dx-y\,dy=0$ 4. $y\,dy-x\,dy=xy\,dx$

5. $\frac{dy}{dx}=\frac{x(1+y^2)}{y(1+x^2)}$ 6. $x\sqrt{1-y^2}dx+y\sqrt{1-x^2}\,dy=0$

7. $\frac{dx}{x}-\frac{y\,dy}{1+y^2}=0$ 8. $dr+r\tan\theta\,d\theta=0$

9. A curve $y=f(x)$ is such that $\frac{dy}{dx}=5e^x$ and when $x=0$, then $y=6$. Find $f(x)$.

10. Prove that a particular solution of the equation $\frac{dy}{dx}=\frac{2}{y}$ is $y^2=4x$. Find the general solution of the equation.

11. The differential equation of a curve is $(1+y^2)dx-xy\,dy=0$ and it passes through the point $(2, \sqrt{3})$, Prove that the co-ordinates of the foci of the curve are $(\pm\sqrt{2}, 0)$.

12. If $y=\frac{\pi}{4}$, when $x=0$, then find the particular solution of the equation $\cos y\,dx+(1+2e^{-x})\sin y\,dy=0$.

§ 5·7. Solution of differential equations by substitution.

When the variables cannot be separated by inspection, then in certain cases the dependent variables are substituted by a new variable so that the new variable and the independent variables can be seperated. The solution in

terms of the new variable is then expressed in terms of the old variable.

Example 1. Solve : $\log\left(\frac{dy}{dx}\right)=ax+by$

Let, $ax+by=z$

Differentiating both sides with respect to x,

$$a+b\,\frac{dy}{dx}=\frac{dz}{dx}\quad\therefore\quad\frac{dy}{dx}=\frac{1}{b}\left(\frac{dz}{dx}-a\right)$$

Now, from the given equation,

$$\frac{dy}{dx}=e^{ax+by}\quad\text{or,}\quad\frac{1}{b}\left(\frac{dz}{dx}-a\right)=e^{z}$$

or, $\frac{dz}{dx}-a=be^{z}$ or, $\frac{dz}{dx}=a+be^{z}$ or, $\frac{dz}{a+be^{z}}=dx.$

or, $\int\frac{dz}{a+be^{z}}=\int dx\cdots\quad\cdots(1)$

Now, $\int\frac{dz}{a+be^{z}}=\int\frac{e^{-z}dz}{ae^{-z}+b}$

Let, $ae^{-z}+b=u$

$\therefore\quad -ae^{-z}dz=du,\quad$ or, $\quad e^{-z}dz=-\frac{1}{a}du$

$$\therefore\quad\int\frac{dz}{a+be^{z}}=-\frac{1}{a}\int\frac{du}{u}=-\frac{1}{a}\log u$$

$$=-\frac{1}{a}\log\left(ae^{-z}+b\right).$$

Hence from (1) we get,

$$-\frac{1}{a}\log\left(ae^{-z}+b\right)=x+\log c'$$

or, $\log\left(\frac{a}{e^{z}}+b\right)+\log c=-ax.\ [c=a\log c']$

or, $\log\left\{\left(\frac{a+be^{z}}{e^{z}}\right)\cdot c\right\}=-ax$

or, $\frac{a+be^{z}}{e^{z}}c=e^{-ax}$ or, $\frac{a}{e^{z}}+b=\text{A}e^{-ax}\left[c=\frac{1}{\text{A}}\right]$

or, $\frac{a}{e^{ax+by}}+b=\text{A}e^{-ax}$ [as $z=ax+by$]

or, $ae^{-ax-by}+b=\text{A}e^{-ax}$ or, $ae^{-by}+be^{ax}=\text{A}.$

Ex. 2. Solve : $(x+y)^2\dfrac{dy}{dx}=a^2$

Let, $x+y=z$ $\therefore$ $1+\dfrac{dy}{dx}=\dfrac{dz}{dx}$ or, $\dfrac{dy}{dx}=\dfrac{dz}{dx}-1$

$\therefore$ The given equation is $z^2\left(\dfrac{dz}{dx}-1\right)=a^2$

or, $z^2\dfrac{dz}{dx}-z^2=a^2$; or, $z^2\dfrac{dz}{dx}=z^2+a^2$

or, $\dfrac{z^2dz}{z^2+a^2}=dx$. or, $\displaystyle\int\frac{z^2dz}{z^2+a^2}=\int dx$

or, $\displaystyle\int\frac{z^2+a^2-a^2}{z^2+a^2}dz=\int dx.$

or, $\displaystyle\int dz-a^2\int\frac{dz}{z^2+a^2}=\int dx$

or, $z-a\tan^{-1}\dfrac{z}{a}=x-c$

or, $(x+y)-a\tan^{-1}\left(\dfrac{x+y}{a}\right)=x-c$

or, $y=a\tan^{-1}\left(\dfrac{x+y}{a}\right)-c$

or, $\dfrac{y+c}{a}=\tan^{-1}\left(\dfrac{x+y}{a}\right)$ or, $\dfrac{x+y}{a}=\tan\left(\dfrac{y+c}{a}\right)$.

Exercise V C

Determine the general solution of the following equations :—

1. $(x-y)^2\dfrac{dy}{dx}=a^2$. 2. $\dfrac{dy}{dx}=\sqrt{y-x}$.

3. $\dfrac{dy}{dx}+1=e^{x+y}$. 4. $\cos^{-1}\left(\dfrac{dy}{dx}\right)=x+y$

5. $\dfrac{dy}{dx}=f(ax+by+c)$

§ 5·8. Solution of homogeneous equations of the first order and first degree.

In the previous article we have solved certain differential equations by substitution. But we have not indicated any rule for substitution. In the present article and the next

we shall discuss the method of substitution in certain special cases. In the present article we shall discuss solution of homogeneous differential equations of the first order and first degree.

Homogeneous equations. If in an equation of the form $f_1(x, y)dx+f_2(x, y)dy=0$, the power of each term of $f_1(x, y)$ and $f_2(x, y)$ be the same, then the equation is said to be a homogeneous equation of the first order and first degree.

In the equation $(x^2+y^2)dy-xy\ dx=0$, the powers of each of x^2, y^2 and xy is 2 and so the equation is homogeneous.

The equation $\frac{dy}{dx}=\frac{y^2}{x^2}-\frac{y}{x}$ can be expressed in the form

$x^2dy=(y^2-xy)dx$ and the powers of each of x^2, y^2, xy is 2 and so the equation is homogeneous. The equation $(x^3+y^3)dy=x^2y\ dx$ is also a homogeneous equation.

To solve homogeneous equations, put $y=vx$. In the new equation in v and x after substitution, the variables will be easily separated and the solution obtained in terms of v and x should be expressed in terms of y and x.

Example 1. Solve : $2\frac{dy}{dx}=\frac{y}{x}+\frac{y^2}{x^2}$.

The equation is homogeneous. Hence for the determination of solution,

Let $y=vx$, $\therefore$ $\frac{dy}{dx}=v+x\frac{dv}{dx}$. and $\frac{y}{x}=v$, $\frac{y^2}{x^2}=v^2$.

Hence the given equation reduces to

$$2\left(v+x\frac{dv}{dx}\right)=v+v^2. \quad \text{or,} \quad 2v+2x\frac{dv}{dx}=v+v^2$$

or, $2x\frac{dv}{dx}=v^2-v$

or, $\frac{2dv}{v^2-v}=\frac{dx}{x}$; or, $\int\frac{2dv}{v^2-v}=\int\frac{dx}{x}$ $\cdots(1)$

Now, $\int\frac{2dv}{v^2-v}=\int\frac{2dv}{v(v-1)}=2\int\left(\frac{1}{v-1}-\frac{1}{v}\right)dv$

$=2 \log (v-1)-2 \log v=2 \log \dfrac{v-1}{v}.$

$\therefore$ From (1) the required solution is

$$2 \log \frac{v-1}{v}=\log x+\log c$$

or, $\log \left(\dfrac{v-1}{v}\right)^2=\log cx$ or, $\left(\dfrac{\frac{y}{x}-1}{\frac{y}{x}}\right)^2=cx$

or, $\dfrac{(y-x)^2}{y^2}=cx$ or, $(y-x)^2=cxy^2$.

Ex. 2. Solve : $(x^2+y^2)dx-2xy\ dy=0$ [C. U.]

Let, $y=vx$; $\therefore$ $\dfrac{dy}{dx}=v+x\dfrac{dv}{dx}$.

$\therefore$ From the given equation we get,

$(x^2+v^2x^2)-2x.vx\left(v+x\dfrac{dv}{dx}\right)=0$

or, $x^2(1+v^2)-2x^2v\left(v+x\dfrac{dv}{dx}\right)=0$

or, $1+v^2-2v\left(v+x\dfrac{dv}{dx}\right)=0.$

or, $1+v^2-2v^2-2vx\dfrac{dv}{dx}=0$

or, $1-v^2-2vx\dfrac{dv}{dx}=0$ or, $1-v^2=2vx\dfrac{dv}{dx}$

or, $\dfrac{dx}{x}=\dfrac{2v\ dv}{1-v^2}$ or, $\displaystyle\int\frac{dx}{x}=\int\frac{2v\ d_v}{1-v^2}$

or, $\log x=-\log(1-v^2)+\log c$

or, $\log x+\log(1-v^2)=\log c$ or, $\log\{x(1-v^2)\}=\log c$

or, $x.\left(1-\dfrac{y^2}{x^2}\right)=c$ or, $x.\left(\dfrac{x^2-y^2}{x^2}\right)=c$

or, $\dfrac{x^2-y^2}{x}=c$ or, $x^2-y^2=cx.$

Ex. 3. Solve : $x^2y\ dx-(x^3+y^3)\ dy=0$

or, $x^2y\ dx=(x^3+y^3)dy$ or, $\dfrac{dy}{dx}=\dfrac{x^2y}{x^3+y^3}$

Now let, $y=vx \quad \therefore \quad \frac{dy}{dx}=v+x\frac{dv}{dx}$

Hence the equation becomes

$$v+x\frac{dv}{dx}=\frac{x^2.vx}{x^3+v^3x^3}=\frac{v}{1+v^3}$$

or, $x\frac{dv}{dx}=\frac{v}{1+v^3}-v=\frac{-v^4}{1+v^3}$

or, $\frac{(1+v^3)}{v^4}dv+\frac{dx}{x}=0$ or, $\int\frac{1+v^3}{v^4}dv+\int\frac{dx}{x}=\log c$

or, $\int\frac{1}{v^4}dv+\int\frac{1}{v}dv+\int\frac{dx}{x}=\log c$

or, $-\frac{1}{3v^3}+\log v+\log x=\log c$

or, $-\frac{x^3}{3y^3}+\log\frac{y}{x}.x=\log c$

or, $\log y-\log c=\frac{x^3}{3y^3}$ or, $\log\frac{y}{c}=\frac{x^3}{3y^3}$

or, $\frac{y}{c}=e^{\frac{x^3}{3y^3}}$

$\therefore$ Required general solution is $y=ce^{\frac{x^3}{3y^3}}$

Exercise VD

Find the general solution of the following differential equations :—

1. $\frac{dy}{dx}=\frac{y^2}{x^2}$. 2. $(x^2+y^2)\,dy=xy\,dx$

3. $x\frac{dy}{dx}+\frac{y^2}{x}=y$ 4. $\frac{dy}{dx}=\frac{2y-x}{2x-y}$

5. $xy^2dy=(x^3+y^3)\,dx$

6. $\left(x+y\cos\frac{y}{x}\right)dx=x\cos\frac{y}{x}\,dy.$ [C. U. 1964]

§ 5·9. Solution of equations of the form

$$\frac{dy}{dx}=\frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}\left(\frac{a_1}{a_2}\neq\frac{b_1}{b_2}\right)$$

Equations of the above form can be solved by reducing them to homogeneous equations. To reduce the equation in the homogeneous form,

put $x=x'+h$ and $y=y'+k$; where h and k are to be so selected that the new equation after substitution does not contain any term independent of x' and y'.

If $x=x'+h$ and $y=y'+k$, then

$a_1x+b_1y+c_1$ and $a_2x+b_2y+c_2$

reduce respectively to the forms

$a_1x'+b_1y'+a_1h+b_1k+c_1$ and $a_2x'+b_2y'+a_2h+b_2k+c_2$

As in the new form there will be no term independent of x' and y';

so $a_1h+b_1k+c_1=0$ ...(1)

and $a_2h+b_2k+c_2=0$ ....(2)

as $\dfrac{a_1}{a_2}\neq\dfrac{b_1}{b_2}$ or, $a_1b_2-a_2b_1\neq 0$;

So the equations (1) and (2) can be solved simultaneously and solving we get,

$$\frac{h}{b_1c_2-b_2c_1}=\frac{k}{c_1a_2-c_2a_1}=\frac{1}{a_1b_2-a_2b_1}$$

or, $h=\dfrac{b_1c_2-b_2c_1}{a_1b_2-a_2b_1}$ and $k=\dfrac{c_1a_2-c_2a_1}{a_1b_2-a_2b_1}$

For these values of h and k, $a_1x+b_1y+c_1$ and $a_2x+b_2y+c_2$ will reduce to the forms $a_1x'+b_1y'$ and $a_2x'+b_2y'$.

Again as $x=x'+h$ and $y=y'+k$

$\therefore$ $dx=dx'$ and $dy=dy'$.

Hence the equation reduces to the form

$$\frac{dy'}{dx'}=\frac{a_1x'+b_1y'}{a_2x'+b_2y'}.$$

This equation is homogeneous in x' and y' and can be solved by the method of § 5·8 by the substitution $y'=vx'$,

but the solution will be in terms of x' and y'. Hence putting $x'=x-h$ and $y'=y-k$, the general solution should be expressed in terms of x and y.

Example 1. Solve :

$(2x+3y-5)\dfrac{dy}{dx}+(3x+2y-5)=0.$ [C. U. 1962]

$$\text{or,}\quad \frac{dy}{dx}=-\frac{3x+2y-5}{2x+3y-5}$$

Now let, $x=x'+h$ and $y=y'+k$.

$\therefore$ $dx=dx'$ and $dy=dy'$ So, the equation reduces to

$$\frac{dy'}{dx'}=\frac{3x'+2y'+3h+2k-5}{2x'+3y'+2h+3k-5}$$

Now select h and k so that

$$3h+2k-5=0\cdots(1)$$

and $2h+3k-5=0\cdots(2)$

Solving equations (1) and (2)

$$\frac{h}{-10+15}=\frac{k}{-10+15}=\frac{1}{9-4}$$

or, $h=\frac{5}{5}=1$ and $k=\frac{5}{5}=1$

$\therefore$ Let $x=x'+1$ and $y=y'+1$ and the equation reduces to

$$\frac{dy'}{dx'}=-\frac{3x'+2y'}{2x'+3y'}$$

Now this is a homogeneous equation. To solve it

let, $y'=vx'$; $\therefore$ $\dfrac{dy'}{dx'}=v+x'\dfrac{dv}{dx'}$

$$\therefore\quad v+x'\frac{dv}{dx'}=-\frac{3x'+2vx'}{2x'+3vx'}=-\frac{3+2v}{2+3v}$$

$$\text{or,}\quad x'\frac{dv}{dx'}=-\frac{3+2v}{2+3v}-v=-\frac{3+4v+3v^2}{2+3v}$$

$$\text{or,}\quad \frac{3v+2}{3v^2+4v+3}\,dv=-\frac{dx'}{x'}$$

Now integrating both sides we get,

$$\frac{1}{2}\int\frac{du}{u}=-\int\frac{dx'}{x'}\qquad\left[\text{ Taking } u=3v^2+4v+3,\right]$$

or, $\log u^{\frac{1}{2}} = -\log x' + \log c$

or, $\log (3v^2+4v+3)^{\frac{1}{2}} + \log x' = \log c.$

or, $\log (3v^2+4v+3)^{\frac{1}{2}} x' = \log c$

or, $(3v^2+4v+3)x'^2 = c^2 = A$

or, $\left(3\dfrac{y'^2}{x'^2} + 4\dfrac{y'}{x'} + 3\right)x'^2 = A \quad \left[\because\ y' = vx, \quad \therefore\ v = \dfrac{y'}{x'}\right]$

or, $3y'^2 + 4x'y' + 3x'^2 = A$

or, $3(y-1)^2 + 4(x-1)(y-1) + 3(x-1)^2 = A$

$[\because\ x = x'+1, \quad \therefore\ x' = x-1;$

and $y = y'+1, \quad \therefore\ y' = y-1]$

and this is the required solution.

Ex. 2. Solve : $\dfrac{dy}{dx} = \dfrac{5x-7y}{x-3y+4}$

Let $x = x'+h$ and $y = y'+k$,

$\therefore\ dx = dx'$ and $dy = dy'$

$\therefore\ 5x-7y = 5(x'+h) - 7(y'+k) = 5x' - 7y' + 5h - 7k.$

and $x-3y+4 = (x'+h) - 3(y'+k) + 4 = x' - 3y' + h - 3k + 4$

Now select h and k so that

$5h - 7k = 0 \cdots (1)$ and $h - 3k + 4 = 0 \cdots (2)$

From (1) $k = \frac{5}{7}h$. From (2) $h - \frac{15}{7}h + 4 = 0$

or, $-\dfrac{8h}{7} = -4$ or, $h = \frac{7}{2}$ $\therefore\ k = \frac{5}{7}h = \frac{5}{7} \times \frac{7}{2} = \frac{5}{2}$

Now, for these values of h and k

$$v + x'\frac{dv}{dx'} = \frac{5x' - 7vx'}{x' - 3vx'} \quad [\, y' = vx' \text{ say }]$$

or, $x'\dfrac{dv}{dx'} = \dfrac{5-7v}{1-3v} - v = \dfrac{3v^2 - 8v + 5}{1-3v}$

or, $\dfrac{(1-3v)\,dv}{3v^2-8v+5} = \dfrac{dx'}{x'}$ or, $\displaystyle\int \frac{(1-3v)\,dv}{3v^2-8v+5} = \int \frac{dx'}{x'}$

Now, $\displaystyle\int \frac{(1-3v)\,dv}{3v^2-8v+5} = \int \left\{ \frac{1}{v-1} - \frac{6}{3v-5} \right\} dv.$

$= \log (v-1) - 2\log (3v-5) = \log \dfrac{v-1}{(3v-5)^2}$

$\therefore$ Required general solution is

$$\log \frac{v-1}{(3v-5)^2}=\log x'-\log c$$

or, $\log \dfrac{\frac{y'}{x'}-1}{\left(\frac{3y'}{x'}-5\right)^2}=\log \dfrac{x'}{c}$ or, $\dfrac{(y'-x)}{(3y'-5x')^2}=\dfrac{1}{c}$

or, $\dfrac{(y-\frac{5}{2}-x+\frac{7}{2})}{(3y-\frac{15}{2}-5x+\frac{35}{2})^2}=\dfrac{1}{c}$ $\left[\begin{matrix} x'=y-\frac{5}{2} \\ y'=x-\frac{7}{2}\end{matrix}\right]$

or, $\dfrac{(y-x+1)}{(3y-5x+10)^2}=\dfrac{1}{c}$

or, $(3y-5x+10)^2=c(y-x+1)$

[Note : $\dfrac{1-3v}{3v^2-8v+5}=\dfrac{1-3v}{(3v-5)(v-1)}=\dfrac{A}{3v-5}+\dfrac{B}{v-1}$(say)

$A+3B=-3$ and $-A-5B=1$; $\therefore$ $B=1$; $A=-6$

$\therefore$ $\dfrac{1-3v}{3v^2-8v+5}=-\dfrac{6}{3v-5}+\dfrac{1}{v-1}$]

§ 5·10. Solution of equations of the form

$\dfrac{dy}{dx}=\dfrac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$ when $\dfrac{a_1}{a_2}=\dfrac{b_1}{b_2}$

If $\dfrac{a_1}{a_2}=\dfrac{b_1}{b_2}$, the equation $\dfrac{dy}{dx}=\dfrac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$ cannot be solved by the method of the preceding article. For, as $\dfrac{a_1}{a_2}=\dfrac{b_1}{b_2}$ or, $a_1b_2-a_2b_1=0$, so the values of h and k cannot be determined. In this case follow the method as shown below.

Method : Let $\dfrac{a_1}{a_2}=\dfrac{b_1}{b_2}=\dfrac{1}{m}$.

The equation reduces to

$$\frac{dy}{dx}=\frac{a_1x+b_1y+c_1}{m(a_1x+b_1y)+c_2} \cdots (2)$$

Now, let $a_1x+b_1y=z$.

$\therefore$ $\dfrac{dz}{dx}=a_1+b_1\dfrac{dy}{dx}$ or, $\dfrac{dy}{dx}=\dfrac{1}{b_1}\left(\dfrac{dz}{dx}-a_1\right)$

Hence the equation-(2) will reduce to the form.

$$\frac{1}{b_1}\left(\frac{dz}{dx}-a_1\right)=\frac{z+c_1}{mz+c_2}$$

or, $\dfrac{dz}{dx}=a_1+\dfrac{b_1(z+c_1)}{mz+c_2}=\dfrac{z(a_1m+b_1)+c_2a_1+b_1c_1}{mz+c_2}$

or, $\dfrac{(mz+c_2)dz}{z(a_1m+b_1)+c_2a_1+b_1c_1}=dx.$

Now the variables are separated and the equation can be solved by integrating both sides.

Example 1. Find the general solution :

$$(2x+4y+3)\frac{dy}{dx}=2y+x+1$$

Given equation is $(2x+4y+3)\dfrac{dy}{dx}=2y+x+1$

or, $\dfrac{dy}{dx}=\dfrac{x+2y+1}{2x+4y+3}$

or, $\dfrac{dy}{dx}=\dfrac{x+2y+1}{2(x+2y)+3}$ ...(1)

Now let; $x+2y=v$ $\therefore$ $1+2\dfrac{dy}{dx}=\dfrac{dv}{dx}$

or, $\dfrac{dy}{dx}=\frac{1}{2}\left(\dfrac{dv}{dx}-1\right)$

$\therefore$ from (1); $\dfrac{1}{2}\left(\dfrac{dv}{dx}-1\right)=\dfrac{v+1}{2v+3}$

or. $\dfrac{dv}{dx}-1=\dfrac{2v+2}{2v+3}$; or, $\dfrac{dv}{dx}=\dfrac{2v+2}{2v+3}+1=\dfrac{4v+5}{2v+3}$

or; $\dfrac{(2v+3)dv}{4v+5}=dx.$

or; $\left\{\dfrac{1}{2}\cdot\dfrac{4v+5+1}{4v+5}\right\}dv=dx$; or, $\left\{\dfrac{1}{2}+\dfrac{1}{2}\cdot\dfrac{1}{4v+5}\right\}dv=dx.$

or; $\int\frac{1}{2}\,dv+\frac{1}{2}\int\dfrac{dv}{4v+5}=\int dx.$

or; $\frac{1}{2}v+\frac{1}{8}\log\,(4v+5)=x+c'$

or, $\frac{1}{2}(x+2y)+\frac{1}{8}\log\,(4x+8y+5)=x+c'$

or, $\frac{1}{8}\log(4x+8y+5)=-y+\frac{x}{2}+\frac{c}{2}=\frac{x-2y+c}{2}$

or, $\log(4x+8y+5)=4x-8y+4c=4x-8y+A.$

Ex. 2. Find the general solution of :

$$\frac{dy}{dx}=\frac{x+y+1}{2x+2y+1}.$$

$$\frac{dy}{dx}=\frac{x+y+1}{2(x+y)+1}$$

Let $x+y=z \quad \therefore \quad 1+\frac{dy}{dx}=\frac{dz}{dx}$, or, $\frac{dy}{dx}=\frac{dz}{dx}-1$

Now the equation becomes

$$\frac{dz}{dx}-1=\frac{z+1}{2z+1}$$

or, $\frac{dz}{dx}=\frac{z+1}{2z+1}+1=\frac{3z+2}{2z+1}$

or, $\frac{(2z+1)dz}{3z+2}=dx$

or, $\int\frac{(2z+1)dz}{3z+2}=\int dx. \quad \cdots \quad (1)$

Now, $\int\frac{(2z+1)dz}{3z+2}=\int\left(\frac{2}{3}\,\frac{3z+2}{3z+2}-\frac{1}{3}\,\frac{1}{3z+2}\right)dz.$

$$=\frac{2}{3}z-\frac{1}{9}\log(3z+2).$$

Hence from (1)

$$\frac{2}{3}z-\frac{1}{9}\log(3z+2)=x+c$$

or, $\frac{2}{3}(x+y)-\frac{1}{9}\log(3x+3y+2)=x+c.$

or, $\frac{2}{3}y-\frac{1}{3}x=\frac{1}{9}\log(3x+3y+2)+c.$

or, $6y-3x=\log(3x+3y+2)+9c$

Hence required solution is

$$6y-3x=\log(3x+3y+2)+A.$$

Exercise V E

Find the general solution of the following equations :

1. $\frac{dy}{dx}=\frac{2x-y+1}{x-2y+1}$, 2. $\frac{dy}{dx}=\frac{4x-5y+3}{5x-6y+2}$

3. $(6x-5y+4)dy+(y-2x-1)\,dx=0$

4. $\dfrac{dy}{dx}=\dfrac{6x-2y-7}{3x-y+4}$

5. $(2x+4y+3)\,dy=(2y+x+1)\,dx$ [C. U. 1963]

Examples 5

1. Solve : (i) $\dfrac{dy}{dx}+\dfrac{y(y-1)}{x(x-1)}=0$

(ii) $(1+y^2)dx+(1+x^2)dy=0$

(i) $\dfrac{dy}{dx}+\dfrac{y(y-1)}{x(x-1)}=0$ or, $\dfrac{dy}{y(y-1)}+\dfrac{dx}{x(x-1)}=0$

$$\int\frac{dy}{y(y-1)}+\int\frac{dx}{x(x-1)}=\log c \quad \cdots \quad \cdots(1)$$

Now, $\displaystyle\int\frac{dx}{x(x-1)}=\left(\frac{1}{x-1}-\frac{1}{x}\right)dx=\log\frac{x-1}{x}.$

Similarly, $\displaystyle\int\frac{dy}{y(y-1)}=\log\frac{y-1}{y}.$

So from (1), we obtain,

$$\log\frac{y-1}{y}+\log\frac{x-1}{x}=\log c$$

or, $\log\dfrac{(y-1(x-1)}{xy}=\log c$

or, $(x-1)(y-1)=cxy$ and this is the required solution.

(ii) $(1+y^2)\,dx+(1+x^2)\,dy=0$

or, $\displaystyle\int\frac{dx}{1+x^2}+\int\frac{dy}{1+y^2}=0$

or, $\tan^{-1}x+\tan^{-1}y=\tan^{-1}c.$

or, $\tan^{-1}\dfrac{x+y}{1-xy}=\tan^{-1}c$

or, $\dfrac{x+y}{1-xy}=c$ or, $x+y=c(1-xy)$ and this is the required general solution.

2. Find the general solution of : $x^2\dfrac{dy}{dx}+y=1$

$$x^2\frac{dy}{dx}+y=1 \quad \text{or,} \quad x^2\frac{dy}{dx}=1-y$$

or, $\dfrac{dy}{1-y}=\dfrac{dx}{x^2}$ or, $\int\dfrac{dy}{1-y}=\int\dfrac{dx}{x^2}$

or, $-\log(1-y)=-\dfrac{1}{x}-\log c$

or, $\log(1-y)=\dfrac{1}{x}+\log c$ or, $\log\dfrac{1-y}{c}=\dfrac{1}{x}$

or, $1-y=ce^{\frac{1}{x}}$ or, $y=1-ce^{\frac{1}{x}}$.

3. Solve : $\dfrac{dy}{dx}=e^{x-y}$

$\dfrac{dy}{dx}=e^{x-y}=\dfrac{e^x}{e^y}$ or, $e^x dx=e^y dy$

$\therefore$ $\int e^x dx=\int e^y dy$ or, $e^x=e^y+c$.

4. Determine the general solution of the differential equation :

$$ydx+(1+x^2)\tan^{-1}x\, dy=0.$$

$$y\, dx+(1+x^2)\tan^{-1}x\, dy=0$$

or, $\dfrac{dx}{(1+x^2)\tan^{-1}x}+\dfrac{dy}{y}=0$

or, $\displaystyle\int\frac{dx}{(1+x^2)\tan^{-1}x}+\int\frac{dy}{y}=\log c \quad \cdots(1)$

To determine $\dfrac{dx}{(1+x^2)\tan^{-1}x}$, put $\tan^{-1}x=z$

or, $\dfrac{dx}{1+x^2}=dz$ $\therefore$ $\displaystyle\int\frac{dx}{(1+x^2)\tan^{-1}x}=\int\frac{dz}{z}=\log z$

$=\log(\tan^{-1}x)$.

So, from (1) we get,

$$\log(\tan^{-1}x)+\log y=\log c.$$

or, $\log(y\tan^{-1}x)=\log c$ or, $y\tan^{-1}x=c$

which is the required general solution.

5. Solve. $\dfrac{\log(\sec x+\tan x)}{\cos x}dx=\dfrac{\log(\sec y+\tan y)\, dy}{\cos y}$

From the given equation we get,

$$\int\frac{\log(\sec x+\tan x)\, dx}{\cos x}=\int\frac{\log(\sec y+\tan y)\, dy}{\cos y} \quad \cdots(1)$$

Now, to determine $\displaystyle\int\frac{\log(\sec x+\tan x)}{\cos x}dx$,

put $\log(\sec x+\tan x)=z$

or, $\frac{1}{\sec x+\tan x}\cdot(\sec x\tan x+\sec^2 x)\,dx=dz$

or, $\frac{1}{\sec x+\tan x}\sec x(\tan x+\sec x)\,dx=dz$

or, $\sec x\,dx=dz$ or, $\frac{dx}{\cos x}=dz\cdot$

So, $\int\frac{\log(\sec x+\tan x)}{\cos x}dx$

$$=\int z dz=\frac{z^2}{z}=\frac{1}{2}\Big[\log(\sec x+\tan x)\Big]^2$$

Similarly, $\int\frac{\log(\sec y+\tan y)}{\cos y}dy$

$$=\frac{1}{2}\Big[\log(\sec y+\tan y)\Big]^2$$

So, from (1) we obtain,

$\frac{1}{2}[\log(\sec x+\tan x)]^2=\frac{1}{2}[\log(\sec y+\tan y)]^2+\frac{1}{2}c.$

or, $[\log(\sec x+\tan x)]^2=[\log(\sec y+\tan y)]^2+c.$

6. Find the general solutions of the equations

(i) $(x+y)(dx-dy)=dx+dy$

(ii) $xdx+ydy+\frac{xdy-ydx}{x^2+y^2}=0\cdot$

(i) $(x+y)(dx-dy)=dx+dy=d(x+y)$

or, $dx-dy=\frac{d(x+y)}{x+y}.$

Integrating we get

$$x-y=\log(x+y)-\log c$$

or, $x-y=\log\frac{x+y}{c}$ or, $\frac{x+y}{c}=e^{x-y}.$

or, $x+y=ce^{x-y}$

(ii) $\frac{x\,dy-y\,dx}{x^2+y^2}=\frac{\frac{x\,dy-y\,dx}{x^2}}{\frac{x^2+y^2}{x^2}}=\frac{d\left(\frac{y}{x}\right)}{1+\left(\frac{y}{x}\right)^2}.$

So, from the given equation we get

$$x\,dx+y\,dy+\frac{d\left(\frac{y}{x}\right)}{1+\left(\frac{y}{x}\right)^2}=0$$

or, $\int x\,dx+\int y\,dy+\int\frac{d\left(\frac{y}{x}\right)}{1+\left(\frac{y}{x}\right)^2}=\frac{c}{2}$

or, $\frac{x^2}{2}+\frac{y^2}{2}+\tan^{-1}\frac{y}{x}=c$

or, $x^2+y^2+2\tan^{-1}\frac{y}{x}=c$

and this is the required solution.

7. Find the general solution of the equation

$$\frac{dy}{dx}=\sin(x+y)+\cos(x+y).$$

Let $x+y=v$ $\therefore$ $1+\frac{dy}{dx}=\frac{dv}{dx}$ or, $\frac{dy}{dx}=\frac{dv}{dx}-1$.

Hence from the given equation we get,

$$\frac{dv}{dx}-1=\sin v+\cos v.$$

or, $\frac{dv}{dx}=1+\sin v+\cos v$

or, $\frac{dv}{1+\sin v+\cos v}=dx$

or, $\frac{dv}{2\cos^2\frac{v}{2}+2\sin\frac{v}{2}\cos\frac{v}{2}}=dx$

or, $\frac{\sec^2\frac{v}{2}dv}{2\left(1+\tan\frac{v}{2}\right)}=dx$

or, $\frac{dz}{1+z}=dx$ $\left[\text{Putting } \tan\frac{v}{2}=z\right.$

or, $\left.\frac{1}{2}\sec^2\frac{v}{2}dv=dz\right]$

or, $\log(1+z)=x+\log c$ (Integrating)

or, $\log\dfrac{1+z}{c}=x$ or, $1+z=ce^x$

or, $1+\tan\dfrac{y}{2}=ce^x$ or, $1+\tan\frac{1}{2}(x+y)=ce^x$

and this is the required solution.

8. Solve :

(i) $\dfrac{dy}{dx}=(x+y)^2$; (ii) $\dfrac{dy}{dx}=(4x+y+1)^2$.

(i) Let $x+y=v$ $\therefore$ $1+\dfrac{dy}{dx}=\dfrac{dv}{dx}$ or, $\dfrac{dy}{dx}=\dfrac{dv}{dx}-1$

Hence from the given equation we get,

$$\frac{dv}{dx}-1=v^2 \quad \text{or,} \quad \frac{dv}{dx}=1+v^2 \quad \text{or,} \quad \frac{dv}{1+v^2}=dx.$$

Integrating we get, $\tan^{-1}v=x+c$

or, $\tan^{-1}(x+y)=x+c$ or, $x+y=\tan(x+c)$

which is the required solution.

(ii) Let $4x+y+1=u$

or, $4+\dfrac{dy}{dx}=\dfrac{du}{dx}$ or, $\dfrac{dy}{dx}=\dfrac{du}{dx}-4$

So, from the given equation we get,

$$\frac{du}{dx}-4=u^2 \quad \text{or,} \quad \frac{du}{dx}=4+u^2, \quad \text{or,} \quad \int\frac{du}{4+u^2}=\int dx$$

or, $\frac{1}{2}\tan^{-1}\dfrac{u}{2}=x+c'$ or, $\tan^{-1}\dfrac{(4x+y+1)}{2}=2x+2c'$

or, $4x+y+1=2\tan(2x+c)$ [taking $c=2c'$]

and this is the required solution.

9. Solve :

(i) $\dfrac{dy}{dx}=\dfrac{x^2+y^2}{2x^2}$ (ii) $x+y\dfrac{dy}{dx}=2y$.

(i) The equation is homogeneous and so put $y=vx$

$\therefore$ $\dfrac{dy}{dx}=v+x\dfrac{dv}{dx}$ and $\dfrac{x^2+y^2}{2x^2}=\dfrac{x^2+v^2x^2}{2x^2}=\dfrac{1+v^2}{2}$.

So, from the given equation we get

$$v+x\frac{dv}{dx}=\frac{1+v^2}{2} \quad \text{or,} \quad x\frac{dv}{dx}=\frac{1+v^2}{2}-v=\frac{1+v^2-2v}{2}$$

$$=\frac{(v-1)^2}{2}.$$

or, $\dfrac{2dv}{(v-1)^2}=\dfrac{dx}{x}$ or, $\displaystyle\int\frac{2dv}{(v-1)^2}=\int\frac{dx}{x}$

or, $-\dfrac{2}{v-1}=\log x-\log c$ or, $-\dfrac{2}{\frac{y}{x}-1}=\log\dfrac{x}{c}$

or, $\dfrac{2x}{x-y}=\log\dfrac{x}{c}$ or, $\dfrac{x}{c}=e^{\frac{2x}{x-y}}$

or, $x=c\,e^{\frac{2x}{x-y}}$ and this is the required solution.

(ii) Putting $y=vx$, we get from the given equation,

$$x+vx\left(v+x\frac{dv}{dx}\right)=2vx.$$

or, $1+v^2+vx\,\dfrac{dv}{dx}=2v$

or, $v^2-2v+1+vx\,\dfrac{dv}{dx}=0$

or, $\dfrac{v\,dv}{(v-1)^2}+\dfrac{dx}{x}=0$

or, $\displaystyle\int\frac{v\,dx}{(v-1)^2}+\int\frac{dx}{x}=0$

or, $\displaystyle\int\frac{v-1}{(v-1)^2}dv+\int\frac{dv}{(v-1)^2}+\int\frac{dx}{x}=0$

or, $\log(v-1)-\dfrac{1}{v-1}+\log x=\log c$

or, $\log\left\{\left(\dfrac{y}{x}-1\right)\dfrac{x}{c}\right\}=\dfrac{1}{\frac{y}{x}-1}$ or, $\log\dfrac{y-x}{c}=\dfrac{x}{y-x}$

or, $y-x=c\;e^{\frac{x}{x-y}}$; which is the required solution.

10. Prove that in equations of the form $yf(xy)dx+xg(xy)dy=0$, the variables can be separated by putting $xy=v$.

Let $xy=v$ $\therefore$ $y=\dfrac{v}{x}$ or, $dy=\dfrac{x\,dv-v\,dx}{x^2}$

$\therefore$ $x\,dy=dv-\dfrac{v}{x}dx.$

So, from the given equation we shall obtain,

$$\frac{v}{x}f(v)\,dx+g(v)\left(dv-\frac{v}{x}dx\right)=0$$

or, $g(v)\ dv+\frac{v}{x}\,dx\{f(v)-g(v)\}=0$

or, $\frac{g(v)\ dv}{v\{g(v)-f(v)\}}=\frac{dx}{x}$ and the variables x and v are now separated.

11. Solve :

$$y(2xy+1)dx+x(1+2xy+x^2y^2)dy=0$$

Let $xy=v$ $\therefore$ $y=\frac{v}{x}$. So, $dy=\frac{x\ dv-v\ dx}{x^2}$.

$\therefore$ $x\ dy=dv-\frac{v}{x}dx.$

Now, from the given equation, we obtain

$$\frac{v}{x}(2v+1)dx+(1+2v+v^2)\left(dv-\frac{v}{x}dx\right)=0$$

or, $v(2v+1)dx+(1+v)^2(x\ dv-v\ dx)=0$

or, $dx\{v(2v+1)-v(1+v)^2\}+(1+v)^2x\ dv=0$

or, $-v^3dx+(1+v)^2x\ dv=0$

or, $\frac{dx}{x}=\frac{(1+v)^2}{v^3}=\frac{dv}{v^3}+\frac{2dv}{v^2}+\frac{dv}{v}$

or, $\log x=-\frac{1}{2v^2}-\frac{2}{v}+\log v+\log c$

or, $\log x-\log v-\log c=-\frac{1}{2v^2}-\frac{2}{v}$

or, $\log\frac{x}{cv}=-\frac{1}{2x^2y^2}-\frac{2}{xy}=-\frac{1+4xy}{2x^2y^2}$

or, $\log\frac{x}{cxy}=-\frac{1+4xy}{2x^2y^2}$ or, $\log\frac{1}{cy}=-\frac{1+4xy}{2x^2y^2}$

or, $-\log(cy)=-\frac{1+4xy}{2x^2y^2}$ or, $\log(cy)=\frac{1+4xy}{2x^2y^2}$

or, $cy=e^{\frac{1+4xy}{2x^2y^2}}$.

and this is the required solution.

12. Solve : $\frac{dy}{dx}=\frac{2x+9y-20}{6x+2y-10}$.

Let $x=x'+h,\ y=y'+k$; $\therefore$ $dx=dx',\ dy=dy'$.

$$\therefore \quad \frac{dy'}{dx'}=\frac{2(x'+h)+9(y'+k)-20}{6(x'+h)+2(y'+k)-10}=\frac{2x'+9y'+(2h+9k-20)}{6x'+2y'+(6h+2k-10)}$$

Now if $2h+9k-20=0$ and $6h+2k-10=0$, then $h=1$ and $k=2$ and for these values of h and k, the equation takes the form,

$$\frac{dy'}{dx'}=\frac{2x'+9y'}{6x'+2y'}.$$

Now, let $y'=vx'$ $\therefore$ $\dfrac{dy'}{dx'}=v+x'\dfrac{dv}{dx'}$

$$\therefore \quad v+x'\frac{dv}{dx'}=\frac{2+9v}{6+2v} \quad \text{or,} \quad x'\frac{dv}{dx'}=\frac{2+9v}{6+2v}-v$$

$$\text{or,} \quad x'\frac{dv}{dx'}=\frac{2+3v-2v^2}{6+2v} \quad \text{or,} \quad -\frac{(2v+6)dv}{2v^2-3v-2}=\frac{dx'}{x'}$$

$$\text{or,} \quad \frac{(2v+6)}{(2v+1)(v-2)}\,dv+\frac{dx'}{x'}=0.$$

$$\text{or,} \quad \frac{2}{v-2}\,dv-\frac{2}{2v+1}\,dv+\frac{dx'}{x'}=0$$ [See Appendix]

or, integrating both sides

$$2\log(v-2)-\log(2v+1)+\log x'=\log c$$

$$\text{or,} \quad \log\frac{x'(v-2)^2}{2v+1}=\log c$$

$$\text{or,} \quad x'(v-2)^2=c(2v+1)$$

$$\text{or,} \quad x'\left(\frac{y'}{x'}-2\right)^2=c\left(\frac{2y'}{x'}+1\right)$$

$$\text{or,} \quad (y'-2x)^2=c(x'+2y')$$

$$\text{or,} \quad [(y-2)-2(x-1)]^2=c[(x-1)+2(y-2)]$$

$$\text{or,} \quad (y-2x)^2=c(x+2y-5).$$

13. Solve : $\dfrac{dy}{dx}=\dfrac{x+y}{x+y-2}$.

Let $x+y=u$

$$\therefore \quad 1+\frac{dy}{dx}=\frac{du}{dx}$$

Hence from the given equation we get

$$\frac{du}{dx}-1=\frac{u}{u-2} \quad \text{or,} \quad \frac{du}{dx}=\frac{u}{u-2}+1=\frac{2u-2}{u-2}$$

$$\text{or,} \quad \frac{(u-2)\,du}{2u-2}=dx \quad \text{or} \quad \tfrac{1}{2}\left(1-\frac{1}{u-1}\right)du=dx$$

or, $\frac{1}{2}\{u-\log(u-1)\}=x+\frac{\log c}{2}$ [Integrating both sides]

or, $u-\log(u-1)=2x+\log c$

or, $x+y-\log(x+y-1)=2x+\log c$

or, $y-x=\log(x+y-1)+\log c=\log c(x+y-1)$

$\therefore$ $c(x+y-1)=e^{y-x}$ and this is the required solution.

Ex. 14. Solve : (i) $\tan x\frac{dy}{dx}=1+y^2$; when $x=\frac{\pi}{2}$ then $y=1$

[H. S. 1980]

(ii) $\frac{dy}{dx}=\frac{3x^2+1}{4y+2}$ when $x=1$, then $y=1$ [H. S. 1980]

(i) $\tan x\frac{dy}{dx}=1+y^2$ or $\frac{dy}{1+y^2}=\frac{dx}{\tan x}$

or, $\int\frac{dy}{1+y^2}=\int\cot x dx$

or, $\tan^{-1}y=\log\sin x+c$

$\left[\int\cot x dx=\int\frac{\cos x dx}{\sin x}=\int\frac{d(\sin x)}{\sin x}=\log(\sin x)+c\right]$

Now when $x=\frac{\pi}{2}$, then $y=1$

$\therefore$ $\tan^{-1}1=\log\sin\frac{\pi}{2}+c$

or, $\frac{\pi}{4}=\log 1+c$ or c [$\because$ $\log 1=0$]

$\therefore$ The required solution is $\tan^{-1}y=\log\sin x+\frac{\pi}{4}$.

(ii) $\frac{dy}{dx}=\frac{3x^2+1}{4y+2}$

or, $(3x^2+1)dx=(4y+2)dy$

or, $\int(3x^2+1)dx=\int(4y+2)dy$

or, $x^3+x=2y^2+2y+c.$

Now when $x=1$, then $y=1$

$\therefore$ $1+1=2+2+c$ or $c=-2.$

Hence the required solution is $x^3+x=2y^2+2y-2.$

Ex. 15. Solve : (i) $(1-x^2)\frac{dy}{dx}=2y$, when $x=2$, then $y=1$

[H. S. 1981]

(ii) $\tan x \dfrac{dy}{dx}=\tan y$, when $x=\frac{\pi}{6}$, then $y=\frac{\pi}{3}$.

(i) $(1-x^2)\dfrac{dy}{dx}=2y$.

or, $\dfrac{dy}{2y}=\dfrac{dx}{1-x^2}=-\dfrac{dx}{x^2-1}$

or, $\dfrac{dy}{2y}+\dfrac{dx}{x^2-1}=0$

or, $\int\dfrac{dy}{2y}+\int\dfrac{dx}{x^2-1}=\dfrac{c}{2}$

or, $\frac{1}{2}\log y+\frac{1}{2}\log\dfrac{x-1}{x+1}=\frac{1}{2}\log c$

or, $\log\left(y\dfrac{x-1}{x+1}\right)=\frac{1}{2}\log c$ or, $y\left(\dfrac{x-1}{x+1}\right)=c$

Now, when $x=2$, then $y=1$

or, $1.\dfrac{2-1}{2+1}=c$ $\therefore$ $c=\frac{1}{3}$.

Hence the required solution is $y\left(\dfrac{x-1}{x+1}\right)=\dfrac{1}{3}$ or, $3y(x-1)=1+x$.

(ii) $\tan x\dfrac{dy}{dx}=\tan y$

or, $\dfrac{dy}{\tan y}=\dfrac{dx}{\tan x}$ or, $\cot y\,dy=\cot x\,dx$

or, $\int\cot y\,dy=\int\cot x\,dx$

or, $\log\sin y=\log\sin x+\log c$

or, $\log\sin y=\log(c\sin x)$

or, $\sin y=c\sin x$. Now, when $x=\frac{\pi}{6}$ then $y=\frac{\pi}{3}$.

$\therefore$ $\sin\frac{\pi}{3}=c\sin\frac{\pi}{6}$

or, $\dfrac{\sqrt{3}}{2}=c.\dfrac{1}{2}$ $\therefore$ $c=\sqrt{3}$.

Hence the required solution is $y=\sqrt{3}\sin x$.

Ex. 16. Solve : $\dfrac{dy}{dx}=0$ when $x<1$

$=1$ when $x>1$

$y(2)=3$ and y is continuous at $x=1$

[$y(2)$ means value of y when $x=2$]

$\frac{dy}{dx}=0$ when $x<1$ or, $dy=0$ when $x<1$

or, $y=c_1$ (Integrating) when $x<1$.

$\frac{dy}{dx}=1$ when $x>1$ or, $dy=dx$ or $y=x+c_2$ (Integrating)

when $x=2$, then $y=3$

$\therefore$ $3=2+c_2$ $\therefore$ $c_2=1$

$\therefore$ $y=x+1$ when $x>1$.

Now y is continuous at $x=1$.

$\therefore \lim\limits_{x\to 1+} y = \lim\limits_{x\to 1} y$

or, $\lim\limits_{x\to 1-} (x+1) = \lim\limits_{x\to 1+} (c_1)$

or, $2=c_1$

$\therefore$ The required solution is

$y=2k$ when $x<1$

$y=x+1$ when $x>1$

Ex. 17. Solve : $\frac{dy}{dx}=2x$ when $x>0$

$=1$ when $x<0$

$y=2$ when $x=1$; $y(0)$ is continuous.

$\frac{dy}{dx}=2x$ when $x>0$ or $dy=2x\,dx$

or, $y=x^2+c_1$ (Integrating).

$y=2$ when $x=1$ $\therefore$ $2=1+c_1$ $\therefore$ $c_1=1$ $\therefore$ $y=x^2+1$

when $x<1$, $\frac{dy}{dx}=1$ or $dy=dx$

or, $\int dy=\int dx$ or, $y=x+c_2$

Now $y(0)$ is continuous.

$\therefore \lim\limits_{x\to 0+} y = \lim\limits_{x\to 0-} y$

or, $\lim\limits_{x\to 0+} (x^2+1) = \lim\limits_{x\to 0-} (x+c_2)$

or, $1=c_2$

So, required solution is

$\therefore$ $y=x^2+1$ when $x>0$

$=x+1$ when $x<0$.

Exercise 5

Find the general solution of the following equations.

1. $x^2(y-1)\,dx+y^2(x-1)\,dy=0$.
2. $x\cos^2 y\,dx-y\cos^2 x\,dy=0$.
3. $(x^2-yx^2)\,dy+(y^2+xy^2)\,dx=0$.
4. $x^2(x\,dx+y\,dy)+2a\,(x\,dy-y\,dx)=0$.
5. $(x+y)^2\dfrac{dy}{dx}=a^2$.
6. $\dfrac{dy}{dx}=\sqrt{x+y}$.
7. $\dfrac{dy}{dx}=\sin\,(x+y)$.
8. $\dfrac{dy}{dx}=\dfrac{3x+2y}{2x-3y}$.
9. $\dfrac{dy}{dx}=\dfrac{y(x-2y)}{x(x-3y)}$.
10. $\dfrac{dy}{dx}=\dfrac{y(y+x)}{x(y-x)}$.
11. $y^2\,dx+(x^2+xy)dy=0$.
12. $\dfrac{dy}{dx}=\dfrac{y}{x}+\tan\dfrac{y}{x}$.
13. $(x-3y+4)\,dy+(7y-5x)\,dx=0$.
14. $(x+y+1)\,dx-(3x+3y+1)\,dy=0$.
15. $RI+L\dfrac{dI}{dt}=0$ and when $t=0$, $I=I_0$

    [R and L are constants].
16. $r\dfrac{dp}{dr}+2p=2A$.
17. $\dfrac{dy}{dx}=\dfrac{x+1}{y+1}$; when $x=1$, then $y=1$.
18. $\dfrac{dy}{dx}=\dfrac{x^2+x+1}{y^2+y+1}$ when $x=0$, then $y=0$.
19. $\sec^2 x\tan y\,dx+\sec^2 y\tan x\,dy=0$.

    when $x=\dfrac{\pi}{3}$, then $y=\dfrac{\pi}{6}$.
20. $\dfrac{dy}{dx}+\dfrac{y^2+y+1}{x^2+x+1}=0$ when $x=1$ and $y=1$.
21. $x^2(y-1)\,dx+y^2(x-1)\,dy=0$ when $x=2$, then $y=2$.

22. $\dfrac{dy}{dx}=\dfrac{3x+2y}{2x-3y}$ when $x=1$, then $y=1$.

23. $y^2\,dx+(x^2+xy)\,dy=0$, when $x=2$ and $y=1$.

24. $y=2$ when $x>1$

$=1$ when $x<1$

when $x=2$, then $y=5$; $y(1)$ is continuous at x.

25. $\dfrac{dy}{dx}=-2$ when $x<\frac{1}{2}$

$=2$ when $x>\frac{1}{2}$.

$y(-1)=3$; $y(\frac{1}{2})$ is continuous.

26. $\dfrac{dy}{dx}=6x$ when $x>2$

$=7$ when $x<2$

$y(3)=29$; $y(2)$ is continuous.

APPENDIX

Integration of Algebraic Rational Functions

§ A·1. Algebraic Rational Functions

Functions of the form $\dfrac{f(x)}{g(x)}=\dfrac{a_0x^n+a_1x^{n-1}+\cdots+a_{n-1}x+a_n}{b_0x^m+b_1x^{m-1}+\cdots+b_{m-1}x+b_m}$ (where $a_0, a_1, \cdots, a_n, b_0, b_1, \cdots, b_m$ are real numbers and m and n are positive integers) are called **Algebraic Rational Functions.** So So functions of the form $\dfrac{1}{x^2-5x+6}$, $\dfrac{1}{x^2-9}$, $\dfrac{1}{x^2+4}$, $\dfrac{ax^2+bx+c}{x^4}$, $\dfrac{x^2+x-1}{x^3+x^2-6x}$, $\dfrac{x}{(x-1)(x^2-4)}$, $\dfrac{x^2}{(x+1)(x+2)^2}$ etc. are algebraic rational functions.

We have already indicated different methods of integration of algebraic rational functions. Presently we propose to make a more detailed discussion of these methods. But before such discussion it is necessary to discuss different methods of breaking a rational algebraic function into partial fractions.

§ A·2. Partial Fractions. In the present section we are discussing different methods of breaking a rational algebraic function into partial fractions when the degree of the numerater of the function is less than that of the denominator of the function.

1. When the denominator can be resolved into product of two or more linear factors.

Example 1. To express $\dfrac{x}{x^2-5x+6}$ as the sum of partial fractions.

Let $\dfrac{x}{x^2-5x+6}=\dfrac{x}{(x-2)(x-3)}=\dfrac{A}{x-2}+\dfrac{B}{x-3}$

$\therefore\ \dfrac{x}{(x-2)(x-3)}=\dfrac{A(x-3)+B(x-2)}{(x-2)(x-3)}$

$\therefore\ x=A(x-3)+B(x-2)\ \cdots\ \cdots\ (1)$

$\quad =x(A+B)-(3A+2B).$

Now the coefficients of x and the constant terms on both sides of (1) are equal.

$\because$ $A+B=1$ and $3A+2B=0$.

Solving we get $A=-2$ and $B=3$.

So $\dfrac{x}{x^2-5x+6}=\dfrac{3}{x-3}-\dfrac{2}{x-2}$.

From the above example we get the following rule :—

If $\alpha_1, \alpha_2, \cdots, \alpha_n$ be different from one another and the degree of $f(x)$ be less than n, then $\dfrac{f(x)}{(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_n)}$ can be expressed as the sum of n partial fractions of the form $\dfrac{A_1}{x-\alpha_1}, \dfrac{A_2}{x-\alpha_2}, \cdots, \dfrac{A_n}{x-\alpha_n}$.

The values of $A_1, A_2, \cdots, A_n$ can be determined by equating the coefficients of $x^{n-1}, x^{n-2}, \cdots$ and x^0 (i e., the constant term) on both sides.

Ex. 2. Express $\dfrac{x^2}{(x-1)(x-2)(x-3)}$ into sum of partial fractions and hence determine $\displaystyle\int\frac{x^2\,dx}{(x-1)(x-2)(x-3)}$.

Let $\dfrac{x^2}{(x-1)(x-2)(x-3)}=\dfrac{A}{x-1}+\dfrac{B}{x-2}+\dfrac{C}{x-3}$.

Now, $\dfrac{A}{x-1}+\dfrac{B}{x-2}+\dfrac{C}{x-3}$

$$=\frac{A(x-2)(x-3)+B(x-3)(x-1)+C(x-1)(x-2)}{(x-1)(x-2)(x-3)}$$

$$=\frac{x^2(A+B+C)-x(5A+4B+3C)+6A+3B+2C}{(x-1)(x-2)(x-3)}$$

$\therefore$ $\dfrac{x^2}{(x-1)(x-2)(x-3)}$

$$=\frac{x^2(A+B+C)-x(5A+4B+3C)+6A+3B+2C}{(x-1)(x-2)(x-3)}$$

$\therefore$ $x^2=x^2(A+B+C)-x(5A+4B+3C)+6A+3B+2C$

Now, the coefficients of x^2, x and the constant terms on both sides are equal. $\therefore$ $A+B+C=1\ldots$(i) $5A+4B+3C=0\ldots$(2) and $6A+3B+2C=0\ldots$(3).

Solving the three equations we get $A=\frac{1}{2}$, $B=-4$, $C=\frac{9}{2}$.

$$\therefore \quad \frac{x^2}{(x-1)(x-2)(x-3)}=\frac{1}{2(x-1)}-\frac{4}{x-2}+\frac{9}{2(x-3)}$$

$$\text{So} \quad \int\frac{x^2\,dx}{(x-1)(x-2)(x-3)}=\int\left\{\frac{1}{2(x-2)}-\frac{4}{x-2}+\frac{9}{2(x-3)}\right\}dx$$

$$=\tfrac{1}{2}\int\frac{dx}{x-1}-4\int\frac{dx}{x-2}+\tfrac{9}{2}\int\frac{dx}{x-3}$$

$$=\tfrac{1}{2}\log(x-1)-4\log(x-2)+\tfrac{9}{2}\log(x-3)+c.$$

2. Rule of expressing a rational algebraic function when the denominator of the function can be expressed as the product of linear factors, some of which are repeated.

Rule. When the denominator is of the form $(x-a)(x-b)^m (x-c)^n$, then the function can be expressed as the sum of $(1+m+n)$ partial fractions in the form $\frac{A}{x-a}+\frac{B_1}{x-b}+\frac{B_2}{(x-b)^2}+\cdots+\frac{B_m}{(x-b)^m}+\frac{C_1}{x-c}+\frac{C_2}{(x-c)^2}+\cdots+\frac{C_n}{(x-c)^n}$. The constants $A, B_1, B_2, \cdots B_m, C_1, C_2, \cdots C_n$ can be determined from the equality of the coefficients of the different powers of x on both sides.

Example 3. Integrate $\frac{x^2}{(x+1)(x+2)^2}$ by expressing it as the sum of partial fractions.

$$\text{Let} \quad \frac{x^2}{(x+1)(x+2)^2}=\frac{A}{x+1}+\frac{B}{x+2}+\frac{C}{(x+2)^2}$$

$$\text{Now} \quad \frac{A}{x+1}+\frac{B}{x+2}+\frac{C}{(x+2)^2}$$

$$=\frac{A(x+2)^2+B(x+1)(x+2)+C(x+1)}{(x+1)(x+2)^2}$$

$$=\frac{x^2(A+B)+x(4A+3B+C)+4A+2B+C}{(x+1)(x+2)^2}$$

$$\therefore \quad \frac{x^2}{(x+1)(x+2)^2}=\frac{x^2(A+B)+x(4A+3B+C)+4A+2B+C}{(x+1)(x+2)^2}$$

$$\therefore \quad x^2=x^2(A+B)+x(4A+3B+C)+4A+2B+C$$

Now the coefficients of x^2, x and the constant terms on both the sides are equal.

So $A+B=1 \cdots (1)$, $4A+3B+C=0 \cdots (2)$ and $4A+2B+C=0 \cdots (3)$

Solving equations (1), (2) and (3) we get $A=1$, $B=0$ and $C=-4$.

$$\therefore \quad \frac{x^2}{(x+1)(x+2)^2}=\frac{1}{x+1}-\frac{4}{(x+2)^2}$$

So, $$\int\frac{x^2dx}{(x+1)(x+2)^2}=\int\left\{\frac{1}{x+1}-\frac{4}{(x+2)^2}\right\}dx$$

$$=\int\frac{1}{x+1}dx-4\int\frac{dx}{(x+2)^2}=\log(x+1)-4\left(-\frac{1}{x+2}\right)+c$$

$$=\log(x+1)+\frac{4}{x+2}+c.$$

Ex. 4. Integrate : $\int\frac{dx}{x(x+1)^2}$

Let, $\frac{1}{x(x+1)^2}=\frac{A}{x}+\frac{B}{x+1}+\frac{C}{(x+1)^2}$

Now, $\frac{A}{x}+\frac{B}{x+1}+\frac{C}{(x+1)^2}=\frac{A(x+1)^2+Bx(x+1)+Cx}{x(x+1)^2}$

$$=\frac{x^2(A+B)+x(2A+B+C)+A}{x(x+1)^2}$$

$$\therefore \quad \frac{1}{x(x+1)^2}=\frac{x^2(A+B)+x(2A+B+C)+A}{x(x+1)^2}$$

Now, the coefficients of x^2, x and the constant terms on both the sides are equal.

$A+B=0 \cdots (1)$, $2A+B+C=0 \cdots (2)$, $A=1 \cdots (3)$.

Solving equations (1), (2) and (3) we obtain,

$A=1$, $B=-1$, $C=-1$.

$$\therefore \quad \frac{1}{x(x+1)^2}=\frac{1}{x}-\frac{1}{x+1}-\frac{1}{(x+1)^2}$$

$$\therefore \quad \int\frac{dx}{x(x+1)^2}=\int\frac{dx}{x}-\int\frac{dx}{x+1}-\int\frac{dx}{(x+1)^2}$$

$$=\log x-\log(x+1)+\frac{1}{x+1}+c$$

$$=\log\frac{x}{x+1}+\frac{1}{x+1}+c.$$

Ex. 5. Integrate : $\int\frac{x^2+x-1}{x^3+x^2-6x}dx.$ [P. P. 1931]

$x^3+x^2-6x=x(x^2+x-6)=x(x+3)(x-2)$

Let $\dfrac{x^2+x-1}{x^3+x^2-6x}=\dfrac{A}{x}+\dfrac{B}{x+3}+\dfrac{C}{x-2}$.

Now, $\dfrac{A}{x}+\dfrac{B}{x+3}+\dfrac{C}{x-2}=\dfrac{A(x+3)(x-2)+Bx(x-2)+Cx(x+3)}{x(x+3)(x-2)}$

$\therefore\quad \dfrac{x^2+x-1}{x(x+3)(x-2)}=\dfrac{A(x+3)(x-2)+Bx(x-2)+Cx(x+3)}{x(x+3)(x-2)}$

$\therefore\quad x^2+x-1=A(x+3)(x-2)+Bx(x-2)+Cx(x+3)$

Now this equality is true for all values of x. So putting $x=0$, -3 and 2 successively on both sides we get $A=\frac{1}{6}$, $B=\frac{1}{3}$, $C=\frac{1}{2}$.

$$\therefore\quad \int\frac{x^2+x-1}{x^3+x^2-6x}dx=\tfrac{1}{6}\int\frac{dx}{x}+\tfrac{1}{3}\int\frac{dx}{x+3}+\tfrac{1}{2}\int\frac{dx}{x-2}$$

$$=\tfrac{1}{6}\log x+\tfrac{1}{3}\log(x+3)+\tfrac{1}{2}\log(x-2)+c.$$

Note. In the first four examples, the values of A, B, C etc. were determined by equating coefficients of different powers of x on both sides. In example 5 above we have used an alternative method. In example 6 below, we shall use both the methods. But the method used in the first four examples (*i.e.*, the method of equating coefficients) in the general method. But solving the corresponding equations frequently becomes trouble some.

Ex. 6. Integrate : $\displaystyle\int\frac{dx}{(x-a)^2(x-b)}$

Let $\dfrac{1}{(x-a)^2(x-b)}=\dfrac{A}{(x-a)^2}+\dfrac{B}{x-a}+\dfrac{C}{x-b}$

$$=\frac{A(x-b)+B(x-a)(x-b)+C(x-a)^2}{(x-a)^2(x-b)}$$

$\therefore\quad 1=A(x-b)+B(x-a)(x-b)+C(x-a)^2$

Putting $x=a$ on both sides we get $1=A(a-b)$

or, $A=\dfrac{1}{a-b}$.

Again putting $x=b$ on both sides we get $1=C(b-a)^2$

$\therefore\quad C=\dfrac{1}{(b-a)^2}$.

Also coefficients of x^2 on both sides are equal

$\therefore\quad B+C=0\quad\therefore\quad B=-C=-\dfrac{1}{(b-a)^2}$.

$$\therefore \quad \frac{1}{(x-a)^2(x-b)}=\frac{1}{a-b}\cdot\frac{1}{(x-a)^2}-\frac{1}{(b-a)^2}\frac{1}{x-a}+\frac{1}{(b-a)^2}\cdot\frac{1}{x-b}.$$

$$\therefore \quad \int\frac{dx}{(x-a)^2(x-b)}=\frac{1}{a-b}\int\frac{dx}{(x-a)^2}-\frac{1}{(b-a)^2}\int\frac{dx}{x-a}+\frac{1}{(b-a)^2}\int\frac{dx}{x-b}$$

$$=\frac{1}{a-b}\left(-\frac{1}{x-a}\right)-\frac{1}{(b-a)^2}\log(x-a)+\frac{1}{(b-a)^2}\log(x-b)+k$$

$$=\frac{1}{(b-a)(x-a)}+\frac{1}{(b-a)^2}\log\frac{x-b}{x-a}+k.$$

3. If the denominator contains one or more different quadratic factors, the algebraic rational function will have corresponding to every quadratic factor x^2+bx+c $(c\neq 0)$,

a partial fraction of the form $\dfrac{Ax+B}{x^2+bx+c}$.

Ex. 7. Integrate : $\displaystyle\int\frac{x\,dx}{x^3-1}$.

Let $\dfrac{x}{x^3-1}=\dfrac{x}{(x-1)(x^2+x+1)}=\dfrac{A}{x-1}+\dfrac{Bx+C}{x^2+x+1}$;

$\therefore \quad x=A(x^2+x+1)+(Bx+C)(x-1).$

Putting $x=1$ on both sides we gett $1=3A$ $\therefore$ $A=\frac{1}{3}$.

Again coefficients of x^2 on both sides are equal.

$\therefore$ $A+B=0$. $\therefore$ $B=-A=-\frac{1}{3}$.

Again constant terms on both sides are equal.

$\therefore$ $A-C=0$ $\therefore$ $C=A=\frac{1}{3}$.

$$\therefore \quad \int\frac{x\,dx}{x^3-1}=\tfrac{1}{3}\int\left(\frac{1}{x-1}-\frac{x-1}{x^2+x+1}\right)dx.$$

$$=\tfrac{1}{3}\int\frac{dx}{x-1}-\tfrac{1}{3}\int\frac{x-1}{x^2+x+1}dx$$

$$=\tfrac{1}{3}\int\frac{dx}{x-1}-\tfrac{1}{6}\int\frac{2x+1-3}{x^2+x+1}dx=\tfrac{1}{3}\int\frac{dx}{x+1}-\tfrac{1}{6}\int\frac{2x+1}{x^2+x+1}dx+\tfrac{1}{2}\int\frac{dx}{x^2+x+1}$$

$$=\tfrac{1}{3}\log(x-1)-\tfrac{1}{6}\log(x^2+x+1)+\frac{1}{\sqrt{3}}\tan^{-1}\frac{2x+1}{\sqrt{3}}+c$$

[See Ex. 2. § 2·8]

Ex. 8. Integrate : $\int \frac{x\,dx}{(1+x)(1+x^2)}$.

Let $\frac{x}{(1+x)(1+x^2)}=\frac{A}{1+x}+\frac{Bx+C}{1+x^2}$

$\therefore\ x=A(1+x^2)+(Bx+C)(1+x)$.

Putting $x=-1$ on both sides we get $-1=2A$, $\therefore\ A=-\frac{1}{2}$.

Equating coefficients of x^2 on both sides we get,

$0=A+B\ \ \therefore\ \ B=-A=\frac{1}{2}$.

Again the constant terms on both sides are equal.

$\therefore\ 0=A+C\ \ \therefore\ \ C=-A=\frac{1}{2}$.

$$\therefore\ \frac{x}{(1+x)(1+x^2)}=\tfrac{1}{2}\left(\frac{x+1}{1+x^2}-\frac{1}{1+x}\right)$$

$$\therefore\ \int\frac{x\,dx}{(1+x)(1+x^2)}=\int\tfrac{1}{2}\left(\frac{x+1}{1+x^2}-\frac{1}{1+x}\right)dx.$$

$$=\tfrac{1}{4}\int\frac{2x\,dx}{1+x^2}+\tfrac{1}{2}\int\frac{dx}{1+x^2}-\tfrac{1}{2}\int\frac{dx}{1+x}$$

$$=\tfrac{1}{4}\log(1+x^2)+\tfrac{1}{2}\tan^{-1}x-\tfrac{1}{2}\log(1+x)+c.$$

§A·3. Integration of rational algebraic function when the degree of the numerator is greater than or equal to the degree of the denominator.

Let $\frac{f(x)}{g(x)}$ be an algebraic rational function. To integrate $\frac{f(x)}{g(x)}$ with respect to x divide $f(x)$ by $g(x)$ untill the degree of the remainder is less than the degree of the denominator. Let in such a division $q(x)$ and $r(x)$ be respectively the quotient and the remainder where $q(x)$ and $r(x)$ are polynomials.

$$\therefore\ \frac{f(x)}{g(x)}=q(x)+\frac{r(x)}{g(x)}\text{ and }\int\frac{f(x)}{g(x)}dx$$

$$=\int\left\{q(x)+\frac{r(x)}{g(x)}\right\}dx=\int q(x)dx+\int\frac{r(x)}{g(x)}dx$$

Now, as $q(x)$ is a polynomial, so $\int q(x)\,dx$ can be easily determined. Also $\int\frac{r(x)}{g(x)}\,dx$ can be determined following the rules discussed in § A·2.

Example 1. Integrate : $\int \frac{x^3\,dx}{x^2+7x+12}$

$$\begin{array}{r|l|l}
x^2+7x+12 & x^3 & x-7 \\
 & \underline{x^3+7x^2+12x} & \\
 & -7x^2-12x & \\
 & \underline{-7x^2-49x-84} & \\
 & 37x+84 &
\end{array}$$

$$\therefore \quad \frac{x^3}{x^2+7x+12}=x-7+\frac{37x+84}{x^2+7x+12}=x-7+\frac{37x+84}{(x+3)(x+4)}$$

Now, let $\quad \frac{37x+84}{(x+3)(x+4)}=\frac{A}{x+3}+\frac{B}{x+4}.$

$\therefore \quad 37x+84=A(x+4)+B(x+3)=x(A+B)+4A+3B$

$\therefore \quad A+B=37$ and $4A+3B=84$

Solving we get $A=-27,\ B=64$

$$\therefore \quad \frac{37x+84}{(x+3)(x+4)}=-\frac{27}{x+3}+\frac{64}{x+4}$$

$$\therefore \quad \int\frac{x^3dx}{x^2+7x+12}=\int\left(x-7-\frac{27}{x+3}+\frac{64}{x+4}\right)dx$$

$$=\int x\,dx-7\int dx-27\int\frac{dx}{x+3}+64\int\frac{dx}{x+4}$$

$$=\frac{x^2}{3}-7x-27\log(x+3)+64\log(x+4)+c.$$

Ex. 2. Integrate : $\int\frac{x^4+x^2+1}{(x^2+1)(x+1)}dx.$

$(x^2+1)(x+1)=x^3+x^2+x+1.$

$$\begin{array}{r|l|l}
x^3+x^2+x+1 & x^4+x^2+1 & x-1 \\
 & \underline{x^4+x^3+x^2+x} & \\
 & -x^3-x+1 & \\
 & \underline{-x^3-x^2-x-1} & \\
 & x^2+2 &
\end{array}$$

$$\therefore \quad \frac{x\ +x^2+1}{(x^2+1)(x+1)}=x-1+\frac{x^2+2}{(x^2+1)(x+1)}$$

Now, let $\frac{x^2+2}{(x^2+1)(x+1)}=\frac{Ax+B}{x^2+1}+\frac{C}{x+1}$

$\therefore \quad x^2+2=(Ax+B)(x+1)+C(x^2+1)$

Putting $x=-1$ on both sides we get,

$2C=3, \quad \therefore \quad C=\frac{3}{2}.$

Again as the constant terms on both sides are equal, $2 = B + C$.

$\therefore \quad B = 2 - C = 2 - \frac{3}{2} = \frac{1}{2}.$

Also the coefficients of x^2 on both sides are equal.

$\therefore \quad 1 = A + C \quad \therefore \quad A = 1 - C = 1 - \frac{3}{2} = -\frac{1}{2}.$

$$\therefore \quad \frac{x^2+2}{(x^2+1)(x+1)} = \frac{3}{2(x+1)} - \frac{x-1}{2(x^2+1)}$$

So, $$\int \frac{x^4+x^2+1}{(x^2+1)(x+1)}dx = \int \left\{x - 1 + \frac{3}{2(x+1)} - \frac{x-1}{2(x^2+1)}\right\}dx$$

$$= \int x\,dx - \int dx + \tfrac{3}{2}\int \frac{dx}{x+1} - \tfrac{1}{4}\int \frac{2x\,dx}{x^2+1} + \tfrac{1}{2}\int \frac{dx}{x^2+1}$$

$$= \frac{x^2}{2} - x + \tfrac{3}{2}\log(x+1) - \tfrac{1}{4}\log(x^2+1) + \tfrac{1}{2}\tan^{-1}x + c.$$

§ A·4. A special technique in a special case.

If both the numerator and denominator contain only even powers of x, then it is convenient to express the function as the sum of partial fractions by putting $x^2 = t$. Note that here the variable t does not replace the variable x. Before integration, replace t by x^2.

Example. 1. Integrate : $\displaystyle\int \frac{x^2dx}{(x^2+a^2)(x^2+b^2)}$

$$\frac{x^2}{(x^2+a^2)(x^2+b^2)} = \frac{t}{(t+a^2)(t+b^2)} \quad [\text{Putting } x^2 = t]$$

$$= \frac{1}{a^2-b^2}\left(\frac{a^2}{t+a^2} - \frac{b^2}{t+b^2}\right) = \frac{1}{a^2-b^2}\left(\frac{a^2}{x^2+a^2} - \frac{b^2}{x^2+b^2}\right)$$

$$\therefore \quad \int \frac{x^2dx}{(x^2+a^2)(x^2+b^2)} = \int \frac{1}{a^2-b^2}\left\{\frac{a^2}{x^2+a^2} - \frac{b^2}{x^2+b^2}\right\}dx$$

$$= \frac{a^2}{a^2-b^2}\int \frac{dx}{x^2+a^2} - \frac{b^2}{a^2-b^2}\int \frac{dx}{x^2+b^2}$$

$$= \frac{a^2}{a^2-b^2}\,\frac{1}{a}\tan^{-1}\frac{x}{a} - \frac{b^2}{a^2-b^2}.\ \frac{1}{b}\tan^{-1}\frac{x}{b} + c$$

$$= \frac{1}{a^2-b^2}\left(a\tan^{-1}\frac{x}{a} - b\tan^{-1}\frac{x}{b}\right) + c.$$

Ex. 2. Integrate : $\displaystyle\int \frac{(x^2+1)dx}{x^4-3x^2+2}.$

Let, $x^2 = t$.

$$\therefore \quad \frac{x^2+1}{x^4-3x^2+2} = \frac{t+1}{t^2-3t+2} = \frac{t+1}{(t-1)(t-2)} = \frac{A}{t-1} + \frac{B}{t-2}$$

$\therefore \quad t + 1 = A(t-2) + B(t-1)$

Putting $t=1$ and 2 successively on both sides we get $A=-2$, $B=3$.

$$\because \quad \frac{x^2+1}{x^4-3x^2+2}=\frac{3}{t-2}-\frac{2}{t-1}=\frac{3}{x^2-2}-\frac{3}{x^2-1}$$

$$\therefore \quad \int\frac{x^2+1}{x^4-3x^2+2}dx=3\int\frac{dx}{x^2-2}-2\int\frac{dx}{x^2-1}$$

$$=\frac{3}{2\sqrt{2}}\log\left|\frac{x-\sqrt{2}}{x+\sqrt{2}}\right|-\log\left|\frac{x-1}{x+2}\right|+c.$$

§ A·5. **A special substitution.**

If in an algebraic rational fraction the numerator and denominator contain respectively only odd powers of x and only even powers of x, then the substitution $x^2=t$ is frequently found convenient. In this case the integrand is expressed in terms of t. This new integrand is then broken up as sums of partial fractions.

Example 1. Integrate : $\int\frac{x^3dx}{x^4+x^2+1}$

Let $x^2=t$; $\therefore$ $2xdx=dt$ and $x^3dx=x^2.xdx=\frac{1}{2}tdt$

Also $x^4+x^2+1=t^2+t+1$.

$$\therefore \quad \int\frac{x^3dx}{x^4+x^2+1}=\tfrac{1}{2}\int\frac{tdt}{t^2+t+1}=\tfrac{1}{4}\int\frac{2t+1}{t^2+t+1}dt-\tfrac{1}{4}\int\frac{dt}{t^2+t+1}$$

$$=\tfrac{1}{4}\log(t^2+t+1)-\tfrac{1}{4}.\frac{2}{\sqrt{3}}\tan^{-1}\frac{2t+1}{\sqrt{3}}+c$$

$$=\tfrac{1}{4}\log(x^4+x^2+1)-\frac{1}{2\sqrt{3}}\tan^{-1}\frac{2x^2+1}{\sqrt{3}}+c$$

Ex. 2. Integrate : $\int\frac{x^5dx}{1+x^4}$.

$$\frac{x^5}{1+x^4}=x-\frac{x}{1+x^4}.$$

$$\therefore \quad \int\frac{x^5dx}{1+x^4}=\int\left(x-\frac{x}{1+x^4}\right)dx=\int xdx-\int\frac{xdx}{1+x^4}.$$

Now $\int xdx=\frac{x^2}{2}+C_1$

For determination of $\int\frac{xdx}{1+x^4}$,

Put $x^2=t$ $\therefore$ $2xdx=dt$ or, $xdx=\frac{dt}{2}$.

Also $1+x^4=1+t^2$.

$$\therefore \quad \int\frac{x\,dx}{1+x^4}=\tfrac{1}{2}\int\frac{dt}{1+t^2}=\tfrac{1}{2}\tan^{-1}t+C_2=\tfrac{1}{2}\tan^{-1}x^2+C_2$$

$$\therefore \quad \int\frac{x^5dx}{1+x^4}=\frac{x^2}{2}+C_1-\tfrac{1}{2}\tan^{-1}x^2+C_2=\frac{x^2}{2}+\tfrac{1}{2}\tan^{-1}x^2+C.$$

Miscellaneous Examples

Example 1. Integrate : $\displaystyle\int\frac{x^3}{(x-a)(x-b)(x-c)}dx$

[C. U. 1954]

Let $\displaystyle\frac{x^3}{(x-a)(x-b)(x-c)}=1+\frac{A}{x-a}+\frac{B}{x-b}+\frac{C}{x-c}.$

$$\therefore \quad x^3=(x-a)(x-b)(x-c)+A(x-b)(x-c)$$
$$+B(x-c)(x-a)+C(x-a)(x-b).$$

Putting $x=a$, $x=b$, $x=c$ successively in both sides we get

$$A=\frac{a^3}{(a-b)(a-c)},\ B=\frac{b^3}{(b-c)(b-a)} \text{ and } \frac{c^3}{(c-a)(c-b)}$$

$$\therefore \quad \int\frac{x^3}{(x-a)(x-b)(x-c)}dx=\int\left(1+\frac{A}{x-a}+\frac{B}{x-b}+\frac{C}{x-c}\right)dx$$

$$=\int dx+A\int\frac{dx}{x-a}+B\int\frac{dx}{x-b}+C\int\frac{dx}{x-c}.$$

$$=x+A\log(x-a)+B\log(x-b)+C\log(x-c)+k$$

$$=x+\frac{a^3}{(a-b)(a-c)}\log(x-a)+\frac{b^3}{(b-c)(b-a)}\log(x-b)$$
$$+\frac{c^3}{(c-a)(c-b)}\log(x-c)+k.$$

[Putting the values of A, B and C]

Note. Notice that here the numerator and denominator are both of degree 3. So, the numerator is divided by the denominator. The quotient $q(x)$ is 1 and the remainder is written in the form $\dfrac{A}{x-a}+\dfrac{B}{x-b}+\dfrac{C}{x-c}$.

Ex. 2. Integrate : $\displaystyle\int\frac{(x-1)dx}{(x+2)(x-3)}$ [C. U. 1924]

Let $\displaystyle\frac{x-1}{(x+2)(x-3)}=\frac{A}{x+2}+\frac{B}{x-3}$

$$\therefore \quad x-1=A(x-3)+B(x+2)$$

Putting $x=-2$ on both sides we get $-3=-5A$ or, $A=\frac{3}{5}$.

Putting $x=3$ on both sides we get $2=5B$ $\therefore$ $B=\frac{2}{5}$.

So, $\dfrac{x-1}{(x+2)(x-3)}=\dfrac{3}{5(x+2)}+\dfrac{2}{5(x-3)}$

$\therefore$ $\displaystyle\int\frac{x-1}{(x+2)(x-3)}dx=\frac{3}{5}\int\frac{dx}{x+2}+\frac{2}{5}\int\frac{dx}{x-3}$

$$=\frac{3}{5}\log(x+2)+\frac{2}{5}\log(x-3)+c.$$

Ex. 3. Integrate : $\displaystyle\int\frac{xdx}{(x+a)^2(x+b)}$.

Let $\dfrac{x}{(x+a)^2(x+b)}=\dfrac{A}{(x+a)^2}+\dfrac{B}{x+a}+\dfrac{C}{x+b}$.

$\therefore$ $x=A(x+b)+B(x+a)(x+b)+C(x+a)^2$.

Putting $x=-a$ on both sides we get,

$$-a=A(b-a) \quad \therefore \quad A=\frac{a}{a-b}.$$

Putting $x=-b$ on both sides we get.

$$-b=C(a-b)^2 \quad \text{or,} \quad C=-\frac{b}{(a-b)^2}.$$

Again the coefficients of x^2 on both sides we get

$$0=B+C \quad \therefore \quad B=-C=\frac{b}{(a-b)^2}.$$

Now $\displaystyle\int\frac{xdx}{(x+a)^2(x+b)}=\int\left\{\frac{A}{(x+a)^2}+\frac{B}{x+a}+\frac{C}{x+b}\right\}dx.$

$$=A\int\frac{dx}{(x+a)^2}+B\int\frac{dx}{x+a}+C\int\frac{dx}{x+b}.$$

$$=-\frac{A}{x+a}+B\log(x+a)+C\log(x+b)+k.$$

$$=-\frac{a}{(a-b)(x+a)}+\frac{b}{(a-b)^2}\log(x+a)-\frac{b}{(a-b)^2}\log(x+b)+k.$$

$$=\frac{a}{(b-a)(x+a)}+\frac{b}{(a-b)^2}\log\frac{x+a}{x+b}+k.$$

Ex. 4. Integrate : $\displaystyle\int\frac{dx}{(x^2+a^2)(x^2+b^2)}$. [C. U. '28, '31, '37]

$$\frac{1}{(x^2+a^2)(x^2+b^2)}=\frac{1}{a^2-b^2}\left[\frac{1}{x^2+b^2}-\frac{1}{x^2+a^2}\right].$$

$$\therefore \int \frac{dx}{(x^2+a^2)(x^2+b^2)} = \int \frac{1}{a^2-b^2}\left[\frac{1}{x^2+b^2} - \frac{1}{x^2+a^2}\right]dx.$$

$$= \frac{1}{a^2-b^2}\int \frac{dx}{x^2+b^2} - \frac{1}{a^2-b^2}\int \frac{dx}{x^2+a^2}.$$

$$= \frac{1}{a^2-b^2}\cdot\frac{1}{b}\tan^{-1}\frac{x}{b} - \frac{1}{a^2-b^2}\cdot\tan^{-1}\frac{x}{a} + c.$$

$$\frac{1}{a^2-b^2}\left[\frac{1}{b}\tan^{-1}\frac{x}{b} - \frac{1}{a}\tan^{-1}\frac{x}{a}\right] + c.$$

Ex. 5. Integrate : $\int \frac{dx}{x^3+1}$.

$$\frac{1}{x^3+1} = \frac{1}{(x+1)(x^2-x+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2-x+1} \quad \text{[say]}.$$

$\therefore \quad 1 = A(x^2-x+1) + (Bx+C)(x+1).$

Putting $x=-1$ on both sides we get $1=3A$ or, $A=\frac{1}{3}$.

Coefficients of x^2 and x on both sides are equal.

$\therefore \quad 0 = A+B$ and $0=-A+B+C$.

$\therefore \quad B=-A=-\frac{1}{3}$ and $C=A-B=2A=\frac{2}{3}$.

$$\therefore \quad \frac{1}{x^3+1} = \frac{1}{3}\left[\frac{1}{x+1} + \frac{-x+2}{x^2-x+1}\right].$$

$$\therefore \quad \int \frac{dx}{x^3+1} = \frac{1}{3}\left\{\int \frac{dx}{x+1} - \int \frac{x-2}{x^2-x+1}\,dx\right\}.$$

$$= \frac{1}{3}\left\{\int \frac{dx}{x+1} - \frac{1}{2}\int \frac{2x-1}{x^2-x+1} + \frac{3}{2}\int \frac{dx}{x^2-x+1}\right\}$$

$$= \frac{1}{3}\left\{\int \frac{dx}{x+1} - \frac{1}{2}\int \frac{2x-1}{x^2-x+1}\,dx + \frac{3}{2}\int \frac{dx}{(x-\frac{1}{2})^2 + \left(\frac{\sqrt{3}}{2}\right)^2}\right\}$$

$$= \frac{1}{3}\log(x+1) - \frac{1}{6}\log(x^2-x+1)\frac{1}{2}\cdot\frac{1}{\frac{\sqrt{3}}{2}}\tan^{-1}\frac{x-\frac{1}{2}}{\frac{\sqrt{3}}{2}} + c.$$

$$= \frac{1}{3}\log(x+1) - \frac{1}{6}\log(x^2-x+1) + \frac{1}{\sqrt{3}}\tan^{-1}\frac{2x-1}{\sqrt{3}} + c.$$

Ex. 6. Integrate : $\int \frac{dx}{x^4+x^2+1}$.

Let $\frac{1}{x^4+x^2+1} = \frac{1}{(x^2+x+1)(x^2-x+1)} = \frac{Ax+B}{x^2+x+1} + \frac{Cx+D}{x^2-x+1}$.

$\therefore \quad 1 = (Ax+B)(x^2-x+1) + (Cx+D)(x^2+x+1).$

As the corresponding coefficients of x^3, x^2, x and the constant terms on both sides are equal, so we get $A+C=0$; $B-A+C+D=0$; $-B+A+C+D=0$ and $B+D=1$.

Solving we get $A=B=D=\frac{1}{2}$ and $C=-\frac{1}{2}$.

$$\therefore \int\frac{dx}{x^4+x^2+1}=\int\left\{\frac{x+1}{2(x^2+x+1)}+\frac{-x+1}{2(x^2-x+1)}\right\}dx.$$

$$=\tfrac{1}{2}\left\{\int\frac{x+1}{x^2+x+1}dx-\int\frac{x-1}{x^2-x+1}dx\right\}.$$

$$=\tfrac{1}{2}\left\{\tfrac{1}{2}\int\frac{2x+1}{x^2+x+1}+\tfrac{1}{2}\int\frac{dx}{x^2+x+1}-\tfrac{1}{2}\int\frac{2x-1}{x^2-x+1}+\tfrac{1}{2}\int\frac{dx}{x^2-x+1}\right\}$$

$$=\tfrac{1}{2}\left\{\tfrac{1}{2}\log(x^2+x+1)\tfrac{1}{2}.\frac{1}{\frac{\sqrt{3}}{2}}\tan^{-1}\frac{2x+1}{\sqrt{3}}\right.$$

$$\left.-\tfrac{1}{2}\log(x^2-x+1)+\tfrac{1}{2}\frac{1}{\frac{\sqrt{3}}{2}}\tan^{-1}\frac{2x-1}{\sqrt{3}}\right\}+c$$

$$=\tfrac{1}{4}\log\frac{x^2+x+1}{x^2-x+1}+\frac{1}{2\sqrt{3}}\left(\tan^{-1}\frac{2x+1}{\sqrt{3}}+\tan^{-1}\frac{2x-1}{\sqrt{3}}\right)+c.$$

Ex. 7. Integrate : $\int\frac{(x-2)(x-4)}{(x-1)(x-5)}dx.$

$$\frac{(x-2)(x-4)}{(x-1)(x-5)}=\frac{x^2-6x+8}{x^2-6x+5}=1+\frac{3}{x^2-6x+5}=1+\frac{A}{x-5}+\frac{B}{x-1}\text{(say)}.$$

$\therefore\quad x^2-6x+8=(x-1)(x-5)+A(x-1)+B(x-5).$

Putting $x=5$ on both sides we get, $3=4A \quad \therefore \quad A=\frac{3}{4}$.

Putting $x=1$ on both sides we get, $3=-4B \quad \therefore \quad B=-\frac{3}{4}$.

$$\therefore\quad\int\frac{(x-2)(x-4)}{(x-1)(x-5)}dx=\int\left\{1+\frac{3}{4(x-5)}-\frac{3}{4(x-1)}\right\}dx$$

$$=\int dx+\tfrac{3}{4}\int\frac{dx}{x-5}-\tfrac{3}{4}\int\frac{dx}{x-1}$$

$$=x+\tfrac{3}{4}\log(x-5)-\tfrac{3}{4}\log(x-1)+c.$$

$$=x+\tfrac{3}{4}\log\frac{x-5}{x-1}.$$

Ex. 8. Integrate : $\int\frac{x^2}{1-x^4}dx.$

Let $x=t \quad \therefore \quad \frac{x^2}{1-x^4}=\frac{t}{1-t^2}=\frac{A}{1+t}+\frac{B}{1-t}$ (say).

$\therefore\quad t=A(1-t)+B(1+t).$

Putting $t=-1$ on both sides we get $-1=2A \quad A=-\frac{1}{2}$.

Putting $t=1$ on both sides we get $1=2B \quad \therefore \quad B=\frac{1}{2}$.

$$\therefore\quad\frac{t}{1-t^2}=-\frac{1}{2(1+t)}+\frac{1}{2(1-t)}$$

or, $\frac{x^2}{1-x^4}=-\frac{1}{2(1+x^2)}+\frac{1}{2(1-x^2)}$

$$\therefore \quad \int\frac{x^2dx}{1-x^4}=-\tfrac{1}{2}\int\frac{dx}{1+x^2}+\tfrac{1}{2}\int\frac{dx}{1-x^2}$$

$$=\tfrac{1}{2}\left[\tfrac{1}{2}\log\frac{1+x}{1-x}-\tan^{-1}x\right]+c.$$

Ex. 9. Integrate : $\int\frac{x^3dx}{(x^2+a^2)(x^2+b^2)}$

Let $x^2=t \quad \therefore \quad 2xdx=dt$

Now, $x^3dx=x^2.xdx=t.\frac{dt}{2}=\frac{1}{2}tdt.$

$$(x^2+a^2)(x^2+b^2)=(t+a^2)(t+b^2)$$

$$\therefore \quad \int\frac{x^3dx}{(x^2+a^2)(x^2+b^2)}=\frac{1}{2}\int\frac{tdt}{(t+a^2)(t+b^2)}$$

Now let $\frac{t}{(t+a^2)(t+b^2)}=\frac{A}{t+a^2}+\frac{B}{t+b^2}$

$t=A(t+b^2)+B(t+a^2).$

Putting $t=-a^2$ and $-b^2$ successively on both sides we get

$$-a^2=A(b^2-a^2) \text{ and } -b^2=B(a^2-b^2)$$

$$\therefore \quad A=\frac{a^2}{a^2-b^2},\ B=-\frac{b^2}{a^2-b^2}$$

$$\therefore \quad \int\frac{x^3dx}{(x^2+a^2)(x^2+b^2)}=\frac{1}{2}\int\frac{A}{t+a^2}\,dt+\frac{1}{2}\int\frac{B}{t+b^2}\,dt$$

$$=\frac{1}{2}A\log(t+a^2)+\frac{1}{2}B\log(t+b^2)+c$$

$$=\frac{a^2}{2(a^2-b^2)}\{\log(x^2+a^2)-\frac{b^2}{2(a^2-b^2)}\log(x^2+b^2)\}+c.$$

$$=\frac{1}{2(a^2-b^2)}\{a^2\log(x^2+a^2)-b^2\log(x^2+b^2)\}+c.$$

Ex. 10. Integrate : $\int\frac{x^2dx}{(x-a)(x-b)(x-c)}.$

Let $\frac{x^2}{(x-a)(x-b)(x-c)}=\frac{A}{x-a}+\frac{B}{x-b}+\frac{C}{x-c}$

$$\therefore \quad x^2=A(x-b)(x-c)+B(x-c)(x-a)+C(x-a)(x-b).$$

Putting $x=a$, b and c successively on both sides we obtain

$$A=\frac{a^2}{(a-b)(a-c)},\ B=\frac{b^2}{(b-c)(b-a)} \text{ and } C=\frac{c^2}{(c-a)(c-b)}$$

Now, $\int \frac{x^2 dx}{(x-a)(x-b)(x-c)} = \int \left\{ \frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c} \right\} dx$

$= A\int \frac{dx}{x-a} + B\int \frac{dx}{x-b} + C\int \frac{dx}{x-c}$

$= A \log (x-a) + B \log (x-b) + C \log (x-c) + k$

$= \frac{a^2}{(a-b)(a-c)} \log (x-a) + \frac{b^2}{(b-c)(b-a)} \log (x-b)$

$+ \frac{c^2}{(c-a)(c-b)} \log (x-c) + k.$

[Putting the values of A, B and C].

Exercise

Integrate :

1. $\int \frac{dx}{x^2-3x+2}$.
2. $\int \frac{dx}{(3x+2)(4x+3)}$
3. $\int \frac{(x-1)dx}{(x-1)(x-1)}$. [C. U. '37]
4. $\int \frac{3x dx}{x^2-x-2}$. [C. U. '38]
5. $\int \frac{x dx}{(x-a)(x-b)}$. [C. U. '23]
6. $\int \frac{x dx}{(x+1)^2(x+2)}$.
7. $\int \frac{x^2 dx}{(x+1)^2(x+2)}$.
8. $\int \frac{dx}{(x-1)^2(x-3)}$.
9. $\int \frac{(x+2)dx}{(1-x)(4+x^2)}$.
10. $\int \frac{dx}{1-x^3}$.
11. $\int \frac{x dx}{1+x^3}$.
12. $\int \frac{x^3+2}{(x-1)(x-2)} dx$.
13. $\int \frac{x^3 dx}{(x+a)(x^2+a^2)}$.
14. $\int \frac{x^4+x^2+1}{(x^2+1)(x-1)} dx$.
15. $\int \frac{(x^2-3)dx}{(x-1)(x-2)(x+3)}$.
16. $\int \frac{x^2 dx}{x^4+x^2-12}$.
17. $\int \frac{dx}{(x^2+1)(x^2+2)}$.
18. $\int \frac{dx}{(x^2+a^2)(x^2+b^2)}$.
19. $\int \frac{x^2 dx}{x^4+x^2+12}$.
20. $\int \frac{2x^4+3}{x^4+5x^2+6} dx$.
21. $\int \frac{x^3 dx}{1-x^2}$.
22. $\int \frac{x dx}{x^4-1}$.
23. $\int \frac{t^3 dt}{t^4+5t^2+6}$.
24. $\int \frac{x dx}{x^4-x^2-2}$.

ANSWERS

[In the first three chapters add an arbitary constant of integration with every integral]

Exercise IA

1. $\dfrac{x^{101}}{101}$ 2. $\dfrac{x^8}{8}$ 3. $-\dfrac{1}{x}$ 4. $-\dfrac{1}{2x^2}$

5. $-\dfrac{4}{\sqrt[4]{x}}$ 6. $\frac{2}{7}x^3\sqrt{x}$

Exercise IB

1. $\dfrac{e^{2x}}{2}$ 2. $\dfrac{e^{17x}}{17}$ 3. $\dfrac{e^{cx}}{c}$ 4. $\frac{5}{4}.e^{\frac{4}{5}x}$

5. $2e^{\frac{x}{2}}$ 6. $-\dfrac{e^{-70x}}{70}$ 7. $-\dfrac{5}{\sqrt[5]{e^x}}$ 8. $\dfrac{x^2}{2}$

Exercise IC

1. $\dfrac{3^x}{\log\ {}^3}$ 2. $\dfrac{-2^{-x}}{\log^2{}_e}$ 3. $\dfrac{a^x}{\log^a{}_e}$

4. $\frac{1}{2}\dfrac{6^{2x}}{\log_e{}^6}$ 5. $\dfrac{10^x}{\log_e{}^{10}}$ 6. $\dfrac{1}{10}\dfrac{6^{10x}}{\log_e{}^6}$

Exercise ID

1. $-\dfrac{\cos 7x}{7}$ 2. $\dfrac{\cos 2x}{2}$ 3. $\dfrac{\sin 6x}{6}$

4. $\dfrac{\sin 4x}{4}$ 5. $\dfrac{-\cot(3x)}{3}$ 6. $\dfrac{\operatorname{cosec} 2x}{2}$

Exercise IE

1. $\dfrac{x^3}{3}+\dfrac{3^x}{\log_e{}^3}$ 2. $8x+18x^2+18x^3+\frac{27}{4}x^4$

3. $\dfrac{x^3}{3}-\dfrac{x^2}{2}+x$ 4. $x+\dfrac{e^{2x}}{2}$ 5. $\dfrac{x^8}{8}+e^x+\dfrac{a^x}{\log_e{}^a}$

6. $\frac{2}{3}x^{\frac{3}{2}}-2x^{\frac{1}{2}}-\dfrac{e^{2x}}{2}$ 7. $-\cot x-x$

8. $2\sin x+\tan x-x$

Exercise IF

1. $\frac{1}{2}\left(\sin x-\frac{\sin 3x}{3}\right)$

2. $\frac{1}{2}\left(-\frac{\cos 16x}{16}-\frac{\cos 4x}{4}\right)$

3. $\frac{\sin 10x}{10}+\frac{\sin 2x}{2}$

4. $\frac{1}{2}\left(x-\frac{\sin 2x}{2}\right)$

5. $\frac{1}{2}\left(x+\frac{\sin 4x}{4}\right)$

6. $\frac{1}{2}(x+\sin x)$

7. $\frac{1}{12}\sin 3x+\frac{3}{4}\sin x$

8. $-\frac{\cos 3x}{4}+\frac{\cos 9x}{36}$

9. $\frac{1}{4}\left(x+\frac{\sin 2x}{2}+\frac{\sin 4x}{4}+\frac{\sin 6x}{6}\right)$

10. $\frac{1}{4}\left(\cos x-\frac{\cos 3x}{3}+\frac{\cos 7x}{7}-\frac{\cos 5x}{5}\right)$

Exercise 1

1. (i) $2\sqrt{x}$ (ii) $-\frac{1}{x}$ (iii) $n\sqrt[n]{x}$

2. (i) $\frac{2}{3}x^3-\frac{5}{2}x^2+2x$ (ii) $\frac{3}{2}2^{\frac{3}{2}}x^{\frac{2}{3}}+\frac{3}{\sqrt[3]{x^2}}$

3. (i) $\frac{x^2}{2}-3x$ (ii) $\frac{x^3}{3}+\frac{x^2}{2}-6x$

4. (i) $x-2e^{-x}-\frac{e^{-2x}}{2}$ (ii) $e^x+4e^{-x}-\frac{e^{-2x}}{2}$

[Read e^{2x} in the denominator]

5. $\frac{e^{4x}}{4}+\frac{e^{2x}}{2}+x$ 6. $\frac{180}{\pi}\sin x$ 7. $\frac{1}{2}x-\frac{\sin 2ax}{4a}$

8. $\frac{1}{2}x+\frac{\sin 12x}{24}$ 9. $\frac{1}{2}x-\frac{3}{4}\sin\frac{2x}{3}$ 10. $-\frac{\cot 2x}{2}-x$

11. $\frac{\sin 2x}{2}$ 12. $-\operatorname{cosec}\theta$ 13. $\sin x-\frac{\sin 7x}{7}$

14. $\frac{\sin 9x}{18}+\frac{\sin x}{2}$ 15. $-\left\{\frac{\cos(m+n)x}{2(m+n)}+\frac{\cos(m-n)}{2(m-n)}\right\}$

$\left[\text{ if } m\neq n\,;\ \frac{-\cos 2mx}{4m} \text{ if } m=n\right]$

16. $-\frac{3}{2}\cos\frac{x}{2}+\frac{1}{6}\cos\frac{3x}{2}$ 17. $\tan x-x$

18. $\sec x+\operatorname{cosec} x$ 19. (i) $\tan x+x$

(ii) $2x-2\cot x+3\tan x$ 20. $\sin x-\cos x$.

21. (i) $\dfrac{a^x}{\log_e a}+2x-\dfrac{a^{-x}}{\log_e a}$ (ii) $\dfrac{a^x}{\log_e a}-\dfrac{a^{-x}}{\log_e a}$

22. (i) $\frac{1}{2}\tan x$ (ii) $-\cot\dfrac{x}{2}$ (iii) $\tan x-\sec x.$

23. $2(\tan x+\sec x+\cos x-\frac{3}{2}x)$

24. (i) $-\cos x+\sin x$ (ii) $\sqrt{2}\sin x$

25. $2\sin x+2x\cos\theta.$ 26. $\dfrac{20x+3\sin 4x}{32}$

27. (i) $\frac{3}{8}x+\dfrac{\sin 2x}{4}+\dfrac{\sin 4x}{32}$ (ii) $\frac{1}{32}\sin 4x-\frac{1}{4}\sin 2x+\frac{3}{8}x$

28. $$\frac{1}{4}\left\{\frac{\sin(a-b-c)x}{a-b-c}-\frac{\sin(a+b+c)x}{a+b+c}+\frac{\sin(a-b+c)x}{a-b+c}-\frac{\sin(a+b-c)x}{a+b-c}\right\}$$

Exercise IIA

1. (i) $\dfrac{1}{12a}(ax+b)^{12}$ (ii) $\dfrac{1}{28}(4x-5)^7$ (iii) $\dfrac{1}{a-x}$

(iv) $\dfrac{1}{b}\log\dfrac{1}{a-bx}$

2. (i) $\dfrac{1}{a}\sin(ax+b)$ (ii) $\frac{1}{2}\tan(2x+3)$

(iii) $\frac{1}{2}t-\dfrac{\sin(4t+6)}{8}$ (iv) $\frac{1}{3}\cot(2-3t)-t$

3. $\dfrac{a^{p+qt}}{q\log_e a}$

4. (i) $\frac{1}{6}\log\dfrac{x-3}{x+3}$ (ii) $\frac{1}{4}\log\dfrac{2+x}{2-x}$ (iii) $\frac{1}{20}\log\dfrac{2x-5}{2x+5}$

5. $\dfrac{2}{3(a-b)}\left\{(x+a)^{\frac{3}{2}}-(x+b)^{\frac{3}{2}}\right\}$

6. $\dfrac{1}{b}x-\dfrac{a}{b^2}\log(a+bx)$

7. (i) $\frac{1}{2}\log\dfrac{x-6}{x-4}$ (ii) $\log\dfrac{x-4}{x-3}$

Exercise II B

1. (i) $\log(ax^2+bx+c)$ (ii) $\dfrac{(x^3+6x^2+5x+2)^2}{2}$

2. $\log(e^x+e^{-x})$ 3. $\log(1+x^2)$

4. (i) $\frac{1}{2}(\sin^{-1}x)^2$ (ii) $\log(\tan^{-1}x)$

5. (i) $\frac{1}{6}(x^4+a^4)^{\frac{3}{2}}$ (ii) $\frac{1}{2}\sqrt{2x^2+3}$

6. $\frac{1}{2}\log(1+\sin^2 x)$ 7. $\dfrac{(\tan x+\sin x)^3}{3}$

8. (i) $-\dfrac{1}{1+\tan x}$ (ii) $\log\dfrac{1}{1+\cot x}$ (iii) $2\sqrt{\tan x-1}$

9. (i) $(x^2-a^2)^{\frac{1}{2}}$ (ii) $\log(1+x^4)$ (iii) $\dfrac{a}{n}\log(x^n+b)$

10. (i) $\log(1+\log x)$ (ii) $\log\{\log(\log x)\}$

11. (i) $\log\{\log(\sin x)\}$ (ii) $\log\{\log(\sec x)\}$

12. $\log(x\sin x)$ 13. $\log(\log\tan x)$

Exercise II C

1. $\tan(e^x)$ 2. $\dfrac{a^{\sin^{-1}x}}{\log_e^a}$

3. $\log\sin(e^x)$ 4. $2\sin\sqrt{x}$

5. $e^{\sin^{-1}x}$ 6. $\sin(\log x)$ 7. $e^{\sin x}$

8. $\frac{1}{3}\sin x^3$ 9. $\dfrac{1}{n}\sin x^n$

10. (i) $\dfrac{1}{4b}(a+bx^2)^2$ (ii) $-\dfrac{1}{bn}\cos(a+bx^n)$

11. $\dfrac{(\tan x-x)^2}{2}$ 12. $\dfrac{1}{202m}(2x^m+11)^{101}$

Exercise II D

1. $\dfrac{1}{3}\tan^{-1}\dfrac{x}{3}$ 2. $\dfrac{1}{ab}\tan^{-1}\dfrac{bx}{a}$ 3. $\frac{1}{4}\tan^{-1}\dfrac{x^2}{4}$

4. $\frac{1}{2}\tan^{-1}\dfrac{\tan x}{2}$ 5. $\frac{1}{4}\tan^{-1}\dfrac{e^{2x}}{2}$

6. $\frac{1}{2}\tan^{-1}\dfrac{\sin x}{2}$ 7. $\tan^{-1}(\tan^{-1}x)$

8. $\dfrac{1}{\sqrt{3}}\tan^{-1}\dfrac{\log x}{\sqrt{3}}$ 9. $\tan^{-1}(e^x)$

Exercise II E

1. $\dfrac{2}{2\sqrt{2}}\log\dfrac{x-\sqrt{2}}{x+\sqrt{2}}$ 2. $\frac{1}{2}\log\dfrac{1+x}{1-x}$

3. $\dfrac{1}{2a}\log\dfrac{x-2a}{x}$ 4. $\dfrac{1}{2a}\log\dfrac{x}{2a-x}$

5. $\frac{1}{2}\log\frac{1+\log x}{1-\log x}$ if $\log x<1$

$\frac{1}{2}\log\frac{\log x+1}{\log x-1}$ if $\log x>1$

6. $\frac{1}{4}\log\frac{1+e^{2x}}{1-e^{2x}}$

7. $\frac{1}{2}\log\frac{\tan\theta-1}{\tan\theta+1}$ when $\tan\theta>1$

$\frac{1}{2}\log\frac{1-\tan\theta}{1+\tan\theta}$ when $\tan\theta<1$

8. $\log\tan\frac{\theta}{2}$

Exercise II F

1. $\log(x+\sqrt{x^2+9})$

2. $\frac{1}{b}\log(bx+\sqrt{b^2x^2+a^2})$

3. $\log(x^2+\sqrt{1+x^4})$

4. $\frac{1}{b}\sin^{-1}\frac{bx}{a}$

5. $\sin^{-1}(\tan^{-1}x)$

6. $\sin^{-1}(\tan x)$

Exercise II G

1. $\frac{1}{7}\log\frac{x-3}{x+4}$

2. $\frac{1}{13}\log\frac{2+x}{3x+7}$

3. $\tan^{-1}(x+2)$

4. $\frac{1}{\sqrt{5}}\log\frac{2x+\sqrt{5}+1}{\sqrt{5}-1-2x}$

5. $\frac{1}{2}\tan^{-1}\frac{\sin x+1}{2}$

6. $\frac{2}{\sqrt{3}}\tan^{-1}\frac{2\log x+1}{\sqrt{3}}$

7. $\frac{1}{4}\log\frac{x^2+1}{x^2+3}$

8. $\frac{1}{5}\log\frac{1+2e^x}{2-e^x}$

9. $\frac{1}{8}\log\left|\frac{2-\sin x}{2-5\sin x}\right|$

Exercise II H

1. $\frac{1}{2}\log(x^2+4x+5)-\tan^{-1}(x+2)$

2. $x+\log\frac{x-2}{x+2}$

3. $\log(x^2+2x+3)-\frac{3}{\sqrt{2}}\tan^{-1}\frac{x+1}{\sqrt{2}}$

4. $-\frac{4}{\sqrt{5}}\log\frac{\sqrt{5}+2x+1}{\sqrt{5}-2x-1}-\log(1-x-x^2)$

5. $-\frac{1}{8}\log(4x^2-4x-3)+\frac{1}{16}\log\frac{2x-3}{2x+1}$

Exercise II I

1. $2 \log (\sqrt{x+2}+\sqrt{x-1})$

2. $2 \log (\sqrt{x-2}+\sqrt{x-3})$ 3. $2 \sin^{-1} \sqrt{x-2}$

4. $\frac{1}{\sqrt{3}} \sin^{-1} \frac{3x-4}{\sqrt{22}}$

Exercise II J

1. $\log \{(x+1)+\sqrt{x^2+2x+6}\}$

2. (i) $\sin^{-1} (2x-5)$

(ii) $\frac{1}{\sqrt{2}} \log \{4x+3+4\sqrt{x^2+\frac{3x}{2}+2}\}$ (iii) $\sin^{-1} \frac{2x+1}{\sqrt{5}}$

3. $\frac{1}{\sqrt{3}} \sin^{-1} \frac{6x-1}{5}$ 4. $\frac{1}{\sqrt{3}} \log (x-\frac{1}{6}+\sqrt{x^2-\frac{x}{3}-1})$

5. $\log \{(x+1)+\sqrt{x^2+2x+5}\}$

6. $\frac{2}{\sqrt{5}} \log (\sqrt{\tan x-\frac{2}{5}}+\sqrt{\tan x-2})$

7. $\log \{(\log x+1)+\sqrt{(\log x)^2+2 \log x+5}\}$

Exercise II K

1. $2\{\sqrt{x^2+x+1}+\log (2x+1+2\sqrt{x^2+x+1})\}$

2. $\frac{1}{2} \sqrt{2x^2-8x+5}$

3. $2\sqrt{x^2+3x+1}+2 \log (2x+3+2\sqrt{x^2+3x+1})$

4. $\sqrt{x^2+x+1}-\frac{1}{2} \log \{(2x+1)+2\sqrt{x^2+x+1}\}$

5. $\sqrt{x^2+2x+2}+2 \log (x+1+\sqrt{x^2+2x+2})$

6. $2\sqrt{3} \sin^{-1} \sqrt{\frac{3x+1}{4}}+2\sqrt{1+2x-3x^2}$

7. $-2\sqrt{3x-x^2-2}+16 \sin^{-1} \sqrt{x-1}$.

Exercise II L

1. $2 \tan^{-1} \sqrt{1+x}$

2. (i) $\frac{1}{\sqrt{10}} \log \frac{\sqrt{2(3x+4)}-\sqrt{5}}{\sqrt{2(3x+4)}+\sqrt{5}}$ (ii) $\log \frac{\sqrt{x+3}-1}{\sqrt{x+2}+1}$

3. (i) $\sin^{-1} \frac{1+3x}{\sqrt{5(1+x)}}$ (ii) $-\frac{1}{\sqrt{5}} \log \left\{\frac{2-x+\sqrt{5(1+x^2)}}{(1+2x)}\right\}$

4. $\log \dfrac{2x}{2+x+2\sqrt{x^2+x+1}}$

5. $-\dfrac{1}{a}\sqrt{\dfrac{x+a}{x-a}}$ 6. $-\sqrt{\dfrac{1-x}{1+x}}$ 7. $\log \dfrac{\sqrt{2x+1}-\sqrt{3}}{\sqrt{2x+1}+\sqrt{3}}$

Exercise II M

1. $\dfrac{1}{3}\log \dfrac{3+\tan\frac{x}{2}}{3-\tan\frac{x}{2}}$ 2. $\dfrac{2}{3}\tan^{-1}\dfrac{1}{3}\left(5\tan\frac{1}{2}x+4\right)$

3. $\dfrac{1}{3}\log\left|\dfrac{\tan\frac{x}{2}-2}{2\tan\frac{x}{2}-1}\right|$ 4. $\dfrac{1}{5}\log\dfrac{1+2\tan\frac{x}{2}}{\tan\frac{x}{2}-1}$

5. $\sqrt{2}\tan^{-1}(\sqrt{2}-1)\tan\left(\dfrac{x}{2}-\dfrac{\pi}{8}\right)$

Exercise 2

1. $\dfrac{(1+x)^6}{6}$ 2. $\dfrac{\tan^4 x}{4}$ 3. $\frac{1}{8}\log(3+4\sin 2x)$

4. $\dfrac{a^{x^2}}{2\log^{a}e}$ 5. $\dfrac{e^{x^3}}{3}$

6. (i) $\frac{1}{4}\log(3+4e^x)$. (ii) $\frac{1}{2}e^{x^2+6x+9}$

7. $\dfrac{\{\log(\log x)\}^2}{2}$ 8. $\frac{1}{12}(\log x)^2$ 9. $\frac{1}{2}\sec^{-1}(x)^2$

10. $\log\dfrac{(x+2)^2}{x+1}$ 11. $\dfrac{\tan^3 x}{3}-\tan x+x$

12. $\frac{1}{2}\log(x^2-3x+4)+\dfrac{3-2\alpha}{\sqrt{7}}\tan^{-1}\dfrac{2x-3}{\sqrt{7}}$

13. (i) $x+\frac{1}{2}\log(x^2+x+1)+\sqrt{3}\tan^{-1}\dfrac{2x+1}{\sqrt{3}}$

(ii) $\dfrac{x^2}{2}-x+\frac{1}{3}\log\{(x+2)(x-1)^2\}$

14. $\frac{1}{3}\log\dfrac{\sqrt{1+x^3}-1}{\sqrt{1+x^3}+1}$ 15. $\log\{(x-\frac{7}{2})+\sqrt{x^2-7x+12}\}$

16. $2\log(\sqrt{x-1}+\sqrt{x-2})$

17. (i) $-\sqrt{ax-x^2}+a\sin^{-1}\sqrt{\dfrac{x}{a}}$

(ii) $a\sin^{-1}\sqrt{\dfrac{x}{a}}+\sqrt{ax-x^2}$ 18. (i) $\frac{4}{3}(1+\sqrt{x})^{\frac{3}{2}}$

(ii) $4\sqrt{1+\sqrt{x}}$

19. (i) $\dfrac{1}{2ab}\log\dfrac{ax-b}{ax+b}$ (ii) $\dfrac{1}{a}\left[\log\left(ax+\sqrt{a^2x^2+b^2}\right)\right]$

20. $\log\left[e^x-1+\sqrt{e^{2x}-2e^x+2}\right]$

21. (i) $\dfrac{\sin^7 x}{7}-\dfrac{\sin^9 x}{9}$

(ii) $\frac{1}{32}\left[2x-\dfrac{\sin 2x}{2}-\dfrac{\sin 4x}{2}+\dfrac{\sin 6x}{6}\right]$

(iii) $\dfrac{\tan^3 x}{3}-2\tan x+\frac{5}{2}x-\frac{1}{4}\sin 2x$

22. $(x+a)\cos a-\sin a\log\{\sin(x+a)\}$

23. $\sin^{-1}\sqrt{\dfrac{x}{2}}-\dfrac{\sqrt{2x-x^2}}{2}(1-x)$

24. $\sqrt{x^2+2x}+2\log(\sqrt{x}+\sqrt{x+2})$

25. $-\dfrac{1}{\sqrt{2x+x^2}}$ 26. $\log\{(x+2)+\sqrt{x^2+4x}\}-2\sqrt{\dfrac{x+4}{x}}$

27. $\frac{1}{2}(\tan 2x-\sec 2x)$

28. $\dfrac{1}{\sqrt{a^2+b^2}}\log\tan\left(\dfrac{x}{2}+\frac{1}{2}\tan^{-1}\dfrac{a}{b}\right)$

29. (i) $\dfrac{2}{\sqrt{5}}\tan^{-1}\left(\dfrac{3\tan\dfrac{x}{2}+2}{\sqrt{5}}\right)$

(ii) $\dfrac{2}{\sqrt{5}}\tan^{-1}\left(\dfrac{1}{\sqrt{5}}\tan\dfrac{x}{2}\right)$

30. (i) $\frac{1}{4}\log\dfrac{2+\tan x}{2-\tan x}$ (ii) $\log\dfrac{2\tan x-1}{2\tan x+1}$

31. (i) $\dfrac{1}{2\sqrt{3}}\tan^{-1}\dfrac{\sqrt{3}\tan x}{2}$ (ii) $-\dfrac{1}{3(4+3\tan x)}$.

32. (i) $\frac{1}{2}\{x+\log(\sin x+\cos x)\}$

(ii) $\frac{11}{25}x-\frac{2}{25}\log(3\cos x+4\sin x)$

33. $2x+\log(3+4\sin x+5\cos x)$

34. (i) $\frac{1}{3}\log(\sec 3x+\tan 3x)$
(ii) $\frac{1}{4}[\operatorname{cosec} x-\log(\sec x+\tan x)]$

35. (i) $\sin 2x-x$ (ii) $\dfrac{1}{2\sqrt{3}}\log\dfrac{1+\sqrt{3}\tan x}{1-\sqrt{3}\tan x}$

36. (i) $x+\sin 2x$ (ii) $\dfrac{1}{2\sqrt{3}}\log\dfrac{\sqrt{3}+\tan x}{\sqrt{3}-\tan x}$

38. (i) $-\cos(\log x)$ (ii) $-\cos(e^x)$ (iii) $\cos(xe^x)$

(iv) $-\cos(\tan x)$ 39. (i) $e^{\sin x}$

(ii) $\dfrac{\sin^6 x}{6}$ (iii) $-\dfrac{1}{2\sin^2 x}$

(iv) $\tan^{-1}(\sin x)$ (v) $-\cos(\sin x)$

40. (i) $-\dfrac{1}{\sqrt{2}}\left[\log\left\{\dfrac{3-\sin x}{4(1+\sin x)}+\dfrac{\sqrt{2+\sin x+\sin^2 x}}{\sqrt{2}(1+\sin x)}\right\}\right]$

(ii) $\dfrac{1}{\sqrt{6}}\log\dfrac{\sqrt{e^x+2}-\sqrt{\frac{3}{2}}}{\sqrt{e^x+2}+\sqrt{\frac{3}{2}}}$

Exercise III A

1. $-x^3\cos x+3x^2\sin x+6x\cos x-6\sin x$

2. $-\dfrac{x^2}{2}\cos 2x+\dfrac{x}{2}\sin 2x+\dfrac{\cos 2x}{4}$

3. $(x+5)\tan x+\log\cos x$

4. $\frac{1}{4}\left[\cos 3x\left(\dfrac{2x}{9}+\dfrac{1}{3}\right)+\sin 3x\left(\dfrac{x^2}{3}+x-\dfrac{2}{27}\right)\right.$
$\left.+\cos x\,(6x+9)+\sin x\,(3x^2+9x-6)\right]$

5. $e^x(x-1)$ 6. $\dfrac{e^{2x}}{2}(x^2-x-\frac{3}{2})$

Exercise III B

1. $x\log ax-x$
2. $x(\log x)^3-3x(\log x)^2+6x\log x-6x$
3. $\dfrac{x^2}{2}\log x-\dfrac{x^2}{4}$ 4. $-\left(\dfrac{1+\log x}{x}\right)$
5. $\log x\left(x+\dfrac{x^3}{3}\right)-\left(x+\dfrac{x^3}{9}\right)$
6. $\frac{2}{3}x^3\log x-\frac{2}{9}x^3-\frac{5}{2}x^2\log x+\frac{5}{4}x^2+2(x\log x-x)$
7. $\sin x(\log\sin x-1)$

Exercise III C

1. $x\tan^{-1}(ax)-\frac{1}{2a}\log(1+a^2x^2)$

2. $2(x\sin^{-1}x+\sqrt{1-x^2})$ 3. $2\{x\tan^{-1}x-\frac{1}{2}\log(1+x^2)\}$

4. $2x\tan^{-1}x-\log(1+x^2)$

5. $(x-\frac{1}{2})\sin^{-1}\sqrt{x}+\frac{1}{2}\sqrt{x(1-x)}$

6. $x(\sin^{-1}x)^3+3\sqrt{1-x^2}(\sin^{-1}x)^2-6(x\sin^{-1}x+\sqrt{1-x^2})$

7. $x\cos^{-1}x-\sqrt{1-x^2}$ 8. $x\cot^{-1}x+\frac{1}{2}\log(1+x^2)$

9. $x\operatorname{cosec}^{-1}x+\log(x+\sqrt{x^2-1})$

10. $x\cos^{-1}\frac{1}{x}-\log(x+\sqrt{x^2-1})$

Exercise III D

1. $\frac{1}{2}e^x(\sin x+\cos x)$ 2. $\frac{e^x(\sin ax+a\sin ax)}{1+a^2}$

3. $\{1-\frac{1}{5}(\cos 2x+2\sin 2x)\}\frac{1}{2}e^x$

4. $\frac{e^x}{2}\left[\frac{1}{17}(\sin 4x-4\cos 4x)+\frac{1}{5}(\sin 2x-2\cos 2x)\right]$

5. $\frac{e^{2x}}{4}\left(\frac{2\cos 3x+3\sin 3x}{13}+\frac{6\cos x+3\sin x}{5}\right)$

6. $\frac{e^{2x}}{4}\left(\frac{6\sin x-3\cos x}{5}-\frac{2\sin 3x-3\cos 3x}{13}\right)$

Exercise III E

1. $e^x\cos x$ 2. $e^x\tan x$ 3. $e^x\sec x$ 4. $e^x.x^2$

5. $e^x\tan^{-1}x$ 6. $e^x\log\sin x$ 7. $e^x\frac{x-1}{x+1}$

Exercise III F

1. $\frac{x\sqrt{x^2+9}}{2}+\frac{9}{2}\log(x+\sqrt{x^2+9})$

2. $\frac{x\sqrt{16-9x^2}}{2}+\frac{8}{3}\sin^{-1}\frac{3x}{4}$

3. $\frac{x\sqrt{1-a^2x^2}}{2}+\frac{1}{2a}\sin^{-1}(ax)$

4. $\frac{x^2}{2}+\frac{x\sqrt{x^2-1}}{2}+\frac{1}{2}\log(x+\sqrt{x^2-1})$

5. $\frac{1}{2}(\sin^{-1}x-x\sqrt{1-x^2})$

6. $-\frac{1}{2}\{\log(x+\sqrt{x^2+1})-x\sqrt{x^2+1}\}$

7. $\frac{1}{8}(4x+3)\sqrt{4-3x-2x^2}+\frac{41}{32}\sqrt{2}\sin^{-1}\dfrac{4x+3}{\sqrt{41}}$

8. $\frac{1}{2}(x-1)\sqrt{5-2x+x^2}+2\log(x-1+\sqrt{5-2x+x^2})$

Exercise 3

1. (i) $\log(x+\cos x)$ (ii) $x(\sec x+\tan x)$

2. (i) $\log(x-\sin x)$ (ii) $-x\cot\dfrac{x}{2}$

3. (i) $x\tan\dfrac{x}{2}+2\log\cos\dfrac{x}{2}$

(ii) $x(\tan x-\sec x)+\log(1+\sin x)$

(iii) $-x\cot\frac{1}{2}x+2\log\sin\frac{1}{2}x$

(iv) $x(\sec x+\tan x)-\log(\sec x+\tan x)-\log\sec x$

(v) same as (iv)

4. (i) $\dfrac{(2e^x-3)\sqrt{e^{2x}-3e^x+1}}{4}-\frac{5}{8}\log(e^x-\frac{3}{2}+\sqrt{e^{2x}-3e^x+1})$

(ii) $\frac{1}{2}[e^{-2x}(e^x-1)\sqrt{2e^{2x}-2e^x+1}$
$-\log(1-e^x+\sqrt{2e^{2x}-2e^x+1}+x)]$

(iii) $\frac{1}{3}\left[\dfrac{(2x^3+1)\sqrt{x^6+x^3+1}}{4}\right.$
$\left.+\frac{3}{8}\log\left(\dfrac{2x^3+1}{2}+\sqrt{x^6+x^3+1}\right)\right]$

(iv) $\dfrac{x^4}{4}+a\dfrac{x^3}{3}+\dfrac{b^2x^2}{2}+ax^2x$

(v) $-\frac{1}{2}\left[\dfrac{x+2}{2x^2}\sqrt{1+x+x^2}+\frac{3}{4}\log\left(\dfrac{x+2+2\sqrt{1+x+x^2}}{2x}\right)\right]$

5. (i) $\dfrac{3^x}{(\log 3)^2+16}\{\log 3\cos 4x+4\sin 4x\}$

(ii) $\dfrac{e^{2x}(\sin 2x-\cos 2x)}{8}$ (iii) $\dfrac{1}{\sqrt{2}}e^x\sin x$

(iv) $\dfrac{3e^{mx}}{4(1+m^2)}(m\sin x-\cos x)$
$-\dfrac{e^{mx}}{4(9+m^2)}(m\sin 3x-3\cos 3x)$

(v) $-\dfrac{4e^{-2x}}{17}(2\cos\frac{1}{2}x-\frac{1}{2}\sin\frac{1}{2}x)$

6. (i) $\dfrac{(\tan^{-1}x)^2}{2}(x^2+1)-x\tan^{-1}x+\frac{1}{2}\log(1+x^2)$

(ii) $\frac{1}{3}x^3\tan^{-1}x+\frac{1}{6}[\log(x^2+1)-x^2]$

7. (i) $\dfrac{e^{m\tan^{-1}x}}{\sqrt{m^2+1}}\cos(\tan^{-1}x-\cot^{-1}m)$

(ii) $\frac{1}{4}e^{2\theta}\left[\dfrac{2\cos 3\theta+3\sin 3\theta}{13}+\frac{3}{5}(2\cos\theta+\sin\theta)\right]$

where, $\tan^{-1}x=\theta$

(iii) $\dfrac{e^{\theta}}{\sqrt{2}}\sin\left(\theta-\dfrac{\pi}{4}\right)$ where $\sin^{-1}x=\theta$

(iv) $\dfrac{e^{\theta}}{8}[3(\sin\theta-\cos\theta)-\frac{1}{5}(\sin 3\theta-3\cos 3\theta)]$

where $\sin^{-1}x=0$

8. (i) $\dfrac{x}{\sqrt{1-x^2}}\sin^{-1}x+\log\sqrt{1-x^2}$

(ii) $\frac{1}{3}\left[\left(\dfrac{x}{\sqrt{1-x^2}}\right)^3\sin^{-1}x-\frac{1}{2}\dfrac{x^2}{1-x^2}-\frac{1}{2}\log(1-x^2)\right]$

9. (i) $\dfrac{x}{(\log x)^2}$ (ii) $\dfrac{x}{(\log x)^n}$ (iii) $x\left\{\log(\log x)-\dfrac{1}{\log x}\right\}$

10. (i) $3(x\sin^{-1}x+\sqrt{1-x^2})$

(ii) $3\{x\tan^{-1}x-\frac{1}{2}\log(1+x^2)\}$

(iii) $x\tan^{-1}\sqrt{\dfrac{x}{x+1}}+\frac{1}{2}\tan^{-1}\sqrt{\dfrac{x}{1+x}}$

$-\frac{1}{4}\log\dfrac{\sqrt{1+x}+\sqrt{x}}{\sqrt{1+x}+\sqrt{x}}$

(iv) $-\frac{1}{2}[-x\cos^{-1}x+\sqrt{(1-x^2)}]$

11. (i) $\sqrt{x(x+a)}-a\log(\sqrt{x}+\sqrt{x+a})$

(ii) $\sqrt{x^2+ax}+a\log(\sqrt{x}+\sqrt{x+a})$

(iii) $(a\sin^{-1}\sqrt{\dfrac{x}{a}}+\sqrt{ax-x^2})$

(iv) $\left(-\sqrt{ax-x^2}+a\sin^{-1}\sqrt{\dfrac{x}{a}}\right)$

12 (i) $-\dfrac{1}{x}\tan^{-1}x+\log\dfrac{x}{\sqrt{x^2+1}}$

(ii) $\frac{x^7}{7}\sin^{-1}x+\frac{1}{7}[\sqrt{1-x^2}-(1-x^2)^{\frac{3}{2}}$

$+\frac{3}{5}(1-x^2)^{\frac{5}{2}}-\frac{1}{7}(1-x^2)^{\frac{7}{2}}]$

13. $\frac{\sin x\cos^5 x}{6}+\frac{5}{24}\sin x\cos^2 x+\frac{5}{16}(x+\sin x\cos x)$

15. $e^x \log\sec x$ 16. $\frac{1}{3}\{(\log\sqrt{x})^2\}=\frac{1}{12}(\log x)^2$

17. $xe^x[\log(xe^x)-1]$ 18. $e^x\tan x$

19. $x\log(x+\sqrt{x^2+a^2})-\sqrt{x^2+a^2}$

20. $\frac{1}{2}x^2\log[x+\sqrt{a^2+x^2}$

$-\frac{1}{4}[x\sqrt{x^2+a^2}-a^2\log(x+\sqrt{x\ \ +a})]$

21. $\frac{2}{3(a-b)}\left[(x+a)^{\frac{3}{2}}-(x+b)^{\frac{3}{2}}\right]$

22. (i) $(x+1)\log(x+1)-x$; (ii) $\frac{x^3}{3}\log x-\frac{x^3}{9}$

23. $(x+1)\tan^{-1}\sqrt{x}-\sqrt{x}$

Exercise IV A

1. (i) $\frac{1}{9}(10^9-1)=111111111$ 2. 24 3. (i) $\frac{1}{3}$

4. 0 5. $\frac{\pi}{4}$ 6. $\frac{\pi}{4}-\frac{1}{2}\log 2$ 7. $\frac{1}{4}$ 8. $\frac{26}{3}$

9. $-\frac{7}{288}$ 10. 1 11. $2-\sqrt{2}$ 12. 0

13. $-\frac{1}{2}\left[\frac{(-1)^{m+n}}{m+n}+\frac{(-1)^{m-n}}{m-n}-\frac{2m}{m^2-n^2}\right]$

14. 0 15. $\frac{\pi}{2}$ 16. $\frac{1}{m}\left[e^{mb}-e^{ma}\right]$

17. $8\log 2-3$ 18. $\frac{\pi}{12}+\frac{\sqrt{3}}{2}-1$ 19. π 20. $\frac{1}{16}(\pi^2+4)$

21. $\frac{1}{2}\log 4+\frac{5}{4}-\frac{9}{2}\log\frac{4}{3}$ 22. $\frac{\pi}{12}-\frac{1}{6}+\frac{1}{6}\log 2$

23. $e^{\frac{\pi}{2}}$ 24. $\frac{3}{8}\pi^2-1$

Exercise IV B

1. $\frac{14}{3}$ 2. $\frac{\pi}{4}$ 3. $\frac{\pi}{6}$ 4. $\frac{7}{18}$ 5. $\frac{\pi^2}{8}$ 6. $(\beta-\alpha)^2\frac{\pi}{8}$

7. $e^2(\sqrt{e}-1)$ 8. π 9. π

10. $\frac{2}{35}$ 11. $\frac{1}{2}\log(2+\sqrt{3})$ 12. $\log\frac{4}{3}$

13. $\frac{\pi}{4}+\log\frac{8}{9}$ 14. $2\tan^{-1}2-\frac{\pi}{2}$ 15. $1-\frac{1}{e}$ 16. $\frac{x^2}{32}$

17. $\log 2$ 18. $\frac{2}{a^2-b^2}\log\frac{a}{b}$ 19. $\frac{2}{3}$ 20. $\frac{14}{15}$

Exercise IV C

1. (i) $\frac{1}{3}$ (ii) $\frac{a}{3}+\frac{b}{2}$ (iii) $\frac{2}{3}$ (iv) 4 (v) $\frac{14}{3}$

2. (i) $\frac{\pi}{2}$ (ii) $\frac{3}{8}$ (iii) $\frac{\pi}{4}+\frac{1}{2}\log 2$ (iv) $\frac{1}{m+1}$

Exercise IV D

1. $\frac{9}{2}$ square units 2. $\frac{3}{2}$ square units 3. 80 square units
4. 1 square unit 5. (*a*) $\frac{1}{2}$ square unit
6. $\frac{1}{4}$ square unit 7. 1 square unit 8. $\frac{1}{6}$ square unit.
9. (*a*) $\frac{8}{3}\sqrt{a}h^{\frac{3}{2}}$ square units 10. $\frac{1}{6}$ square unit
12. $(\frac{1}{2}\pi+\frac{4}{3})$ square units
13. $\frac{2}{\sqrt{ab}}\tan^{-1}\frac{2\sqrt{ab}}{b-a}$ square units
14. $\frac{2}{3}(\sqrt{8}-1)$ square units
15. (*a*) 12 square units 16. 2 square units
17. (i) 1 square unit (ii) 1 square unit (iii) 2 square units
18. (i) 2 square units (ii) 2 square units (iii) 4 square units

Exercise 4

1. $\frac{1}{2}\log(2+\sqrt{3})$ 2. $\frac{\pi}{8}$ 3. $\frac{2}{35}$ 4. $\frac{\pi}{6}$ 5. $\frac{\pi}{1-a^2}$ 6. $\frac{\pi}{4}$

7. $\frac{\pi}{4}$ 8. $\frac{\pi^2}{4}$ 9. 0 10. $\frac{1}{5}$ 11. $\frac{\pi}{2}$ 12. $\frac{4}{e}$

13. $\frac{9}{4}$ sq. units 14. 36 sq. units

Exercise V A

1. (i) first degree and first order (ii) second degree and first order (iii) second order and first degree (iv) first degree and second order (v) first order and second degree

2. (i) $x\frac{dy}{dx}+y=0$ (ii) $\frac{d^2y}{dx^2}-m^2y=0$

(iii) $x\left\{y\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2\right\}=y\frac{dy}{dx}$

(iv) $x^2\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+2y=0$ (v) $\frac{d^2r}{d\theta^2}-\frac{dr}{d\theta}\cot\theta=0$

3. $8a\left(\frac{dy}{dx}\right)^3=27y$

Exercise V B

1. $\frac{1}{2}(y^2-x^2)+(y-x)=c$
2. $\frac{y^3}{3}+\frac{y^2}{2}+y=\frac{x^3}{3}+\frac{x^2}{2}+x+c$
3. $x^2-y^2=a^2$ 4. $ye^x=cx$
5. $1+x^2=c(1+y^2)$ 6. $\sqrt{1-x^2}+\sqrt{1-y^2}=c$
7. $x=c\sqrt{1+y^2}$ 8. $r=c\cos\theta$ 9. $y=5e^x+1$
10. $y^2=4x+c$ 12. $(e^x+2)\sec y=3\sqrt{2}$

Exercise V C

1. $y=\frac{a}{2}\log\frac{c(x-y-a)}{x-y+a}$
2. $\sqrt{y-x}+\log(\sqrt{y-x}-1)=\frac{1}{2}x+c$
3. $-e^{-x(x-y)}=xc$ 4. $\tan\left(\frac{x+y}{2}\right)=x+c$
5. $x=\int\frac{dz}{a+bf(z)}+c$ where $z=ax+by+c$

Exercise V D

1. $y=x+cxy$ 2. $\log y=\frac{x^2}{2y^2}+c$
3. $cx=e^{\frac{x}{y}}$ 4. $(x+y)^3=c(y-x)$
5. $y^3=x^3\log cx^3$ 6. $\log x=\sin\left(\frac{y}{x}\right)+c$

Exercise V E

1. $x^2+y^2-xy+x-y=c$
2. $5xy-3y^2-2x^2-2y-3x=c$
3. $(5y-2x-3)^4=c(4y-4x-3)$
4. $2x-y-15\log(3x-y+19)=c$
5. $\log(4x+8y+5)=4x-8y+c.$

Exercise 5

1. $(x+1)^2+(y+1)^2+2\log(x-1)(y-1)=c$
2. $x\tan x-\log\sec x=y\tan y-\log\sec y+c$
3. $\log\frac{x}{y}-\frac{x+y}{xy}=c$ 4. $(x^3+xy^2 4ay)=kx$
5. $y=a\tan^{-1}\frac{x+y}{a}+c$
6. $2\{\sqrt{x+y}-\log(1+\sqrt{x+y})\}=x+c$
7. $\tan(x+y)-\sec(x+y)=c+x$
8. $3\log(x^2+y^2)=4\tan^{-1}\frac{y}{x}+c$ 9. $y^3e^{\frac{x}{y}}=cx^2$
10. $xy=ce^{\frac{y}{x}}$ 11. $xy^2=c(x+2y)$
12. $x=c\sin\frac{y}{x}$ 13. $(3y-5x+10)^2c=(y-x+1)$
14. $\log(4x+4y+2)=6y-2x+c$
15. $\frac{I}{I_0}=e^{-\frac{R}{L}t}$ 16. $p=A+cr^{-2}$
17. $y^2-x^2=2(x-y)$
18. $\frac{y^3}{3}+\frac{y^2}{2}+y=\frac{x^3}{2}+\frac{x^2}{2}+x.$
19. $\tan x\tan y=1$; 20. $xy=1$
21. $(x+1)^2+(y+1)^2+2\log(x-1)(y-1)=18.$
22. $3\log\frac{(x^2+y^2)}{2}=4\tan^{-1}\frac{y}{x}-\pi$
23. $2xy^2=x+2y.$
24. $y=2x+1$ when $x>1$
 $=x+2$ when $x<1$
25. $y=-2x+1$ when $x<\frac{1}{2}$
 $=2x-1$ when $x>\frac{1}{2}$
26. $y=3x^2+2$ when $x>2$
 $=7x$ when $x<2.$

APPENDIX

1. $\log\frac{x-2}{x-1}$ 2. $\log\frac{3x+2}{4x+3}$ 3. $2\log(x-3)-\log(x-2).$
4. $\log\{(x-2)^2(x+1)\}$ 5. $\frac{1}{a-b}\{a\log(x-a)-b\log(x-b)\}$

6. $\dfrac{1}{x+1}+\log\left(\dfrac{x+1}{x+2}\right)^2$ 7. $4\log(x+2)-3\log(x+1)-\dfrac{1}{x+1}$

8. $\dfrac{1}{2(x-1)}+\frac{1}{4}\log\dfrac{x-3}{x-1}$

9. $\frac{3}{10}\{\log(x^2+4)-2\log(1-x)\}-\frac{1}{5}\tan^{-1}\dfrac{x}{2}$

10. $\dfrac{1}{\sqrt{3}}\tan^{-1}\dfrac{2x+1}{\sqrt{3}}-\frac{1}{3}\log\dfrac{1+x}{\sqrt{1+x+x^2}}$

11. $\dfrac{1}{6}\log\dfrac{x^2-x+1}{x^2+2x+1}-\dfrac{1}{\sqrt{3}}\tan^{-1}\dfrac{2x-1}{\sqrt{3}}$

12. $\dfrac{x^2}{2}+3x+10\log(x-2)-3\log(x-1)$

13. $x-\frac{a}{2}\log(x+a)-\frac{a}{4}\log(x^2+a^2)-\frac{a}{2}\tan^{-1}\dfrac{x}{a}$

14. $\dfrac{x^2}{2}+x+\frac{3}{2}\log(x-1)-\frac{1}{4}\log(x^2+1)-\frac{1}{2}\tan^{-1}x$

15. $\frac{1}{2}\log(x-1)+\frac{1}{5}\log(x-2)+\frac{3}{10}\log(x+3)$

16. $\dfrac{\sqrt{3}}{14}\log\dfrac{x-\sqrt{3}}{x+\sqrt{3}}+\frac{2}{7}\tan^{-1}\dfrac{x}{2}$

17. $\tan^{-1}x-\dfrac{1}{\sqrt{2}}\tan^{-1}\dfrac{x}{\sqrt{2}}$

18. $\dfrac{1}{a^2-b^2}\left[\dfrac{1}{b}\tan^{-1}\dfrac{x}{b}-\dfrac{1}{a}\tan^{-1}\dfrac{x}{a}\right]$

19. $\frac{1}{7}\log\dfrac{x-2}{x+2}+\dfrac{\sqrt{3}}{7}\tan^{-1}\dfrac{x}{\sqrt{3}}$

20. $2x+\dfrac{11}{\sqrt{2}}\tan^{-1}\dfrac{x}{\sqrt{2}}-7\sqrt{3}\tan^{-1}\dfrac{x}{\sqrt{3}}$

21. $-\frac{1}{2}x^2-\frac{1}{2}\log(1-x^2)$

22. $\frac{1}{4}\log\dfrac{x^2-1}{x^2+1}$ 23. $\frac{3}{2}\log(t^2+3)-\log(t^2+2)$

24. $\frac{1}{6}\{\log(x^2-2)-\log(x^2+1)\}$.

Objective and Short Answer Type Questions

1. Which of the following is true ?

(*a*) There is only one real number between 1 and 3.

(*b*) There are finite number of real numbers between 1 and 3.

(*c*) There are an infinite number of real numbers between 1 and 3. [Ans. (*c*)]

2. Between $\frac{1}{2}$ and 1.

(i) How many real numbers are there ?

(ii) Is there a rational number which is not a whole number ?

(iii) Is there a rational number which is not a real number ?

[Ans. (i) Infinite (ii) Yes (iii) No]

3. Fill up the gaps with appropriate words from the brackets.

(i) The sum of two...... is always a......

[rational numbers ; irrational numbers.]

(ii) The decimal expression of an irrational number is...... decimal. [terminating ; non-terminating recurring ; non-terminating non-recurring.]

(iii) The decimal expression of a number is terminating, the number is...... [rational ; irrational.]

[Ans. (i) rational numbers (ii) non-terminating non-recurring (iii) rational.]

4. The sum of two surds is (i) a surd ; (ii) not a surd ; (iii) not necessarily a surd. Which of the above is true ? [Ans. (iii)]

5. Express the inequation $-13<x<3$ in the modulus—from.

[Ans. $|x+5|<8$]

6. For which values of x the following functions are undefined ?

(i) $\sin^{-1}\sqrt{x}$; (ii) $\sin x$; (iii) $\frac{x^2}{x}$; (iv) $\frac{x^2-4}{x-2}$;

(v) $\frac{x^2+4x+1}{x^2-2x+1}$; (vi) $\sqrt{5x-6-x^2}$

[Ans. (i) all real numbers other than those in $0<x\leqslant1$. (ii) For no values of x. (iii) $x=0$ (iv) $x=2$ (v) $x=1$ (vi) all real numbers other than $2\leqslant x\leqslant3$.]

7. Show that

(*a*) If $f(x)=\log \frac{1-x}{1+x}$, then $f(a)+f(b)=f\left(\frac{a+b}{1+ab}\right)$

(*b*) If $\phi(x)=\log_e^x$ and $f(x)=e^x$, then $f\{\phi(x)\}=\phi\{f(x)\}$.

8. a, b, c, d are four real numbers no two of which are equal ; If $f(a)=f(b)$ and $f(c)=f(d)$. Which of the following is true ?

(i) $f(x)$ is a constant (ii) $f(x)$ is not a constant. [Ans. (ii)]

9. (*a*) State the domain of definition of the following functions.

(i) $f(x)=\sqrt{1-x^2}$ (ii) $f(x)=\frac{a+x}{a-x}$ (iii) $f(x)=x^{-\frac{1}{2}}$

(iv) $y=\log(zx+1)$.

[Ans. (i) $-1\leqslant x\leqslant 1$ (ii) all real numbers other than a.

(iii) $0<x<\infty$ (iv) $-\frac{1}{2}<x<\infty$.]

(*b*) Are the two functions $f(x)=\frac{x^2}{x}$ and $g(x)=x$ identical ?

[Ans. No]

10. Which of the following are true ?

(i) The sum of two even functions is an even function.

(ii) The sum of two odd functions is an odd function.

(iii) The product and quotient of two odd functions are odd functions.

(iv) The product and quotient of two even functions is an even function. [Ans. (i) and (iv)]

11. (*a*) For any function $f(x)$ show that

(i) $f(x)+f(-x)$ is an even function.

(ii) $f(x)-f(-x)$ is an odd function.

(*b*) Which is true ?

If $\underset{x\to a}{\text{Lt}}\; f(x)=\text{L}$, then

(i) $\text{L}=f(a)$ (ii) $\text{L}\neq f(a)$ (iii) L is not necessarily $f(a)$.

[Ans. (iii)]

12. Are the following statements always true ?

(i) $\underset{x\to a}{\text{Lt}}\;\frac{x^2}{x-a}-\underset{x\to a}{\text{Lt}}\;\frac{a^2}{x-a}=\underset{x\to a}{\text{Lt}}\;\frac{x^2-a^2}{x-a}$.

(ii) $\underset{x\to 2}{\text{Lt}}\ (x^2-4)\times \underset{x\to 2}{\text{Lt}}\ \dfrac{1}{x-2}=\underset{x\to 2}{\text{Lt}}\ \dfrac{x^2-4}{x-4}.$

[Both are incorrect]

13. Are the following statements correct ?

(i) $\underset{x\to 0}{\text{Lt}}\ \dfrac{\sin x^{\circ}}{x}=1$ (ii) $\underset{x\to 0}{\text{Lt}}\ \dfrac{1}{x}\log_{10}(1+x)=1$

(iii) $\underset{x\to 2}{\text{Lt}}\ \sqrt{2-x}=0.$ [Ans. All are wrong]

14. $\underset{x\to a}{\text{Lt}}\ f(x)=\underset{x\to a}{\text{Lt}}\ g(x)$ say $f(x)=g(x)$. [Ans. wrong ;

Hints. $\underset{x\to 2}{\text{Lt}}\ x^2=4=\underset{x\to 2}{\text{Lt}}\ 2x.$

But $f(x)=x^2$ and $g(x)=2x$ are different functions.]

15. Which of the following statements correct.

(i) $f(x)=\dfrac{|x-1|}{x-1}$ is not continuous everywhere.

(ii) $f(x)=[x]$, where $[x]$ is the greatest integer less than or equal to x.

$\underset{x\to 1}{\text{Lt}}\ [x]=1.$ [Ans. (i) correct (ii) incorrect]

16. Which of the following statements correct ?

(i) If a function $f(x)$ is continuous at $x=a$, then it is differentiable at $x=a$.

(ii) If a function $f(x)$ is differentiable at $x=a$, then it is continuous at $x=a$. [Ans. (i) is not always correct (ii) correct]

17. Are the following statements correct ?

(i) If $\dfrac{d}{dx}\{f(x)\}=\dfrac{d}{dx}\{g(x)\}$, then $f(x)=g(x)$.

(ii) If $f'(x)=0$ for all values of x, then $f(x)$ is a constant.

[Ans. (i) Incorrect (ii) Correct]

[Hints for (i) Let $f(x)=x^2$ and $g(x)=x^2+3$]

18. Are the following statement correct.

(i) $f'(x)>0$ everywhere and $a>b$ $\therefore$ $f(a)>f(b)$.

(ii) $f'(2{\cdot}5)<0$; so $f(2{\cdot}6)<f(2{\cdot}5)$.

[Ans. (i) correct (ii) not always correct]

19. If $\int f(x)\,dx=\int \phi(x)\,dx$, then $f(x)=\phi(x)$ always.
Is the statement true ? [Ans. No]

20. Which of the following are correct ?

(i) $\int x\ dx=\dfrac{x^{x+1}}{x+1}+c$ (ii) $\int a^x\,dx=\dfrac{a^{x+1}}{x+1}+c$

(iii) $\int e^{\log x}\,dx=\dfrac{x^2}{2}+c$

[Ans. (i) & (ii) Correct ; (iii) Incorrect]

21. $\int \dfrac{\cos\theta\,d\theta}{\sqrt{\sin^2\theta-4}}=\log(\sin\theta+\sqrt{\sin^2\theta-4})+c$ is the statement correct.

[Ans. No ; for $\sqrt{\sin^2\theta-4}$ is imaginarly]

22. $\int \dfrac{dx}{x^2-a^2}=-\int \dfrac{dx}{a^2-x^2}$

Is the statement correct ? [Ans. No.]

23. If $C=\int e^x\cos x\,dx$ and $S=\int e^x\sin x\,dx$.
Show that $C+S=e^x\sin x$.

24. Are the following statements correct ?

(i) If $\int_a^b f(x)\,dx=\int_a^b g(x)\,dx$, then $f(x)=g(x)$

(ii) $\int_a^b f(a)\,dx=\int_a^b f(z)\,dz.$

[Ans. (i) in correct ; (ii) correct.]

25. $\int_0^2 x(x-1)(x-2)dx=0$

So, the area enclosed by the curve $y=x(x-1)(x-2)$, the $x-$axis and the ordinates $x=0$ and $x=2$ is 0.

Is the statement correct ! [Ans. No.]

26. Comment on the validity of the following statements.

(i) In $y=Ae^{a+x}$ there are two arbitrary constants of integration. So, it is the general solution of a second order differential equation.

(ii) Every differential equation possesses a finite number of solutions.

(iii) The equation $\dfrac{d^2y}{dx^2}-a\left(\dfrac{dy}{dx}\right)^2=0$ is a differential equation of the second order and second degree. [Ans. All are incorrect.]

27. If $\int_0^a f(x)\,dx=\int_0^a f(a-x)$; then $f(x)=f(a-x)$ always.

[Ans. No.]

Solu. $\int_0^{\frac{\pi}{2}} \sin x\,dx=\Big[-\cos x\Big]_0^{\frac{\pi}{2}}=1$

$\int_0^{\frac{\pi}{2}} \sin\left(\frac{\pi}{2}-x\right)dx=\int_0^{\frac{\pi}{2}} \cos x\,dx=\Big[\sin x\Big]_0^{\frac{\pi}{2}}=1$

$\therefore\ \int_0^{\frac{\pi}{2}} \sin x\,dx=\int_0^{\frac{\pi}{2}} \sin\left(\frac{\pi}{2}-x\right)dx$

But $\sin x \neq \sin\left(\frac{\pi}{2}-x\right)$ always.

28. $\int_{-1}^{1} |x|\,dx=$ (i) 1 (ii) -1 (iii) $\frac{1}{2}$ (iv) 0.

which is correct ?

[Ans. $\int_{-1}^{1} |x|\,dx=\int_{-1}^{0} |x|\,dx+\int_0^1 |x|\,dx$

$=\int_{-1}^{0} (-x)\,dx+\int_0^1 x\,dx=\left[-\frac{x^2}{2}\right]_1^0+\left[\frac{x^2}{2}\right]_0^1=\frac{1}{2}+\frac{1}{2}=1$

So, the answer (i) is correct.

29. $\int_0^4 |x-1|\,dx=$ (i) 5 (ii) 7 (iii) 12 (iv) 20.

which is correct ?

[Ans. (i)]

DYNAMICS

CHAPTER I

INTRODUCTION

§ 1·1. **Space, Time, Matter—Ideas and definitions :**

In mechanics one deals with three fundamental concepts namely, *space*, *time* and *matter*. We think we know about these things but they are extremely difficult to define. Our knowledge about them is mostly intuitive.

In Dynamics, *space* generally refers only to length or distance from point to point.

In everyday use a watch or clock provides the means of measuring *time*. *Matter* is considered to be anything that occupies space and is perceptible to our senses. Quantity of matter is called *mass*. Any limited quantity of matter having a finite shape and size and occupying some definite space is called a *body*.

A *particle* is a portion of matter whose size is so small that it may be ragarded as a mass occupying a single point in space. A body is said to be *rigid* when the distance between any two of the particles of which the body is composed of remains unchanged under the action of external forces.

Definition of *force* comes from Newton's laws of motion. It is defined to be something which changes or tends to change the state of rest or of uniform motion of a body.

N. B. The theory of relativity, discovered by *Einstein* (1879-1955) led to the disappearance of a clear cut distinction between a *three dimensional* space and independent time. Space and time are now considered together to form a four dimensional *space time continuum*.

§1·2. Mechanics is the study of the motion of particles and masses and the effects of forces on them. It is generally divided into two parts namely *Dynamics* and *Statics*. *Dynamics*, which treats of moving bodies is again divided into two subgroups.

(i) *Kinematics*, which deals with the properties of motion itself. It does not enquire into the causes, i. e, the forces behind these motions.

(ii) *Kinetics*, which investigates the properties of the forces, the laws of motions and the relations existing between the forces.

Statics, is the subject which deals with the bodies at rest under the action of forces.

§1·3. History of Mechanics :

Mechanics is one of the oldest branehes of science. Of the great mathematicians of antiquity who enquired into the nature of the forces of the universe, *Archimedes* (287-212 B. C.) of Syracuse, Sicily, stands out as first among his peers. He ennunciated the principles of lever, discovered the first law of hydrostatics [It is said that when he diseovered it, he ran, naked from his bath, through the streets of Syracuse shouting "Eureka, Eureka" ("I have found it, I have found it")] and launched, single handed, with levers and other devices, a fully laden ship against the Romans. It is to Archimedes that *Galileo* and *Stevinus* owe their techniques and their methods of reasonings. In fact Galileo did for dynamics what Archimedes had done for statics.

The laws of motion were first discovered by Galileo (1564-1642). His experiments on falling bodies dropped from the leaning tower of Pisa, in Italy, are well known.

The laws of motion, in a form which are now universally-accepted, were formulated by *Sir Issac Newton* (1642—1727) in his famous book '*Principia Mathematica*'. These three laws, known as Newton's laws of motion, form the basis of classical mechanics, (in the first half of this century 'Quantum Mechanics' was developed to deal with motions of sub atomic particles such as electron and proton.)

§1·4. Units—Unit is a quantity or dimension adopted as a standard of measurement. For example, when we speak of six meters of rope, the unit of length is a metre and the number of *units of* such units in the rope we are speaking of is six.

Generally two kinds of units are in vogue in everyday use, namely, the *F. P. S* and the *C. G. S.* units.

The F. P. S. *or the Foot-Pound-Second system* :—This system of units was in common use in England until recently.

A foot is one third of a *yard*. The yard being defined as the distance between two marks on a certain bar of bronze kept at the standard office of the Board of Trade in London, at a temperature of 62° Fahrenheit. Other popular units related to a foot are a *mile* and an *inch*. A mile is equal to 1760 yards and an inch is equal to $\frac{1}{12}$ of a foot.

The unit of mass in the F. P. S. system is the *pound* and is the mass of a piece of platinum preserved in the same office in London. Unit of time is the mean solar second which is defined in terms of a mean solar day. A mean solar day is equal to 86400 seconds.

The C. G. S. *or the Centimetre-Gramme-Second system.*

The *metre* is the metric unit of length. The whole circumference of the Earth measures, 4,00,00,000 metres. The *centimetre* is defined to be hundredth of a metre.

The unit of mass in this system is the gramme (or gram, written as gm.) and was originally defined to be the mass of a cubic centimetre (1 c.c.) of pure water at 4° centigrade. Sometimes larger units like kilogramme or kilometre are used both in Mechanics and in everyday business. One *kiiogramme* (1 kg.) is equal to 1000 grammes and one *kilometre* (1 km.) is equal to 1000 metres.

Relation between the F. P. S. and the C. G. S. units (the figures give approximate values.)

1 foot = 30·4 cms.

1 inch = 2·54 cms.

1 lb = 453·6 gms.

N. B. The C. G. S. system is now accepted in almost every country in the world including United Kingdom.

Before 1960, a certain piece of platinum-iridium bar (called the metre bar) was considered to be the standard of length. But the modern standard of length is defined in terms of the wave lengths of an isotope of Kromium. Also the standard second is defined in terms of certnin properties of Caesium—133 atom.

CHAPTER II

Velocities and Acceleration

2·1. Displacement, Speed and Velocity :

Displacement of a moving particle is its change of position. Let a particle move from a point A to a point B ; then A is called its initial position and B its final position and $\overline{AB}$ is called it' displacement. Displacement is a vector quantity, i.e., it has *magnitude. direction* and *sense*. The *sense* of direction of a vector quantity is normally shown by the *order* of the letters used in naming the line segment representing the vector. Thus $\overline{AB}$ represents a vector whose magnitude is the length of AB and whose direction is along $\overline{AB}$, the sense being *from* A *to* B, whereas $\overline{BA}$ represents a vector having the same magnitude and direction as that of $\overline{AB}$ but having the opposite sense i. e., *from* B *to* A.

Fig. 1

Speed is the rate at which a moving body or point traces out its path. It is determined by the ratio of the distance covered by a moving body to the time taken.

Speed in a specified direction is velocitity, i. e , velocity is a vector quantity in the sense that it has both magnitude and a definite direction and sense whereas speed is a scalar quantity which has only magnitude.

A body is said to have a *uniform* speed if it describes equal lengths in equal intervals of time.

Definition : Velocity of a body is its rate of change of displacement. The velocity of a body is said to be uniform if it does not change its direction (i. e., moves in a straight line) and has uniform speed, i. e., describes equal distances in equal intervals of time. That a body may possess uniform speed but not uniform velocity may be seen from the following example.

Let a moving body describe a circular path with a uniform speed, i. e., it takes equal times in describing arcs of equal lengths. But note that its direction is continuously changing (see fig 2). Hence its velocity is not uniform. So to have uniform velocity it must move in a straight line with uniform speed.

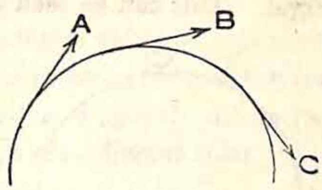


Fig. 2

If the velocity of a particle is not uniform it is said to have a *variable* velocity. To find its velocity at any instant t, suppose that its arcual distance measured from a fixed point in its path be s at the instant t. Let δs be a small distance subsequently described by the particle in an infinitesimal interval of time δt, which is taken to be so small that the velocity of the particle during this short interval can be taken to be uniform. Then the limit, if it exists, of the ratio $\frac{\delta s}{\delta t}$ as $\delta t \to 0$, or, $\lim_{\delta t \to 0} \frac{\delta s}{\delta t}$ or $\frac{ds}{dt}$ in the notation of calculus, is the measure of the velocity of the particle at the instant t.

§2·2. Average speed and velocity :

The average speed of a moving body for any finite interval of time during its motion is that uniform speed with which it would describe the same length in the same interval of time. The average velocity of a moving body during a finite interval of time is that uniform velocity with which a particle would undergo the same displacement in the same interval of time as that of the given body. Let a moving body describe a length AB $(=s)$ in an interval of time t. Then the average speed $= s/t$. Now if the path is not a straight line (as in fig. 3.), then join AB. Let d be the length of the line segment $\overline{AB}$. Then the average velocity $= \frac{d}{t}$ in the direction of $\overline{AB}$.

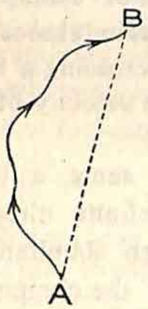


Fig. 3

N. B. In case of curved trajectory the magnitude of the average velocity is different from the average speed during any interval. This can be seen from a simple example. Let a particle

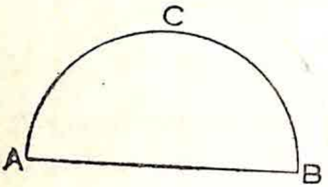


Fig. 4

describe a semicircle ACB of diameter 42 cms. in 11 seconds. Then the total length described by the particle would be $\frac{1}{2} \times \frac{22}{7} \times 42$ (taking $\pi = \frac{22}{7}$) = 66 cms. and the average speed will be 66/11 cms./sec. or, 6 cms./sec wheres its average velocity would be $\frac{42}{11}$ cms./sec. in the direction of $\overline{AB}$.

§2·3. **Representation of uniform velocities by straight lines :**

A directed line segment has both definite magnitude and direction and sense and hence can represent a vector quantity, the length of the segment representing the magnitude and its direction and sense representing the *direction* and *sense* of the vector quantity So uniform velocities, being vector quantities, can be represented by a directed line segment $\overline{AB}$ whose sense is from A to B and is also generally written as $\overline{AB}$. (Actually any velocity, is a vector quantity and can be represented by a line segment but then we would not know about its path).

§2·4. **Composition and Resultant of velocities :**

A moving body may, at a particular instant, possess several velocities of different magnitudes and directions. For example, a man walking inside the compartment of a moving train shares the motion in common with the train ; a swimmer crossing a river flowing along its course possesses, at any instant, the velocity of the current as well as his own velocity.

In all these case if the final result will be the same as if the moving body moves with a single velocity in a definite direction then that velocity is called the *resultant* of the given simultaneous velocities. The simultaneous velocities are called the component velocities of the resultant velocity.

N. B. Generally all motions considered on the Earth are referred to a system in which the Earth is taken to be stationary. Actually, all the earthly objects including us share a velocity

(non uniform) in common with the Earth while the Earth moves round the Sun.

§2·5. Parallelogram of velocities :

If a body possess simultaneously two uniform velocities represented by the directed line segments $\overline{AB}$ *and* $\overline{AD}$ *respectively, then the resultant velocity of the body will be represented by the diagonal (drawn from* A) *of the parallelogram of which* $\overline{AB}$ *and* $\overline{AD}$ *are adjacent sides. Also, the resultant velocity will be uniform.*

Let the body at the beginning of the time under consideration be at the point A. Also let the component velocities be u and v represented by the vectors $\overline{AB}$ and $\overline{AD}$ respectively. At the end of a unit time the body reaches B with velocity u along $\overline{AB}$. While, owing to the velocity v along $\overline{AD}$, each point of the segment $\overline{AB}$ will move with a velocity v along $\overline{AD}$ and as a result $\overline{AB}$ will take up the parallel position $\overline{DC}$. Hence the final position of the body will be at C.

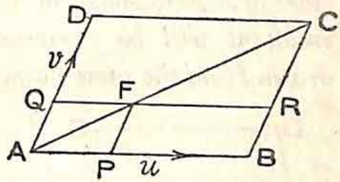


Fig. 5

So far the arguments hold even for non-uniform component velocities, but for non-uniform velocities we would not know anything about the path of the body.

Now let the line $\overline{AB}$ take up the parallel position $\overline{QR}$ in any part of the unit of time under consideration.

If during the same interval the body reaches a point P owing to the velocity u along $\overline{AB}$, then the actual position of the body will be at F where $\overline{FP}$ is equal and parallel to $\overline{AQ}$.

Now for *uniform velocities.*

$$\frac{AP}{AB}=\frac{AQ}{AD}$$

Hence if the parallelogram APFQ be completed, the $\triangle$APF will be similar to the $\triangle$ABC and hence F will be on AC.

$$\text{Also } \frac{AF}{AC}=\frac{AP}{AB}=\frac{AQ}{AD}.$$

Therefore the body moves all along $\overline{AC}$ and describes distances proportional to those described by P or Q. Thus the resultant velocity will be uniform and represented by the diagonal $\overline{AC}$

N. B. In vector notation $\overline{AB}+\overline{AD}=\overline{AC}$.

§2.6. Acceleration :

Definition : Acceleration is the rate of change of velocity. Here 'change' may imply either an alteration in direction or in magnitude or both. A particle describing an arcual path with uniform speed possesses an acceleration because it is continuously changing its direction of motion. A parallelogram law similiar to that of velocity also holds for accelerations.

The parallelogram of accelerations ;

If two component accelerations be represented by two adjacent sides of a parallelogram drawn from an angular point, then their resultant will be represented by the diagonal of the parallelogram drawn from the same point.

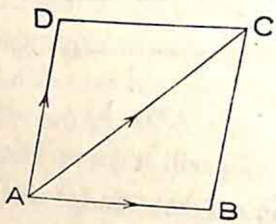


Fig 6

Let $\overline{AB}$, $\overline{AD}$ represent the component accelerations. Complete the parallelogram ABCD. Since $\overline{AB}$, $\overline{AD}$ represent the component accelerations, they represent the component velocities acquired by the moving body per unit time. Hence the diagonal of the parallelogram, by the parallelogram of velocities, represents the resultant velocity acquired per unit time. So the diagonal $\overline{AC}$ represents the resultant acceleration of the body.

§2.7. Triangle of velocities :

If a body possesses three component velocities which can be represented by the sides of a triangle taken in order, then the body will remain at rest.

Let the three velocities be represented by $\overline{AB}$, $\overline{BC}$, $\overline{CA}$, the sides of a triangle ABC. The resultant of $\overline{AB}$ and $\overline{BC}$ is $\overline{AC}$ and this is equal in magnitude and opposite in sense to $\overline{CA}$.

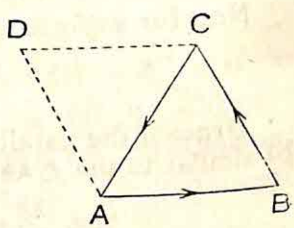


Fig. 7

Hence the body will remain at rest under The three velocities. In vector language, $\overline{AB}+\overline{BC}+\overline{CA}=\overline{O}$. The above proposition can also be stated in the following way.

If a body possesses simultaneously two velocities represented in magnitude, direction and sense successively by the two sides a triangle taken in order then their resultant will be represented by the third side in opposite sense.

Stated in this form, the law holds also for accelerations. For accelerations, the law can also be stated in the following form.

Traingle of accelerations :

If a body simultaneously possesses three accelerations which can be represented by the sides of triangle taken in order, then the body will either remain at rest or move uniformly in a straight line.

§2·8. Polygon of velocities :

If a body has simultaneously a number of component velocities which can be represented by the sides of a closed polygon taken in order, it will remain at rest.

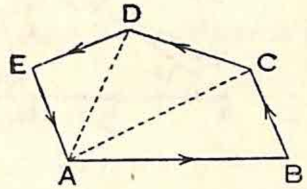


Fig. 8

Let $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DE}$, $\overline{EA}$ be the sides of the polygon representing the component velocities of the body.

Now, $\overline{AB}+\overline{BC}=\overline{AC}$, again $\overline{AC}+\overline{CD}=\overline{AD}$

$\therefore\ \overline{AB}+\overline{BC}+\overline{CD}=\overline{AC}+\overline{CD}=\overline{AD}$

again, $\overline{AD}+\overline{DE}=\overline{AE}$

$\therefore\ \overline{AB}+\overline{BC}+\overline{CD}+\overline{DE}=\overline{AE}$.

Hence four of the component velocities namely $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, DA have a resultant equal to $\overline{AE}$ which is opposite in sense but equal in magnitude to $\overline{EA}$ - representing the fifth velocity. Hencc the resultant of $\overline{AB}$; $\overline{BC}$, $\overline{CD}$, $\overline{DE}$ and $\overline{EA}$ is zero.

So the body will be at rest. The theorem holds for a polygon with any number of sides.

Cor. 1. If a moving body has several component velocities represented in magnitude, direction and sense, successively by a series of lines joined end to end, in the same order, then the resultant velocity is represented by the line closing up the polygon so formed taken in reverse order.

The polygon law stated in this way holds also for accelerations.

Ex. 2. A man walks towards the East a distance of 6 kilometres at the rate of 8 kilometres an hour and then walks towards the North a distance of 5 kilometres at the rate of 10 kilometres an hour. Find the average speed and the the average velocity of the man for his total journey.

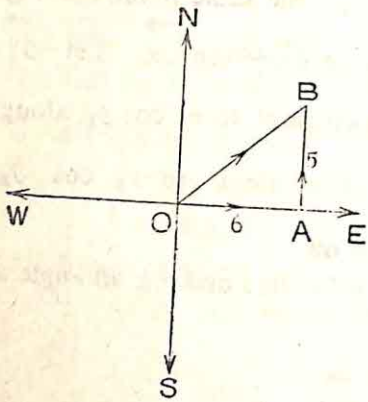


Fig. 13

Let the man start from a position O as shown in fig. His path is $\overline{OA}+\overline{AB}$. The time taken to describe the distance OA($=6$ kms.) is $\frac{6}{8}$ hr. $=\frac{3}{4}$ hr. The time taken to describe the distance AB ($=5$ kms.) along the North at the rate of 10 kms./hr. is $\frac{5}{10}=\frac{1}{2}$ hr.

Here the total distance traversed by the man is 11 kms. Hence his average speed is $\frac{s}{t}=\frac{11}{\frac{1}{2}+\frac{3}{4}}=8{\cdot}8$ kms./hr.

Now $AB=\sqrt{OA^2+AB^2}=\sqrt{6^2+5^2}=\sqrt{61}$

also $\tan BOA=\frac{BA}{OA}=\frac{5}{6}$

hence the average velocity for the total journey was

$$\frac{\sqrt{61}}{\frac{5}{4}}=\frac{4\sqrt{61}}{5}\text{kms./hr. at an angle } \tan^{-1}\tfrac{5}{6} \text{ North of East.}$$

Ex. 3. Find the resultant of two velocities of 5 cms. per second and 4 cms. per second whose directions include an angle of 60°.

The magnitude of the resultant velocity is given by

$$v=\sqrt{4^2+5^2+2\times4\times5\cos60^\circ}$$
$$=\sqrt{16+25+40.\tfrac{1}{2}}=\sqrt{61}\text{ cms./sec.}$$

its direction is given by

$$\tan\theta=\frac{4\sin60^\circ}{5+4\cos60^\circ}=\frac{4.\frac{\sqrt{3}}{2}}{5+4\times\frac{1}{2}}=\tfrac{2}{7}\sqrt{3}$$

hence the resultant velocity is $\sqrt{61}$ cms./sec. making an angle $\tan^{-1}\frac{2}{7}\sqrt{3}$ with the direction of the component velocity of magnitude 5 cms./sec.

Ex. 4. A man can swim a distance of 200 metres down stream in 4 minutes and takes 6 minutes to cover the same distance up stream of a river flowing with a certain velocity. Find the velocity of the current.

Let the velocities of the current and the swimmer be u and v respectively, then the resultant velocity down stream is $u+v$ and up stream is $v-u$.

By the problem $(u+v).4=200$ metres.

$(v-u).6=200$ metres.

Hence $u+v=\frac{200}{4}=50$ metres/minute

and $v-u=\frac{200}{6}=\frac{100}{3}$ metres/minute

solving we get $u=\frac{25}{3}$ metres per minute.

Ex. 5. Resolve the following velocities along the directions $\overrightarrow{OX}$ and $\overrightarrow{OY}$ at right angles to each other. Their directions make the angles with $\overrightarrow{OX}$ given below.

(i) 10 cms /sec., 30° (ii) 20 kms./hr., 45°

(iii) 50 mts /minute, 60°

(i) Component along $\overrightarrow{OX}$ is 10 cos 30°

$$=10\times\frac{\sqrt{3}}{2}=5\sqrt{3} \text{ cms./sec.}$$

component along $\overrightarrow{OY}$ is 10 sin 30° = 5 cms./sec.

(ii) Component along $\overrightarrow{OX}$ is 20 cos 45°

$$=\frac{20}{\sqrt{2}}\text{kms./hr.}=\frac{\frac{20}{\sqrt{2}}\times 1000\times 100}{60\times 60}\text{ cms./sec.}$$

$$=\frac{2500\sqrt{2}}{9}\text{cms./sec.}$$

component along $\overrightarrow{OY}$ is

$$20 \sin 45^\circ=\frac{20}{\sqrt{2}}\text{ kms./hr.}$$

$$=\frac{2500}{9}\sqrt{2}\text{ cms./sec.}$$

When he swims down stream the resultant velocity is $u+v$.

$$\therefore\ t_2=\frac{d}{u+v} \quad \ldots\ldots \quad (2)$$

From (1) and (2), $\dfrac{t_1}{t_2}=\dfrac{d}{\sqrt{u^2-v^2}}\div\dfrac{d}{u+v}=\dfrac{u+v}{\sqrt{u^2-v^2}}$

$$=\frac{\sqrt{u+v}}{\sqrt{u-v}}$$

$$\therefore\ t_1:t_2=\sqrt{u+v}:\sqrt{u-v}.$$

Ex. 9. A body has a velocity of 7 kilometres an hour to the South and also a velocity of $3\sqrt{2}$ kilometres an hour to the North-East. It is brought down to rest by a third velocity. Find the magnitude and direction of this velocity.

The angle between the two given velocities is $90^\circ+45^\circ=135^\circ$.

Let the resultant of these velocities be v, making an angle θ with the South.

Then $v^2=7^2+(3\sqrt{2})^2+2\times7\times3\sqrt{2}\times\cos 135^\circ$

$$=49+18+42\sqrt{2}\times\left(-\frac{1}{\sqrt{2}}\right)$$

$$=49+18-42=25$$

$\therefore\ v=5$ kms./hr. moving at an angle θ with the South which is given by,

$$\tan\theta=\frac{3\sqrt{2}\sin 135^\circ}{7+3\sqrt{2}\cos 135^\circ}=\frac{3\sqrt{2}\times\frac{1}{\sqrt{2}}}{7-3\sqrt{2}\times\frac{1}{\sqrt{2}}}$$

$$=\tfrac{3}{4} \qquad \text{or,}\quad \theta=\tan^{-1}\tfrac{3}{4}.$$

The required velocity will be equal in magnitude but opposite in direction to v, i.e. it is 5kms./hr. at an angle $\tan^{-1}\frac{3}{4}$. West of North.

Ex. 10. A body possesses two velocities in two given directions, having the same magnitude. If one of these velocities be halved, the angle which the resultant makes with the other is halved also. Find the angle between the given directions.

Let 2θ be the angle between the given directions. $\therefore$ Initially the resultant makes an angle θ with each of the directions.

Now if initially, magnitude of each of the velocities be u, then by the problem, when one of the velocities become $\frac{u}{2}$ then

$$\tan \theta/2=\frac{\frac{u}{2}\sin 2\theta}{u+u/2\cos 2\theta}$$

$$=\frac{\sin 2\theta}{2+\cos 2\theta}=\frac{2\sin\theta\cos\theta}{1+2\cos^2\theta}=\frac{2\tan\theta}{\sec^2\theta+2}$$

(dividing numerator and denominator by $\cos^2\theta$)

$$=\frac{2\tan\theta}{3+\tan^2\theta}\cdots(1)$$

Put $\tan\theta/2=x$

$$\text{then }\tan\theta=\frac{2\tan\theta/2}{1-\tan^2\theta/2}=\frac{2x}{1-x^2}$$

We have from (1)

$$x=\frac{2.\frac{2x}{1-x^2}}{3+\left(\frac{2x}{1-x^2}\right)^2}=\frac{4x(1-x^2)}{3(1-x^2)^2+4x^2}.$$

$$\text{or, }\ 1=\frac{4(1-x^2)}{3(1-x^2)^2+4x^2}$$

or, $4(1-x^2)-3(1-x^2)^2-4x^2=0$

or, $3x^4+2x^2-1=0$ or, $(3x^2-1)(x^2+1)=0.$

Taking the positive value of x^2, we have

$3x^2=1$ or, $x^2=\frac{1}{3}$

$x=\frac{1}{\sqrt{3}}$ (assuming $\theta/2$ to be acute)

$$\therefore\ \tan\theta/2=\frac{1}{\sqrt{3}}=\tan 30^\circ$$

$\therefore$ $\theta/2=30^\circ$, giving $2\theta=120^\circ$.

Hence the angle between the given directions is 120°.

Ex. 11. A swimmer can swim in still water at the rate of 4 kilometres an hour. In what direction he must attempt to swim in order to swim directly across a stream, flowing 2 kilo-

metres an hour, to reaeh the directly opposite point on the other bank.

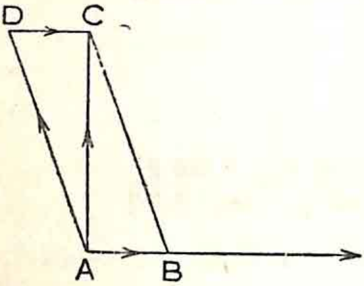


Fig. 16

Let $\overline{AB}$, or $\overline{DC}$ represent the velocity of the current and $\overline{AD}$ that of the man. Let $m\angle DAB=\alpha$.

Now $\overline{AC}$ represents the resultant velocity.

By the problem, $\overline{AC}$ is perpendicular to AB,

$$\cos CBA=\frac{AB}{BC}=\frac{AB}{AD}=\frac{2}{4}=\frac{1}{2}=\cos 60^\circ$$

$$\therefore\ m\angle CBA=60^\circ \quad \text{or,}\quad 180^\circ-\alpha=60^\circ \quad (\alpha=m\angle DAB)$$

$$\therefore\ \alpha=120^\circ.$$

Hence the swimmer should attempt to swim at an angle of 120° with the direction of the current in order to swim directly across the river.

N. B. In the above problem the man takes the shortest route to cross the river but not the shortest possible time. If he wants to cross the river in shortest time then note that after an unit of time the actual width of the river crossed by him is CN.

Then $CN=BC\sin\alpha=4\sin\alpha$.

Hence if t be the total time taken by him to cross the river then $4\sin\alpha\, t=$ width of the river $=b$ say.

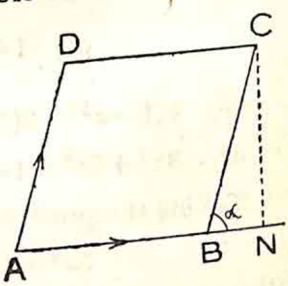


Fig. 17.

$$\therefore\ t=\frac{b}{4\sin\alpha}.$$

Since b is fixed the time will be least when $4\sin\alpha$ will be greatest, i. e., when α will be 90°.

Hence if the man wants to cross the river in shortest time he must attempt to swim in a direction perpendicular to the stream.

Ex. 12. A battle ship sailing North at a speed of 30 kilometres an hour observes a sea-plane carrier due East of itself at

a distance of 20 kilometres; the latter steaming due West at a speed of 40 kilometres an hour. After what time are they at the least distance from each other ? Also find the least distance.

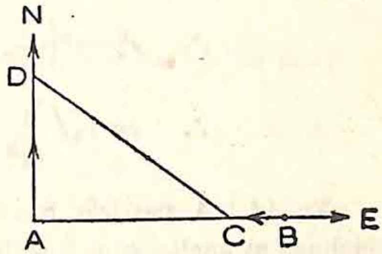


Fig. 16

Let A and B be respectively the initial positions of the battle ship and the sea-plane carrier.

Let after a time t hr. the battle ship describe a distance AD along the North and the sea-plane carrier a distance BC along the West,

$\therefore$ By the problem, $AD=30t$ and
$BC=40t$ $\quad\therefore\quad AC=AB-BC=20-40t$

Now CD, the distance between the two ships, is given by

$$CD^2=AD^2+AC^2=(30t)^2+(20-40t)^2$$
$$=900t^2+400+1600t^2-1600t$$
$$=2500t^2-1600t+400$$
$$=(50t-16)^2+(400-16^2)$$
$$=(50t-16)^2+144.$$

Since CD^2 is always given by a perfect square plus a positive number it can never vanish. The least value of CD is obtained

when $(50t-16)^2=0$ $\quad$ or, $t=\frac{16}{50}$ hr. $=19\frac{1}{5}$ minutes and
then $CD^2=144=12^2$ $\quad\therefore\quad CD=12$ kilometres.
$\therefore$ Least distance $=12$ kms. after $19\frac{1}{5}$ minutes.

Ex. 13. A man can swim directly across a stream of width s metres in t_1 minutes when there is no current and in t_2 minutes when there is current.

Prove that the velocity of the current is

$$s\sqrt{\frac{1}{t_1^2}-\frac{1}{t_2^2}}\text{ mts/min.}$$

Let u and v be the magnitudes of the velocities of the man and the current respeetively. When there is no current the resultant velocity is his own velocity, i.e., u. Hence $u.t_1=s$ $\cdots\cdots$ (1)
and when there is current the resultant velocity (see Ex. 8) is

$$\sqrt{u^2-v^2}. \quad\therefore\quad \sqrt{u^2-v^2}.t_2=s \quad\cdots\quad (2)$$

From (1) and (2) we get

$$\left(\frac{s}{t_1}\right)^2-\left(\frac{s}{t_2}\right)^2=u^2-(u^2-v^2)=v^2$$

$$\therefore \quad v^2=s^2\left(\frac{1}{t_1{}^2}-\frac{1}{t_2{}^2}\right)$$

$$\therefore \quad v=s\sqrt{\frac{1}{t_1{}^2}-\frac{1}{t_2{}^2}}\text{ mts./min.}$$

Ex. 14. A particle has three simultaneous velocities u, v, w inclined at angles α, β, γ with one another ; show that the resultant velocity is

$$\{u^2+v^2+w^2+2uv\cos\alpha+2vw\cos\beta+2wu\cos\gamma\}^{\frac{1}{2}}$$

Let us take two mutually perpendicular direction $\overrightarrow{OX}$ and $\overrightarrow{OY}$, $\overrightarrow{OX}$ being along the direction of the velocity u.

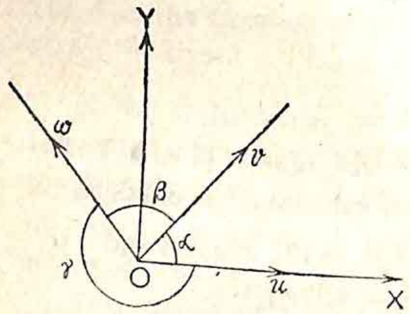


Fig. 17

Let us resolve the velocities along these directions. If the resultant velocity be V making an angle θ with $\overrightarrow{OX}$ then we have,

$$V\cos\theta=u+v\cos\alpha+\omega\cos(\alpha+\beta) \quad \cdots \quad (1)$$

and $V\sin\theta=o+v\sin\alpha+\omega\sin(\alpha+\beta)$

but $\alpha+\beta+\gamma=360°$ $\quad\therefore \cos(\alpha+\beta)=\cos(360°-\gamma)$
$=\cos\gamma$

and $\sin(\alpha+\beta)=\sin(360°-\gamma)=-\sin\gamma$

$$\therefore \quad V\cos\theta=u+v\cos\alpha+\omega\cos\gamma \quad \cdots \quad (3)$$

$$\therefore \quad V\sin\theta=v\sin\alpha-\omega\sin\gamma \quad \cdots \quad (4)$$

Squaring and adding we get

$$V^2=(u+v\cos\alpha+\omega\cos\gamma)^2+(v\sin\alpha-\omega\sin\gamma)^2$$
$$=u^2+v^2(\cos^2\alpha+\sin^2\alpha)+\omega^2(\cos^2\alpha+\sin^2\gamma)$$
$$+2uv\cos\alpha+2uw\cos\gamma+2wv\cos\alpha\cos\gamma-2vw\sin\alpha\sin\gamma.$$
$$=u^2+v^2+w^2+2uv\cos\alpha+2uw\cos\gamma$$
$$+2vw(\cos\alpha\cos\gamma-\sin\alpha\sin\gamma)$$

Now, $\cos\alpha\cos\gamma - \sin\alpha\sin\gamma = \cos(\alpha+\gamma)$
$= \cos(360^\circ - \beta) = \cos\beta$

$\therefore\quad V^2 = u^2 + v^2 + w^2 + 2uv\cos\alpha + 2uw\cos\gamma + 2vw\cos\beta$

$\therefore\quad V = \sqrt{u^2 + v^2 + w^2 + 2uv\cos\alpha + 2uw\cos\gamma + 2vw\cos\beta}$

Exercise 1

1. A body starts from a point A and after describing a complete circle of radius 84 metres in 12 minutes returns to the same point. Find its average speed and average velocity.

2. A point moves in a straight line with a velocity of 3 metres per second. After 3 seconds, it has an additional velocity of 4 metres per second perpendicular to its original direction of motion. Find its distance from the starting point 2 seconds after this.

3. Resolve the following velocities into components along two mutually perpendicular directions $\overrightarrow{OX}$ and $\overrightarrow{OY}$, the angles made by the respective velocities with $\overrightarrow{OX}$ being given.

   (i) 24 metres per second, 90°.
   (ii) 100 kilometres per hour, 120°.
   (iii) 10 cms./sec, 45°.

4. Find the magnitude and directions of the resultant. velocities 5 metres/sec and 10 metres/sec. inclined at an angle 60° with each other.

5. In the following examples u and v are component velocities α is the angle between their directions and w is the resultant.

   (i) Given $u = 20$ kms/hr., $v = 15$kms./hr. $\alpha = 90^\circ$, find w

   (ii) $u = 24$ cms./sec., $w = 25$ cms/sec. $\alpha = 90^\circ$, find v.

   (iii) $u = 3$ mts./min., $v = 3$ mts/min; $\alpha = 30^\circ$ find w.

   (iv) $u = 7$ cms./sec., $v = 8$ cms./sec. $w = 13$ cms/sec., find α.

6. Is it possible for a body having simultaneously three velocities of 20, 10 and 7 units to remain at rest ?

7. Three velocities whose ratios are $(\sqrt{3}+1) : \sqrt{6} : 2$ are simultaneously impressed to a particle and it is found that the particle is at rest. Find the angle at which the directions of the velocities are inclined to each other.

8. The resultant of two velocities is 20 metres per second at right angles to the first component which is 15 metres per sec. Find the magnitude and direction of the other component.

9. A swimer directly crosses and recrosses a river in time t_1. He swims down the stream a distance equal to the breadth of the river and swims back in time t_2

If u and v be respectively the speed of the swimmer in still water and the speed of the current then show that

$$t_1 : t_2 = \sqrt{u^2 - v^2} : u$$

10. Two men having velocities v_1 and v_2 cross a river flowing with a velocity u $<v_1)$. The first person chooses a direction in which the path is shortest while the second person chooses another direction so as to cross the river in shortest time. If they start and reach the opposite bank simultaneously, show that $v_1^2 - v_2^2 = u^2$.

11. In a cricket match, the opening bowler can bowl with a velocity of 90 kms/hr. With what velocity the batsman should strike the ball so that it should travel with the same speed making an angle of 90° with the line of the ball. [Give the answer in meters/sec. unit]

12. A particle possesses simultaneously four velocities whose magnitudes are 8, 5, 10 and 16 metres per second towards North, South, East and West respectively. Find the magnitude and direction of the resultant velocity.

13. Two straight railway lines meet at right angles. A train starts from the junction along one line, and at the same instant another train starts towards the junction from a station on the other line, and they move at the same uniform speed. Show that they are nearest to each other when they are equally distant from the station.

14. How far down the stream will a swimmer reach the opposite bank of a river of width 300 metres if he swims at right angles to the river with double the velocity of the current.

15. An aviator flies round a triangular course each side of which is c kilometres long while the wind blows at u kilometres per hour parallel to a side ; show that he takes

$$c(v+\sqrt{4v^2-3u^2})/(v^2-u^2)$$

hours to complete the circuit in either direction ; v kilometres per hour being his velocity in calm weather.

16. A point has five simultaneous velocity, 5, 10, 15, 20, 25 metres per second respectively. The first three are respectively towards E, N.E. and S. W., the fourth is 150° West of North and the fifth 30° East of South. Find the position of the point 5 secs after start.

17. A train is travelling at the rate of 20 metres per second, and a boy inside projects a ball with a velocity of 40 metres per second at an angle of 120° with the direction of motion of the train. Show that the resultant velocity of the ball will be perpendicular to the direction of motion of the train and determine its magnitude.

18. A man crosses a river of width s metres in time t minutes when there is no current and in time t_1 minutes ($> t$) when there is current. Find the velocity of the current.

19. A point has 3 simultaneous velocities of 2, 4, 4 kilometres/hour parallel to the sides of an equilateral triangle taken in order. Find the resultant velocity. (Give the answers in metres/sec.)

20. An aeroplane has a speed of 100 km./hr. in still air. If the air is flowing from the West at 40 km./hr, find the time taken by the aeroplane to reach a place 250 kms. off to the South West. What is the direction of motion of the plane ?

21. A bus is moving at 12 kilometers per hour along a straight road, and a man, running at 5 kilometres per hour along a

perpendicular road, sees it when it is 200 meters short of the junction and the man himself is 100 meters short. Show that he can never get nearer to the bus than $15\frac{5}{13}$ meters.

22. In a cricket match the field is so arranged by the bowler that the batsman is obliged for slip-cuts to send the moving ball at an angle α to the direction of motion without changing its velocity. Determine how he strikes the ball. Find α so that the striking velocity may be equal to the velocity of the ball (in magnitudes).

CHAPTER III

RELATIVE VELOCITY

§3·1. Relativity of motion :

When we talk of motion of a body we always mean its motion relative to some other body, referred to as the *observer*. In other words, in dynamics, the motion is always referred to a fixed *frame of reference*, i.e., a set of reference axes for defining the position of a point or body in space. In case of motions along a straight line it is a fixed point on the straight line which serves the purpose ; for motions on a plane, two fixed axes act as the frame of reference.

In daily life, you must have observed *the relativity* of motion. For example, as a passenger waiting to catch a train, you see the train approaching towards you with a certain velocity relative to you. But as soon as you board the train, it will be as good as a stationary train to you though it might be moving at a considerable speed relative to an observer standing on the platform. If you observe the behaviour of bodies inside the carriage of a train moving with a uniform velocity and compare it with that in a motionless train you will find no difference. Inside the carriage of a train moving with non-uniform velocity motion will be felt, even with your eyes shut, because of sudden jerks at the instant when the velocity changes, owing to law of inertia. But, sitting in the carriage of a moving train, if you look out side, objects like trees, telegraph posts, nearby buildings etc. will seem to be moving backwards i.e., in a direction opposite to that of the velocity of the train. Also you have perhaps noticed that rain drops falling perpendicularly on the ground, appear to be falling at an angle inclined to the vertical to you if you run or walk very fast. In the light of the above discussions it is easier to apprec ate the following

statement; *"farmes of references in which no external forces act and which are moving with uniform velocities with respect to one another are equivalent to one another."* In fact this is essentially one of the main axioms of Einstein's *special theory of relativity*.

Generally, when we say that an object is at rest we imply that it does not change its position with respect to fixed objects, such as trees, mountains etc, on the surface of the Earth or in other words the Earth itself acts as the frame of referenc. But, as mentioned earlier, the Earth is not fixed relative to the stars in the sky. In fact it moves around the sun with a tremendous speed of nearly 30 kiiometres per second or 10,8000 kilometres per hour. In this book all motions, unless other wise stated, will be referred to fixed objects on the surface of the Earth, i.e., the Earth will be assumed to be at rest relative to the moving bodies under discussions.

§32. Definitions: Relative velocity.

The rate of change of position of a body or a point B as seen from A (which may be in motion itself) is defined to be the *relative velocity* of B with respect to A.

N. B. The position of B in this case is indicated by the line joining A to B.

We shall now show that,

The relative velocity of B with respect to A is obtained by adding with the velocity of B, a velocity equal and opposite to that of A.

Let us first consider the simple case when A and B move with the same velocities u along parallel lines.

After any time, say t_1, their respective displacements $\overline{BB_1}$, AA_1, will be equal and parallel. Similarly the distances traversed by B, viz, B_1B_2, B_2B_3 during the intervals of time say t_2, t_3 will be equal and parallel to the distances covered by A viz, A_1A_2, A_2A_3 respectively during the same intervals of time. Hence (see Figure 20) AB, A_1B_1, A_2B_2, A_3B_3; will be equal and

parallel to each other; i.e., the position of B with respect to A as indicated by the line joining A to B will remain unaltered from A's point of view. Thus B will appear to be at rest to A and their relative velocity will be nil.

Next consider A and B moving with velocities V_A and V_B along two different directions. Let $\overline{AC}$ and $\overline{BG}$ represent the velocities of A and B respectively. Apply, to both A and B, equal and parallel velocities represented by $\overline{AD}$ and $\overline{BE}$ respectively each being equal and opposite to $\overline{AC}$

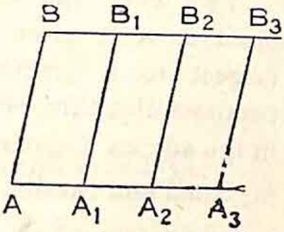


Fig. 20

Now, as shown above, the two equal and parallel velocities applied to A and B will not produce any new relative motion between them, i e., the relative motion of B with respect to A will be unaltered by this application. But now A has two velocities represented by $\overline{AC}$ and $\overline{AD}$ equal and opposite to each other and hence A is now at rest. Again B has now two simultaneous velocities represented by $\overline{BE}$ and $\overline{BG}$ and their resultant is given by the diagonal $\overline{BF}$ which thus represents the relative motion of B with respect to A

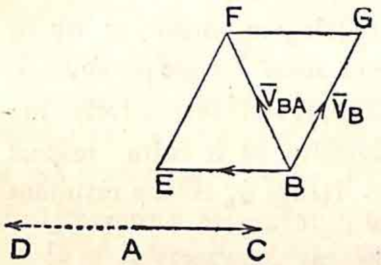


Fig. 21

In vector notation $V_{AB} = \overline{BG} + \overline{BE}$

$$= V_B - V_A$$

Cor. 1. If A and B move along the same straight line with velocities u and v respectively then the relative velocity of B with respect to A is

(i) $v - u$ when u and v have the same direction and sense.

(ii) $u + v$ when u and v have opposite sense, i.e., when they approach each other

Cor. 2. From the relation

$$\overline{V}_{BA}=\overline{V}_B-\overline{V}_A, \text{ we get}$$

$$\overline{V}_B=\overline{V}_{BA}+\overline{V}_A,$$

i e; when the relative velocity of B with respect to an observer A is given then the true velocity (i.e , the velocity with respect to a preassigned frame of reference) is obtained by compounding this velocity with the velocity of A. This is described in the adjoint diagram where $\overline{BF}$ represents $\overline{V}_{BA}$. Now if we draw $\overline{BG}$ equal and parallel to $\overline{V}_A$ in the opposite sense, then in the parallelogram $\overline{BEFG}$, $\overline{BF}$ is the diagonal and $\overline{BG}$, $\overline{BE}$ are two adjacent sides. By the proposition, $\overline{V}_{BA}$ is the resultant of $\overline{BG}$ and the velocity of B. Hence $\overline{BE}$ will represent the velocity of B. Draw $\overline{BH}$ equal and parallel to $\overline{V}_A$ in the same sense, then, as seen from the diagram $\overline{BE}$ is the diagonal of the parallelogram BHEF of which the adjacent sides are are $\overline{BF}$ and $\overline{BH}$. $\overline{BF}$ and $\overline{BH}$ represent respectively the relative velocity of B with respect to A, $\overline{V}_{BA}$ and the velocity of A, $\overline{V}_A$. Hence $\overline{V}_B$ is the resultant of $\overline{V}_{BA}$ and $\overline{V}_A$.

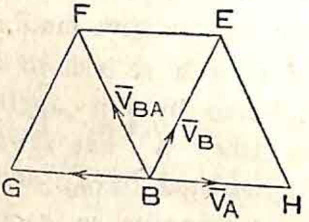


Fig. 22

§ 3·3. **Analytical expression for relative velocity.** Let two bodies A and B be moving with uniform velocities u and v respectively (the directions of which make an angle α with each other) with respect to a frame of reference R. We consider another frame of reference R′ which is at rest with respect to A.

As MN is the change of position of B relative to A in time t_o, the relative velocity of B with respect to A

$$=\frac{\text{relative displacement of B with respect to A}}{\text{time}}$$

$$=\frac{MN}{t_o}=w$$

the magnitude of V_{BA} is $w=\sqrt{u^2+v^2-2uv\cos\alpha}$.

$$\text{Also, } \frac{u}{v}=\frac{BN}{BM}=\frac{\sin(180°-\theta-\alpha)}{\sin\theta}=\frac{\sin(\theta+\alpha)}{\sin\theta}$$

$$=\frac{(\sin\theta\cos\alpha+\cos\theta\sin\alpha)}{\sin\theta}$$

$$=\cos\alpha+\cot\theta\sin\alpha$$

$$\therefore\ \cot\theta=\frac{\frac{u}{v}-\cos\alpha}{\sin\alpha}\quad\therefore\ \tan\theta=\frac{v\sin\alpha}{u-v\cos\alpha}\quad\cdots\quad(2)$$

From equations (1) and (2) we get the magnitude and direction of the relative velocity.

Take rectangular axes $\overrightarrow{OX}$, $\overrightarrow{OY}$ in R and $\overrightarrow{AX'}$ $\overrightarrow{AY'}$ in R′. At any instant t, let A and B be the positions of the bodies A and B respectively. In time t_0, A with the frame of reference R′ moves to L and B in the same interval of time moves to M. Complete the parallelogram ALNB.

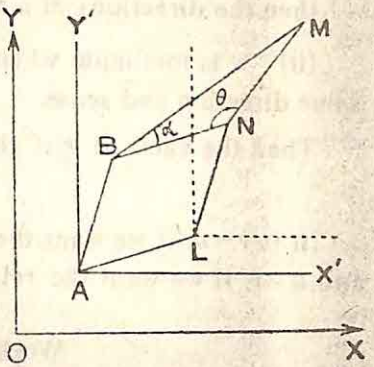


Fig. 23

Since $\overline{LN}$ is equal and parallel to $\overline{AB}$, the point B of R′ when at A is the point N of R′ when at L.

Hence with respect to A or the frame of reference R′, the change of position of B in time t_0 is indicated by $\overline{NM}$

Now $m\angle MBN=\alpha$. Let $m\angle BNM=\theta$.

We have, $AL=BN=ut_0$,

$BM=vt_0$

$\therefore$ from ΔBMN

$$MN^2=BM^2+BN^2-2BM.BN\cos\alpha$$

$$=v^2t_0^2+u^2t_0^2-2uv\cos\alpha\, t_0^2$$

$$=t_0^2(u^2+v^2-2uv\cos\alpha)$$

$\therefore$ $MN=wt_0$

where $w=\sqrt{u^2+v^2-2uv\cos\alpha}$, a constant independent of t_0.

$$\text{Also, } \frac{BM}{\sin\theta}=\frac{MN}{\sin\alpha}$$

$\therefore \quad \sin\theta = \sin\alpha . \dfrac{\mathsf{BM}}{\mathsf{MN}} = \sin\ \alpha . \dfrac{vt_0}{wt_0} = \dfrac{v \sin\alpha}{w}$, a constant independent of time t_0

$\therefore$ $\overline{\mathsf{MN}}$ makes a constant angle with $\overline{\mathsf{AL}}$ and therefore with $\overrightarrow{\mathsf{AX'}}$.

Therefore Relative to the frame R′, B moves with a uniform velocity along a constant direction. By the triangle of velocities, $\overline{\mathsf{MN}}$ represents the resultant of velocities represented by $\overline{\mathsf{NB}}$ and $\overline{\mathsf{BM}}$; so the relative velocity of B with respect to R′ is the resultant of the velocity of B with respect to R and velocity of A reversed in direction also with respect to R.

Cor. (i) w is maximum when $\alpha = \pi$, its value being $u+v$, then the directions of u and v will be opposite to each other.

(ii) w is minimum when $\alpha = 0$, i.e., when u and v will have the same direction and sense.

Then the value of w is given by

$$w = u - v$$

(it is $v-u$, if we want the relative velocity of B with respect to A and $u-v$, if we want the relative velocity of A with respect to B.)

Worked out Examples

1. Two stations A and B are 100 kilometeres apart. A train starts from A and moves towards B at the rate of 40 kms./hr. and at the same instant another train leaves B and moves towards A at the rate of 60 km./hr. When will they meet ?

Since the trains approach each other, they are moving along parallel straight lines but in the opposite directions. Hence the relative velocity of any train with respect to the other is

$$u+v = 40+60 = 100 \text{ km./hr.}$$

So the required time is $100/100 = 1$ hr. after they start.

2. Two trains 200 metres and 250 metres respectively are moving on parallel lines in the same direction. Find how long it takes them to pass each other, if the velocities of the trains are 45 kms/hr. and 30 kms./hr respectively.

Here the relative velocity is

$$45-30=15 \text{ km./hr.}$$

In order to pass each other a train has to cover $(200+250)$

$$=450 \text{ mts.}$$

$\therefore$ the required time is $\dfrac{450}{15\times1000}$ hr.

$$=\frac{450}{15\times1000}\times60\times60 \text{ secs.}=1 \text{ mt. } 48 \text{ secs.}$$

3. A steamer is moving at the rate, of 12 kilometres per hour towards the West and another at the rate of 9 kilometers per hour towards the North. Find the magnitude and the direction of the relative velocity of the second steamer.

The relative velocity of the second steamer = the resultant of its own velocity and a velocity equal and opposite to that of the first steamer.

= the resultant of 9 kilometres per hour towards the North and 12 kilometres per hour towards the East

$$=\sqrt{12^2+9^2}=15 \text{ km./hr}$$

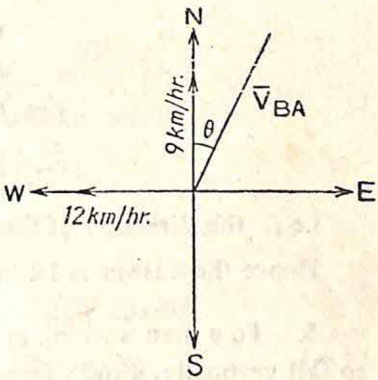


Fig. 24.

Let the relative velocity of the 2nd steamer make an angle θ with the North. Then

$$\tan\theta=\tfrac{12}{9}=4/3$$

$\therefore\ \theta=\tan^{-1}4/3$ East of North.

4. One boat is sailing due North at the rate of 12 kilometres per hour and another boat is sailing North-West at the rate of $12\sqrt{2}$ kilometres per hour. Find the magnitude and direction of the second boat relative to the first.

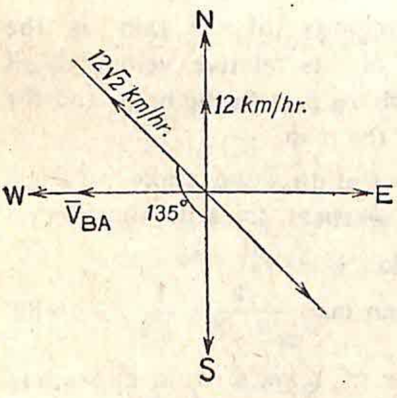


Fig. 25

The relative velocity of the second boat is the resultant of a velocity of 12 km. hr. towards the South and a velocity of $12\sqrt{2}$

km/hr. due North West. The angle between these two velocities being $90^\circ+45^\circ=135^\circ$.

Hence the relative velocity is given by

$$w=\sqrt{12^2+(12\sqrt{2})^2+2.12.12\sqrt{2}\cos(135^\circ)}$$

$$=\sqrt{144+288-2\times12\times12\sqrt{2}\times\frac{1}{\sqrt{2}}}$$

$$=\sqrt{144}=12 \text{ km/hr.}$$

and the angle made by the relative velocity with the North-West direction is given by;

$$\tan\theta=\frac{12\sin 135^\circ}{12\sqrt{2}+12\cos 135^\circ}$$

$$=\frac{\frac{12}{\sqrt{2}}}{12\sqrt{2}-\frac{12}{\sqrt{2}}}=1=\tan 45^\circ$$

$$\therefore\quad \theta=45^\circ.$$

i.e., the direction of the relative velocity is West wards.

Hence the answer is 12 km/hr. West wards.

5. To a man walking at the rate of 3 km/hr the rain appears to fall vertically. Find the actual direction of the rain if its relative velocity is $3\sqrt{3}$ km/hr.

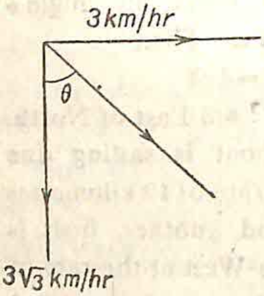


Fig. 26

True velocity of the rain is the resultant of its relative velocity $3\sqrt{3}$ km/hr. with respect to the man and the velocity of the man.

Let the real direction make an angle θ with the vertical towards the mot on of the man.

then $\tan\theta=\dfrac{3}{3\sqrt{3}}=\dfrac{1}{\sqrt{3}}\therefore\ \theta=30^\circ$

6. To a man walking at the rate of 3 km./hr rain appears to fall vertically ; if the man walks at the rate of 5 km/hr. it appears to fall at an angle of 30° with the vertical. Find the actual direction and velocity of the rain.

Let the actual velocity of the rain be v making an angle θ with the vertical towards the motion of the man.

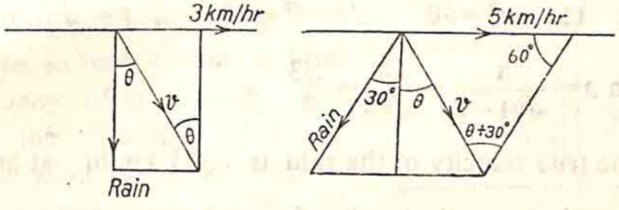


Fig. 27

In the first case, the relative velocity of the rain is

$$\sqrt{v^2-3^2}=\sqrt{v^2-9} \text{ and hence}$$

$$\tan\theta=\frac{3}{\sqrt{v^2-9}} \quad \cdots \quad \text{(i)}$$

In the 2nd case the relative velocity of the rain is

$$\sqrt{v^2+5^2-2v.5\cos(90^\circ-\theta)}$$

$$=\sqrt{v^2+25-10v\sin\theta}.$$

Resolving vertically

$$v\cos\theta=\sqrt{v^2+25-10v\sin\theta}.\cos 30^\circ$$

$$\text{or,}\quad v^2\cos^2\theta=(v^2+25-10v\sin\theta)\tfrac{3}{4} \quad \cdots \quad \text{(ii)}$$

From (i) we have

$$\sec^2\theta=1+\tan^2\theta=1+\frac{9}{v^2-9}=\frac{v^2}{v^2-9}$$

$$\therefore\quad v^2\cos^2\theta=v^2-9$$

$\therefore$ From (ii)

$$v^2-9=(v^2+25-10v\sin\theta)\,3/4$$

$$\text{or,}\quad 4v^2-36=3v^2+75-30v\sin\theta$$

$$\text{or,}\quad v^2+30v\sin\theta-111=0$$

$$\therefore\quad \sin\theta=\frac{111-v^2}{30v}$$

$$\text{and}\quad \cos\theta=\frac{\sqrt{v^2-9}}{v}$$

Now, $\sin^2\theta+\cos^2\theta=1$

$$\therefore\quad \frac{(111-v^2)^2}{900v^2}+\frac{v^2-9}{v^2}=1$$

or, $(111-v^2)^2+900(v^2-9)=900v^2$

or, $(111-v^2)^2=8100$

$\therefore\ 111-v^2=90 \quad \therefore\ v^2=21 \quad \therefore\ v=\sqrt{21}$ km/hr.

$$\therefore\ \tan\theta=\frac{3}{\sqrt{21-9}}=\frac{3}{\sqrt{12}}=\frac{\sqrt{3}}{2}$$

$\therefore$ The true velocity of the rain is $\sqrt{21}$ km/hr. at an angle of $\tan^{-1}\frac{\sqrt{3}}{2}$ with the vertical in the forward sense.

Alt. method

From the diagrams we have.

$$\frac{v}{\sin 90^\circ}=\frac{3}{\sin\theta} \quad \text{or,} \quad v\sin\theta=3 \qquad \text{(i)}$$

again, $\dfrac{v}{\sin 60^\circ}=\dfrac{5}{\sin(30^\circ+\theta)}$

or, $v\sin(30^\circ+\theta)=5\sin 60^\circ$

or, $v\left(\frac{\sqrt{3}}{2}\sin\theta+\frac{1}{2}\cos\theta\right)=5\frac{\sqrt{3}}{2} \quad \ldots \quad$ (ii)

dividing (ii) by (i)

$$\frac{\frac{\sqrt{3}}{2}\sin\theta+\frac{1}{2}\cos\theta}{\sin\theta}=\frac{5\sqrt{3}}{6}$$

or, $\frac{\sqrt{3}}{2}+\frac{1}{2}\cot\theta=\frac{5}{6}\sqrt{3}$

or, $\frac{1}{2}\cot\theta=\frac{5}{6}\sqrt{3}-\frac{\sqrt{3}}{2}=\frac{\sqrt{3}(5-3)}{6}=\frac{\sqrt{3}}{3}=\frac{1}{\sqrt{3}}$

$$\therefore\ \tan\theta=\frac{\sqrt{3}}{2}$$

$$\sin\theta=\frac{1}{\operatorname{cosec}\theta}=\frac{1}{\sqrt{\cot^2\theta+1}}=\frac{1}{\sqrt{\frac{4}{3}+1}}=\frac{\sqrt{3}}{\sqrt{7}}$$

from (i)

$v.\frac{\sqrt{3}}{\sqrt{7}}=3 \quad$ or $\quad v=\frac{3\sqrt{7}}{\sqrt{3}}=\sqrt{21}$.

7. A train is travelling North at 60 km/hr. and the wind is blowing from the South-West at 20 km/hr. Find the direction of the smoke of the steam engine of the train.

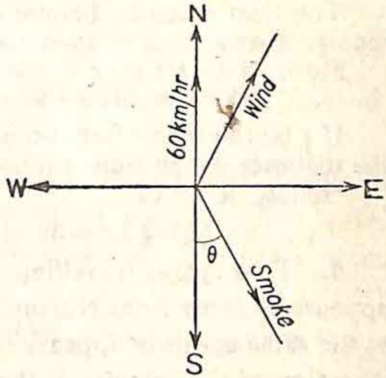


Fig. 28

Let the apparent direction of the smoke, assuming that it loses the velocity of the train after leaving the funnel and moves with the wind, make an angle θ with S East wards.

$$\text{then } \tan\theta=\frac{20\sin 135^\circ}{60+20\cos 135^\circ}$$

$$=\frac{20\sin 45^\circ}{60-20\cos 45^\circ}$$

$$=\frac{20\times\frac{1}{\sqrt{2}}}{60-20\times\frac{1}{\sqrt{2}}}=\frac{20}{60\sqrt{2}-20}=\frac{1}{3\sqrt{2}-1}$$

$\therefore\ \theta=\cot^{-1}(3\sqrt{2}-1)$ East of South.

8. A ship was 20 kilometres North of another ship at noon. The first ship was sailing due South at the rate of 12 kilometres an hour and the second ship due East at 16 kilometres an hour. At what time were they nearest to each other and how far apart were they at that instant?

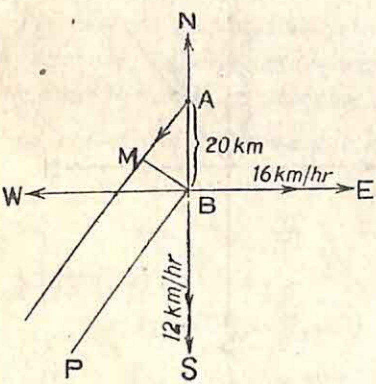


Fig. 29

Let A and B be the positions of the ships at noon.

The relative velocity of A with respect to B is the resultant of its own velocity and the velocity of B in the opposite direction i.e., it is the resultant of two velocities 12 km/hr. due South and 16 km/hr. towards west $=\sqrt{12^2+16^2}$ $=20$ km/hr. at an angle θ with the direction of motion of A where θ is given by $\tan\theta=\frac{16}{12}=\frac{4}{3}$ which gives $\sin\theta=\frac{4}{5}$, $\cos\theta=\frac{3}{5}$;

Now, if $\overrightarrow{AM}$ and $\overrightarrow{BP}$ are parallel,

B will see A moving along $\overrightarrow{AM}$.

The least distance between them is given by BM, the perpendicular drawn from B upon AM.

Now, $BM = AB \sin\theta = 20 \times \frac{4}{5} = 16$ km.

$AM = AB \cos\theta = 20 \times \frac{3}{5} = 12$ km.

If t be the time after noon when they meet, then A describes the distance AM in time t relative to B.

Hence, $20t = 12$

$\therefore\ t = \frac{12}{20} = \frac{3}{5}$ hr. $= 36$ minutes.

9. To a cyclist travelling at 10 miles per hour due East wind appears to come from North-East, but when he travels North-East at the same speed it appears to come from North. Find the true direction of the velocity of the wind. [C. U. 1948]

Let the true velocity of wind make an angle θ with East of South.

The true velocity of wind is the resultant of its apparent velocity with respect to the cyclist and the velocity of the cyclist.

In both cases the angle between the cyclist and the relative velocity is 135°.

In the first case

$$\frac{v}{\sin 135^\circ} = \frac{10}{\sin(45^\circ + \theta)} \quad \dots \quad \text{(i)}$$

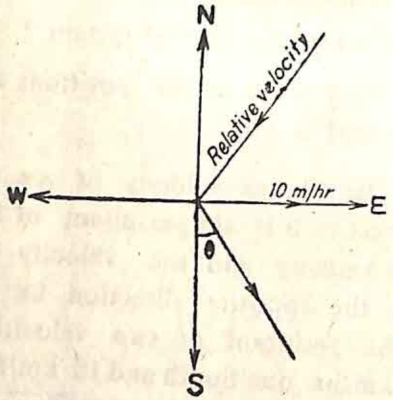


Fig. 30

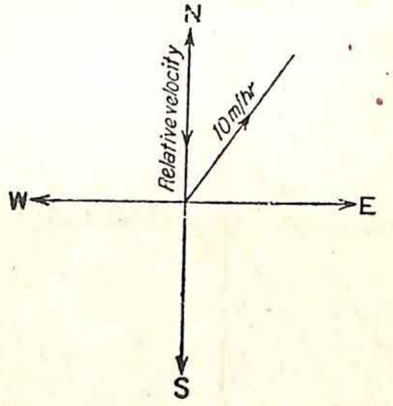


Fig. 31

In the 2nd case

$$\frac{v}{\sin 135^\circ} = \frac{10}{\sin\theta} \quad \text{(ii)}$$

$\therefore$ From (i) and (ii)

$$\sin(180^\circ-\theta)=\sin\theta=\sin(45^\circ+\theta)$$

$$\therefore\ 180^\circ-\theta=45^\circ+\theta$$

$$\therefore\ 2\theta=135^\circ,\quad \therefore\ \theta=67\tfrac{1}{2}^\circ$$

$\therefore$ The true direction of wind makes an angle $67\frac{1}{2}^\circ$ East of South.

10. Two points move with velocities v and $2v$ respectively in opposite directions in the circumference of a circle. In what positions is their relative velocity greatest and least and what values has it then ?

The greatest value is obtained when the angle between their directions of motion is 180°.

i.e. when the two points arrive at the same position from the opposite directions, then their relative velocity is $v+2v=3v$.

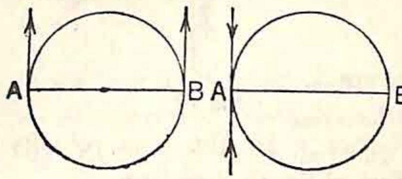


Fig. 32

The least value of the relative velocity is obtained when the angle between their direction is nil i.e. when they are parallel and of the same sense. Hence the least value of the relative velocity occurs when they are at the ends of a diameter and its value is given by $2v-v=v$.

11. Given the relative velocity of A with respect to B and also the relative velocity of B with respect to C. Show how you will determine the relative velocity of C with respect to A.

In vector notations we can write,

$$\overline{V}_{AB}=\overline{V}_A-\overline{V}_B \quad \ldots \quad \text{(i)}$$

also $$\overline{V}_{BC}=\overline{V}_B-\overline{V}_C \quad \ldots \quad \text{(ii)}$$

Adding (i) and (ii)

$$\overline{V}_{AB}+\overline{V}_{BC}=\overline{V}_A-\overline{V}_C$$

Now, $$\overline{V}_{CA}=\overline{V}_C-\overline{V}_A=-(\overline{V}_{AB}+\overline{V}_{BC})$$

Hence the relative velocity of C with respect to A is obtained by compounding the two given relative velocities and then reversing the direction of the resultant.

12. A person travels due east at the rate of 4 miles per hour and observes that the wind seems to blow directly from the North; he then doubles his speed and the wind appears to come from the North-East. Determine the direction and velocity of wind. [C. U. 1943]

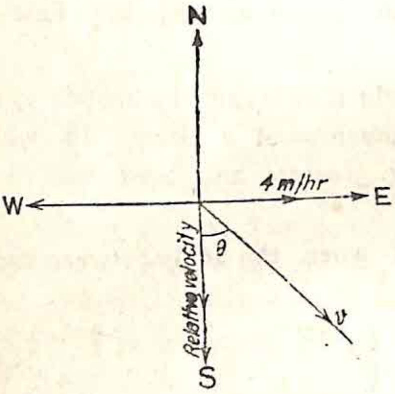


Fig 33

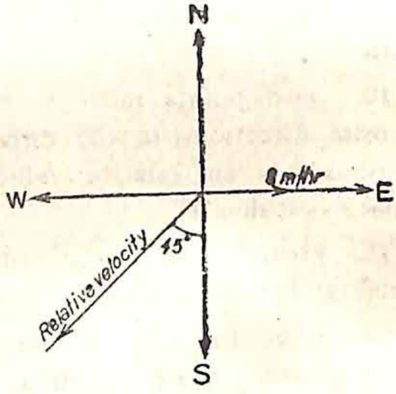


Fig. 34

Let v be the true velocity of wind making an angle θ with the East of South direction.

In the first case, the angle between the relative velocity and the direction of the motion of the man was $90°$.

Hence $\dfrac{v}{\sin 90°}=\dfrac{4}{\sin\theta}$

or, $v\sin\theta=4$ .... (i)

In the 2nd case the angle between the relative velocity of the wind and the velocity of the man was $135°$

$$\frac{v}{\sin 135°}=\frac{8}{\sin(45°+\theta)}$$

or, $v\sin(45°+\theta)=8\sin 135°=\dfrac{8}{\sqrt{2}}=4\sqrt{2}$ ... (ii)

from (i) and (ii) we get

$$\frac{v\sin(45°+\theta)}{v\sin\theta}=\frac{4\sqrt{2}}{4}$$

or, $\dfrac{\sin(45°+\theta)}{\sin\theta}=\sqrt{2}$

or, $\dfrac{\dfrac{1}{\sqrt{2}}\sin\theta+\dfrac{1}{\sqrt{2}}\cos\theta}{\sin\theta}=\sqrt{2}$

or, $\frac{1}{\sqrt{2}}(1+\cot\theta)=\sqrt{2}$

or. $1+\cot\theta=2$ $\therefore$ $\cot\theta=1$ $\therefore$ $\theta=45^\circ$

and from (i) $v\sin 45^\circ=4$ or, $\frac{v}{\sqrt{2}}=4$

$\therefore$ $v=4\sqrt{2}$ miles per hour, hence the true velocity of the wind is $4\sqrt{2}$ m/h. making an angle 45° East of South, i.e., it was blowing from North West.

13. A stone is projected horizontally from the window of a trian moving with a velocity of 90 k.m /hr. If the relative velocity of the stone is 5 metres/sec at right angles to the direction of the motion of the trian, find the actual velocity of the stone at the time of the projection.

Let the actual velocity of the stone be u metres/sec. at an angle of θ with the direction of the motion of the train. By the condition of the problem, actual velocity is inclined at an angle of $90^\circ-\theta$ with the relative velocity.

$\therefore$ actual velocity is the resultant of the relative velocity and the velocity of the train and since 90 kms/hr.$=25$ m/sec.,

$$\frac{u}{\sin 90^\circ}=\frac{5}{\sin\theta}=\frac{25}{\sin(90^\circ-\theta)} \text{ (by § 2·10)}$$

$$\therefore \frac{25}{5}=\frac{\sin(90^\circ-\theta)}{\sin\theta}=\frac{\cos\theta}{\sin\theta}, \text{ or } \cot\theta=5.$$

$\therefore$ $\theta=\tan^{-1}\frac{1}{5}$ and

$\therefore$ $u=5\operatorname{cosec}\theta=5\sqrt{1+\cot^2\theta}=5\sqrt{26}$.

$\therefore$ actual velocity is $5\sqrt{26}$ m./sec making an angle of $\tan^{-1}(1/5)$ with the direction of the motion of the train.

14. A steamer is travelling due East at the rate of u miles an hour. A second streamer is travelling at $2u$ miles an hour in direction θ North of East and appears to be travelling North-East to a passenger on the first steamer.

Prove that $\theta=\frac{1}{2}\sin^{-1}\frac{3}{4}$ [C. U. 1945, '49]

The angle between the velocity of the first steamer and the relative velocity of the second steamer with respect to the first is 45°. Again, the angle between the true volocity and the relative velocity of the 2nd steamer is $45° - \theta$

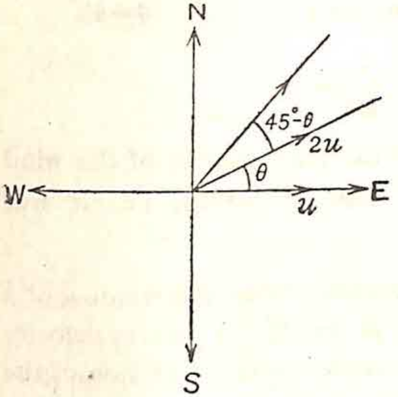


Fig. 35

$\therefore$ We have, $\dfrac{2u}{\sin 45°} = \dfrac{u}{\sin(45° - \theta)}$

or, $\dfrac{\sin(45° - \theta)}{\sin 45°} = \frac{1}{2}$

or, $\sin 45° \cos\theta - \cos 45° \sin\theta$

$= \frac{1}{2}\sin 45° = \dfrac{1}{2\sqrt{2}}$

or; $\cos\theta - \sin\theta = \frac{1}{2}$

Squaring both sides

$\cos^2\theta + \sin^2\theta - 2\cos\theta\sin\theta = \frac{1}{4}$

or, $1 - \sin 2\theta = \frac{1}{4}$

or, $\sin 2\theta = \frac{3}{4}$

$\theta = \frac{1}{2}\sin^{-1}\frac{3}{4}$

15. Two particles start simultaneously from the same point and move along two straight lines at an angle α, one with unifrom velocity u and the other from rest with uniform acceleration f. Show that their relative velocity is least after a time $(u\cos\alpha)/f$ and that the least relative velocity is $u\sin\alpha$.

Since acceleration is rate of change of velocity, the velocity of the 2nd particle after a time t is given by $v = ft$.

$\therefore$ their relative velocity is given by

$$w = \sqrt{u^2 + v^2 - 2uv\cos\alpha}$$
$$= \sqrt{u^2 + f^2t^2 - 2u.ft\cos\alpha}$$
$$= \sqrt{(ft - u\cos\alpha)^2 + u^2 - u^2\cos^2\alpha}$$
$$= \sqrt{(ft - u\cos\alpha)^2 + u^2\sin^2\alpha}$$

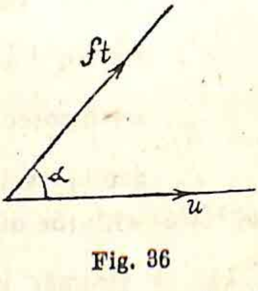


Fig. 36

Since $(ft - u\cos\alpha)^2$ is never negative for any value of t, the least value of w is obtained when

$ft - u\cos\alpha = 0$ or, $t = \dfrac{u\cos\alpha}{f}$

and then $w = u\sin\alpha$.

Alt method.

We have, $w^2 = u^2 + v^2 - 2uv \cos \alpha$
$= u^2 + f^2 t^2 - 2uft \cos \alpha$

least value of w^2 is obtained, when $\frac{dw^2}{dt} = 0$

and $\frac{d^2 w^2}{dt^2} > 0$, but $\frac{d(w^2)}{dt} = 2f^2 t - 2uf \cos \alpha$

i.e., $\frac{d^2(w^2)}{dt^2} = 2f^2 > 0$ always.

$\therefore$ $\frac{dw^2}{dt} = 0$, gives $t = \frac{u \cos \alpha}{f}$ and hence etc.

16 A pistol shot is fired on a running train at an angle α with its direction of motion. The shot enters a carriage at a corner furthest from the engine and passes out at the diagonally opposite corner. If u be the velocity of the train in miles per hour and a and b are the length and breadth of the carriage in feet show that the time the shot takes to pass through the carriage is $15(b \cot \alpha - a)/22u$ seconds.

Let w be the relative velocity of the shot (in ft/sec) with respect to the train.

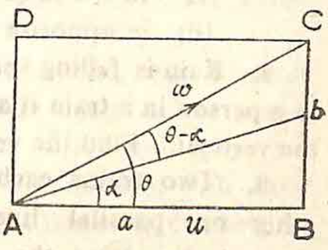


Fig. 37

Now if the angle between w and u is θ,

then we have, $\frac{b}{a} = \tan \theta \cdots$(i)

and also AC = the diagonal of the carriage $= b \operatorname{cosec} \theta$

Also u miles/hr $= \frac{u \times 1760 \times 3}{60 \times 60}$ ft./sec $= \frac{22u}{15}$ ft./sec.

Now since the angle between the true velocity of the shot and the direction of motion of the train is α and the angle between the true velocity and the real velocity is $\theta - \alpha$.

We have

$$\frac{w}{\sin \alpha} = \frac{22u}{15 \sin (\theta - \alpha)}$$

or, $w = \frac{22u}{15} \cdot \frac{\sin \alpha}{\sin (\theta - \alpha)}$ ft./sec. $\cdots$(ii)

Now, the time t taken by the shot to pass through the carriage

= the time taken by the shot to cover the distance AC

$$= \frac{AC}{w} = \frac{b \operatorname{cosec} \theta}{w}$$

$$= b \operatorname{cosec} \theta \cdot \frac{15 \sin(\theta - \alpha)}{22u \sin \alpha} \text{ (from (ii))}$$

$$= \frac{15b \operatorname{cosec} \theta}{22u} \cdot \frac{(\sin \theta \cos \alpha - \cos \theta \sin \alpha)}{\sin \alpha}$$

$$= \frac{15b \operatorname{cosec} \theta}{22u} \cdot (\cot \alpha - \cot \theta) \cdot \sin \theta$$

$$= \frac{15b}{22u} \cdot (\cot \alpha - a/b) \text{ (from (i))}$$

$$= \frac{15}{22u} \cdot (b \cot \alpha - a) \text{ seconds.}$$

Exercises on CHAPTER III

1. Find the relative velocity of two trains travelling at the rate of 30 miles an hour and 66 ft. per sec. respectively when they are moving
   (i) in the same direction
   (ii) in opposite directions.

2. Rain is falling vertically with a velocity of 10 mt/sec. But to a person in a train it appears to be inclined at an angle of 45° to the vertical. Find the velocity of the train in kilometres per hour.

3. Two trains each 250 ms. long are moving towards each other on parallel lines with velocities 20 and 30 km./hr. respectively. Find the time that elapses from the instant when they first meet until they have cleared each other.

4. A man is walking at 8 km./hr. and the rain appears to him to fall with a speed of 16 km/hr. at 30° to the vertical Find the true velocity of the rain.

5. A ship is sailing North-East with a velocity of 10 km/hr. and to a passenger on board the wind appears to come from the North with a velocity of 10 $\sqrt{2}$ km./hr. Find the true velocity of the wind.

6. Two men start to walk simultaneously one East wards at the rate of 4 km./hr. and the other North wards at 3 km./hr. Find the magnitude and direction of the relative velocity of the second man. How far will they be from each other after 5 minutes ?

7. To a man walking at the rate of 6 miles per hour rain appears to fall vertically ; but it appears to meet him at an angle of 45° when he increases his speed to 12 miles per hour. Find the true direction and speed of the rain. (C. U. 1976)

8. At a particular instant two aeroplanes are at a distance of 250 kilometres, one due East of the other. The first one is moving westwards at the rate of 100 km./hr. and the second with a velocity of 75 km./hr. towards the South. Find the time when they are nearest to each other. Also find the least distance between them.

9. A man travelling towards the North-East finds that the wind appears to blow from the North, but when he doubles his speed it seems to come from a direction $\cot^{-1}2$ East of North. Find the true direction of the wind.

10. A railway train is moving at the rate of 60 km/hr., when it is struck by a stone moving horizonally and at right angles to the train with the velocity of 10 mt./sec. Find the magnitude and direction of the velocity with which the stone appears to meet the train.

11. Two motor cars are proceeding, one on each road, towards the point of intersection of two roads which meet at an angle of 60°. If their speeds are $12\frac{1}{2}$ and 20 m/hr and they are respectively 350 and 200 yds. from the cross-roads, find their (i) relative velocity and (ii) their distances from the crossing of the roads when they are nearest to each other.

12. Two points move with speeds u and $2u$ respectively along two straight lines inclined at an angle α to each other. Find the relative velocity of the second point with respect to the first.

13. A submarine leaves A and travels S.W. at 10 kilometres an hour. At the same instant a destroyer leaves B which is 20 kilometers south of A. If the the destroyer can steam at 25 kilometres per hour, find the direction in which it should move to hit the submarine.

14. The direction of an aeroplane makes an angle θ with the direction of the wind. If v be the velocity of the aeroplane relative to the air and V ($< v$ be the velocity of the wind, show that the effective velocity of the aeroplane is $V \cos\theta + \sqrt{v^2 - V^2 \sin^2\theta}$ and find the course to be steered.

15 A cyclist rides at 10 miles/hr due North and the wind which is blowing at 6 miles/hr from a point between North and East appears to the cyclist to come from a point 15° E of N.

(i) Find the true direction of the wind.

(ii) the direction in which the wind will appear to meet him on his return if he rides at the same speed.

16. You are on a ship travelling steadily East at 15 knots. A ship on a steady course whose speed is known to be 26 knots is observed 6 miles due South of you. It is later observed to pass behind you, its distance of closest approach being 3 miles.

(i) What was the course of the other ship?

(ii) What was the time between its position South of you and its position of closest approach? (1 knot = 6080 ft /hr.)

17. A parachutist falling vertically in a steady down pour of rain observes that when his speed is v_1 the rain apears to make an angle α with the vertical. When his speed is v_2 the rain appears to make an angle β with the vertical. Show that the rain actually falls at an angle θ with the vertical given by

$$(v_2-v_1)\cot\theta=v_2\cot\alpha-v_1\cot\beta.$$

18. An Aeroplane which travels at the rate of 80 miles an hour in still air starts from A to go to B which is 200 miles distant N. E. of A. If there is a wind blowing from the North at 20 miles an hour determine the direction in which the aeroplane must move and the time required. If at the end of an hour the wind drops to 5 miles per hour determine the position relative to B of the aeroplane at the time, when it should have arrived at B.

19. A ball is thrown in horizontal line from a window of a train moving with 36 km /hr. If the velocity of the ball is 7 m./sec. at right angle to the motion of the train, find the velocity of the ball relative to the train (at the time of projection).

20. To an observer on a train moving at 30 kilometres per hour due North, wind appears to blow from 15°E of N and from a motor car running at $15(\sqrt{3}-1)$ km./hr. due East, it appears to come from 15°N of E. Find the true direction of wind.

21. A ship steams due West at the rate of 15 km /hr. when the river is flowing at the rate of 6 km./hr. due South. What is the velocity relative to the ship of a train going due North at the rate of 30 km./hr.? (C. U. 1968)

22. A line of men are running along a road at 8 miles an hour behind one another at equal intervals of 20 yards. A line of cyclists are riding in the same direction at 15 miles an hour at equal intervals of 30 yards. At what speed must an observer travel along the road so that whenever he meets a runner he also meets a cyclist?

23. Two particles P and Q, start at the same time from rest from a point O and move in two directions, P with uniform velocity and Q with uniform acceleration. Show that at any time each appears to the other to be moving in a direction parallel to $\overline{QR}$, where R is the middle point of $\overline{OP}$.

24. If d be the distance at any time between two points moving uniformly in one plane, V their relative velocity and u, v the resolved parts of V along and perpendicular to the direction of d, show that their distance when they are nearest to each other is $\frac{dv}{V}$ and that the time of arriving at this nearest distance is $d\frac{u}{V^2}$.

CHAPTER IV

MOTION ALONG A STRAIGHT LINE

§ 4·1. Change of velocity ; motions along a straight line :

In chapter II we have discussed briefly about acceleration. It was defined as the rate of change of velocity and the parallelogram law for accelerations, similar to one for velocities, has been proved previously. It has also been shown that a particle may have uniform speed but non-uniform velocity if it moves in arcous path. For, a particle moving along a straight line, a change in velocity implies a change in its magnitude. In any case a particle with non-uniform velocity is said to be moving with an *acceleration*. If at any instant a particle changes its velocity from $\overline{V}_A$ to $\overline{V}_B$ then we can consider the change in the following way.

At the instant t the particle, in addition to its velocity $\overline{V}_A$, is given another velocity represented both in magnitude and direction by the vector $\overline{V}_B - \overline{V}_A$.

A particle is said to have a *uniform acceleration* if the rate of change of velocity is equal in equal intervals of time, however small. Acceleration, like velocity is a vector quantity, i.e., it has magnitude, direction and sense. A particle moving along a straight line may increase or reduce its speed at uniform rate ; in the former case we say that the particle has uniform acceleration and in the latter case the particle is said to have a uniform *retardation*. In other words negative acceletation is called retardation. In this chapter we will discuss, unless otherwise stated, the motion of point particles along a straight line with uniform acceleration. We will derive various relations between velocity, acceleration, the space described in time t and the interval of time t, but will not discuss about the forces governing the motions of the body. In short, we will discuss the *kinematics* of *motion along a straight line*.

§ 4·2. Analytical expressions for velocity and acceleration.

(a) Velocity at time t.

Suppose a particle moves along a line $\overrightarrow{OX}$. Let O be a fixed point on the line and let us agree to consider O as the initial position of the particle, i.e., its position at time $t=0$.

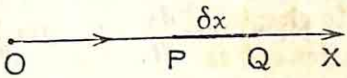


Fig. 38

Let the particle be at P at the instant t where P is at a distance x from O. Also let after a subsequent small interval of time δt its position be at Q at a distance $x+\delta x$ from O.

Here we have $OP=x$, $OQ=x+\delta x$ and hence $PQ=\delta x$.

$\therefore$ The displacement of the particle in time is δx and hence the average velocity in time δt is $\frac{\delta x}{\delta t}$. Th limiting value of this ratio, i.e., $\lim_{\delta t \to 0} \frac{\delta x}{\delta t}$ is defined to be the velocity of the particle at time t,

i.e., $v = \underset{\delta t \to 0}{\text{Lt}} \frac{\delta x}{dt} = \frac{dx}{dt}$, in notation of differential calculus.

Sometimes $\frac{dx}{dt}$ is written as x for the sake of brevity.

Cor. 1. $\frac{dx}{dt}$ always gives the velocity along the positive direction of the x-axis ; a negative value of $\frac{dx}{dt}$ indicates a velocity along the negative direction of the X-axis.

Cor. 2. If $\frac{dx}{dt}$ = a constant, then the particle is said to have a uniform velocity along the X-axis.

(b) Acceleration at time t.

As shown above, $\frac{dx}{dt}$ is the veloeity of the particle at P. Let its velocity be $v+\delta v$ at Q. Hence the change in velocity

during the short interval δt is δv $\therefore$ the average value of the acceleration in the short interval dt is $\frac{\delta v}{\delta t}$. The acceleration f of the particle at time t is defined as

$$f = \underset{\delta t \to 0}{Lt} \frac{\delta v}{\delta t} = \frac{dv}{dt}.$$

Now $v = \frac{dx}{dt}$ $\therefore$ $f = \frac{d}{dt}(v) = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d^2x}{dt^2}$

Sometimes $\frac{d^2x}{dt^2}$ is written as $\ddot{x}$. Also, $f = \frac{dv}{dt} = \frac{dv}{dx} \cdot \frac{dx}{dt} = v\frac{dv}{dx}$

Thus the acceleration of a particle can be expressed as any one of the following ways

(i) $\frac{dv}{dt}$ (ii) $\frac{d^2x}{dt^2}$ (iii) $x\frac{dv}{dx}$

What particular expression is to be used depends on what one wants to derive.

For example, to get a relation between x, v and f, the expression (iii) will be the most convenient.

N. B. If $\frac{d^2x}{dt^2}$ = a constant, then the particle is said to have a uniform acceleration. In particular when $\frac{d^2x}{dt^2} = 0$ always, the particle has uniform velocity.

§ 4·3. **Illustrative Examples.**

1. A particle is moving along a straight line and its distance in cms. from a given point in the line after t seconds from start is given by

$$s = t^3 - 4t - 4$$

Find its velocity at the end of 3 secs. and the acceleration at the end of 5 secs.

Here $v = \frac{ds}{dt} = 3t^2 - 4$ ... (i)

and $f = \frac{d^2s}{dt^2} = 6t$ ... (ii)

(N. B. Here acceleration depends on t and hence is not uniform)

putting $t=3$ in (i) we get the required velocity

$$v=3\times 3^2-4=27-4=23 \text{ cm./sec.}$$

and acceleration at the end of 5 secs. is obtained by putting $t=5$ in (ii)

$\therefore$ it is $6\times 5=30$ cm./sec.^{2}

2 A particle is moving along a straight line OA. Its distance from O in t secs. is given by

$$x=(t^3-2t-16)ft.$$

What is its acceleration when it is at a distance 5 ft. from O ?

The time t when the particle is at a distance 5 ft. from O is given by

$$t^3-2t-16=5$$

$$\text{or,}\quad t^3-2t-21=0$$

$$\text{or,}\quad (t-3)(t^2+3t+7)=0$$

which gives $t=3$ or $t^2+3t+7=0$

$$\text{i.e.,}\ t=\frac{-3\pm\sqrt{-19}}{2}$$

neglecting the imaginary values of t

we get, $t=3$ secs.

$$\text{Now}\quad x=t^3-2t-16$$

$$\therefore\quad \frac{dx}{dt}=3t^2-2$$

$$\text{and}\quad \frac{d^2x}{dt^2}=6t$$

$\therefore$ acceleration at the end of 3 secs. (or when the particle is at a distance 5 ft. from O) is $6\times 3=18$ ft /sec.^{2}

3. If $s=63t-6t^2-t^3$, find the velocity after two seconds and the distance gone before the particle stops, s being the distance covered by the particle in t secs. (C. U. 1958)

$$\text{Here}\quad s=63t-6t^2-t^3$$

$$v=\frac{ds}{dt}=63-12t-3t^2$$

after $t=2$ secs., $v=63-12\times 2-3\times 2^2=27.$

Here the unit of length is not specified.

$\therefore$ $v=27$ units of length/sec.

The particle stops when its velocity is zero, the corresponding time is obtained from

$$o=\frac{ds}{dt}=63-12t-3t^2$$

$$\text{or, } t^2+4t-21=0$$

$$\text{or, } (t+7)\ t-3)=0$$

$$\text{or, } t=-7 \text{ or, } 3$$

neglecting the negative value,

we get $t=3$ secs

$$\therefore \quad s=63\times 3-6\times 3^2-3^3$$

$$=108 \text{ units of length.}$$

4. Find the law of acceleration if,

(i) $v^2=1-x^2$ and (ii) $v^2=6a(x\sin x+\cos x)$

(i) $v^2=1-x^2$

differentiating both sides with respect to x, we get

$$2v\frac{dv}{dx}=-2x$$

$$\therefore \quad v\frac{dv}{dx}=-x$$

$$\therefore \quad f=-x.$$

Now when $x>0, f<0$ i.e., the acceleration is towards O.

Again, when $x<0$, $f>0$ i.e. the acceleration is towards the $+ve$ side of O, when x is on the $-ve$ side of O.

the acceleration is always directed towards O and varies directly as the distance from O

(ii) $v^2=6a\ (x\sin x+\cos x)$

differentiating with respect to x

$$2v\frac{dv}{dx}=6a\ [\sin x+x\cos x-\sin x\ ]$$

$$=6ax\cos x$$

$$\frac{dv}{dx}=3ax\cos x$$

hence $f=3ax\cos x$

here $f>0$ for $x>0$

$f<0$ for $x<0$

$\therefore$ the acceleration is away from O.

5. The law of motion of a body moving along a straight line is $x=\frac{1}{2}vt$, x being its distance from a fixed point on the line at time t and v its velocity there; prove that it moves with a constant acceleration.

We have $= x=\frac{1}{2}vt$

writing $v=\frac{dx}{dt}$ we have. $x=\frac{1}{2}t\frac{dx}{dt}$

$$\text{or,} \quad \frac{2dt}{t}=\frac{dx}{x}$$

Integrating both sides,

$$2\log t=\log x+c', \; c' \text{ is a const.}$$

$$\text{or,} \quad \log x=\log t^2+c'' \; (c''=-c')$$

$$=\log t^2c \; (\text{where } c=\log c'')$$

$$\therefore \quad x=ct^2 \quad \text{or,} \quad \frac{dx}{dt}=2ct, \; \frac{d^2x}{dt^2}=2c$$

Hence acceleration is $2c$, a constant.

6. A particle moves along a straight line and at a distance x from a fixed point on the line, its velocity is $\mu\sqrt{\frac{c-x}{x}}$

Prove that its acceleration is directed towards O and is inversely proportional to the square of its distance.

Here $v=\mu\sqrt{\frac{c-x}{x}}$

$$\therefore \; v^2=\mu^2\frac{(c-x)}{x}=\frac{c\mu^2}{x}-\mu^2$$

Differentiating both sides with respect to x,

$$2v\frac{dv}{dx}=-\frac{2c\mu^2}{x^2}$$

$$\therefore \quad f=v\frac{dv}{dx}=-\frac{c\mu^2}{x^2}\propto\frac{1}{x^2}$$

also f is negative (x is always +ve here for v to be real).

Hence the acceleration is directed towards O and is inversely proportional to the square of the distance.

§ 4.4. Motion along a straight line with uniform acceleration.

In previous chapters we discussed simplest types of motions possible for a particle, i.e., motions along a straight line with

uniform velocity. Now if the velocity is non-uniform, *i.e*, if the particle has acceleration then obviously the next step will be to consider motion with uniform acceleration. For motions with uniform acceleration, we will establish three basic relations between various quantities viz, velocity, acceleration, distance covered during an interval of time t and during the time t.

A particle is moving along a straight line with a untform acceleration f. Let u and v denote its velocity at the beginning and end of any interval of time t, considered during its motion and s, the distance covered by the particle during that interval. Then

(i) $v=u+ft$

(ii) $s=ut+\frac{1}{2}ft^2$

(iii) $v^2=u^2+2fs$

Fig. 39

Let $\overrightarrow{OA}$ be the straight line along which the particle moves. O being the initial position of the particle, i.e., the position at the beginning of the interval t. Let P be its position at the end of the interval t when OP $= s$.

Then the velocity and acceleration of the particle at time t are given by $\frac{ds}{dt}$ and $\frac{d^2s}{dt^2}$ respectively. Since the acceleration of the particle is uniform it has the same value throughout the interval.

$$\therefore \quad \frac{d^2s}{dt^2}=f \qquad \ldots \qquad (i)$$

Integrating both sides of (i) with respect to t we have

$\frac{ds}{dt}=ft+c$ (2) where c is an integrating constant.

Now according to the given conditions when $t=0$,

$\frac{ds}{dt}=$ velocity at $O=u$ $\therefore$ from (2)

$$u=f\times 0+c \qquad \therefore \quad c=u$$

$\therefore$ we get

$$\frac{ds}{dt}=u+ft \quad \cdots \quad (3)$$

writing v for $\frac{ds}{dt}$ we get $\mathbf{v=u+ft}$.

(ii) From (3) we get, $ds=(u+ft)\,dt$. Integrating with respect to t we have,

$s=ut+\frac{ft^2}{2}+c'\cdots(4)$ where c' is a constant to be determined from the initial conditions, $t=o;\ s=o$

$\therefore\quad o=u\times o+f\times o+c' \quad \therefore\quad c'=o$

$\therefore$ (4) gives $\mathbf{s=ut+ft^2}$

The formula (iii) can be proved either by eliminating t between (i) and (ii), or writing $v\frac{dv}{ds}$ in place of $\frac{d^2s}{dt^2}$ in (i)

$v\frac{dv}{ds}=f\cdots(5)$

or, $\frac{d}{ds}\left(\frac{v^2}{2}\right)=f$ or, $d\left(\frac{v^2}{2}\right)=fds$

Integrating we get $v^2/2=f.s+c''$ where c'' is a constant. Initial conditions given are $s=o,\ v=u \quad \therefore\quad u^2/2=f\times o+c''$

$\therefore\quad c''=u^2/2$

$\therefore\quad v^2/2=fs+u^2/2$

$\therefore\quad \mathbf{v^2=u^2+2fs.}$

Cor. (i) For a particle moving with uniform *retardation* along a straight line, the above formulas read as follows :

(1) $v=u-ft$
(2) $s=ut \quad \frac{1}{2}ft^2$
(3) $v^2=u^2-2fs.$

(ii) The *average velocity* of a particle moving along a straight line with uniform acceleration during any interval of time is given by

$$v_{av}=\frac{s}{t}=\frac{ut+\frac{1}{2}ft^2}{t}=u+\frac{1}{2}ft$$

$$=\frac{2u+ft}{2}=\frac{u+u+ft}{2}=\frac{u+v}{2}$$

Hence the average velocity is equal to *the velocity at the middle of the interval* which is the same as the *mean of the initial and final velocity.*

§ 4.5. Distance described in the t-th second.

Let a particle move along a straight line with uniform acceleration f and initial velocity u.

Let s_t be distance described in the t-th second.

Then s_t = distance described in t seconds − distance described in $(t-1)$ seconds

$$= (ut+\tfrac{1}{2}ft^2)-\{u(t-1)+\tfrac{1}{2}f(t-1)^2\}$$
$$= ut-u(t-1)+\tfrac{1}{2}ft^2-\tfrac{1}{2}f(t-1)^2$$
$$= u+\tfrac{1}{2}f[t^2-(t-1)^2]$$
$$= u+\tfrac{1}{2}f(2t-1).$$

For example, the distance described in the 1st second is $u+\frac{1}{2}f$, in the 2nd second is $u+\frac{3}{2}f$, in the 3rd second is $u+\frac{5}{2}f$, etc.

Thus the distances described in the consecutive seconds are in A. P. wiith common difference f.

§ 4·6. Worked out Examples.

1. (In the following examples f is constant)

(i) given, $u=5$, $f=2$, $t=3$, in C. G. S. units, find s.

(ii) $u=2$, $f=1/2$, $t=4$ in F. P. S. units, find v and s.

(iii) $u=10$ mts./sec, $f=-1$ cm /sec^2, $t=5$ secs. find v in km./hr

(iv) $u=6$, $v=4$, $s=10$ in C. G. S. units; find f and t.

(i) In $s=ut+\frac{1}{2}ft^2$

put. $u=5, f=2, t=3,$

$s=5\times 3+\frac{1}{2}\times 2\times 3^2=15+9=24$ cm.

(ii) From $v=u+ft$

we get, $v=2+\frac{1}{2}\times 4=2+2=4$ ft./sec.

$$s=ut+\tfrac{1}{2}ft^2=2\times 4+\tfrac{1}{2}\times\tfrac{1}{2}\times 4^2$$
$$=8+4=12 \text{ ft.}$$

(iii) $v=u+ft$.

Here $u=10$ mts./sec. $=10\times 100=1000$ cms./sec.

$\therefore\ v=1000+(-1)\times 5$

$$=1000-5=995 \text{ cm /sec.}=\frac{995\times 60\times 60}{100\times 1000}\text{ km./hr.}$$

$$=35{\cdot}82 \text{ km./hr.}$$

(iv) $v^2 = u^2 + 2fs$. Here $u=6$, $v=4$, $s=10$

$\therefore$ $4^2 = 6^2 + 2f.10$; or, $16-36=20f$ $\therefore$ $f=-1$.

$\therefore$ Retardation $=1$ cm./sec^2.

Now $v=u+ft$; $\therefore$ $4=6+(-1)t$, $\therefore$ $t=2$ secs.

2. A body starting from rest moves with uniform acceleration of 2 cm./sec^2 ; find the time it takes to describe the first centimetre. Also find its velocity at that instant.

Here $u=0$, $f=2$ cm./sec^2. Let t be the time in seconds to cover the first centimetre.

Then from $s=vt+\frac{1}{2}ft^2$ we have

$$1 = 0\times t + \tfrac{1}{2}.2\times t^2$$
$$= t^2$$

$\therefore$ $t^2 = 1$ $\therefore$ $t=\sqrt{1}=1$ sec.

also $v=u+ft=0+2\times 1=2$ cm./sec.

3. A car has its velocity uniformly increased from 10 ft./sec. to 20 ft./sec. while passing over 50 ft. Find the acceleration.

Here $u=10$ ft./sec , $v=20$ ft./sec., $s=50$ ft.

To find f.

from $v^2=u^2+2fs$

we have, $20^2=10^2+2\times f\times 50$

or, $400=100+100f$

or, $100f=300$ $\therefore$ $f=3$ ft./sec^2.

4. A body starts with an initial velocity 2 cm./sec. and moves with uniform acceleration of 4 cm./sec^2. Find the time it would take to describe 12 cm. after start.

Here $u=2$ cm/sec, $f=4$ cm/sec^2.

Let the required time be t secs.

then from,

$$s=ut+\tfrac{1}{2}ft^2$$

We have; $12=2t+\frac{1}{2}\times 4\times t^2$

or $12=2t+2t^2$

or $2t^2+2t-12=0$

or $t^2+t-6=0$ or $(t+3)(t-2)=0$

$\therefore$ $t=-3$; or $t=2$

Neglecting the negative value for t, the required time is 2 secs.

5. A bullet fired into a target loses half its velocity after penetrating 3 inches. How much further will it penetrte ?

(C. U. 1943)

Let the initial velocity of the bullet be u inch/sec, since the velocity of the bullet decreases, it is subjected to a retardation. Let retardation be f inches/sec^2.

Here $v=\frac{1}{2}u$

$\therefore$ from $v^2=u^2-2fs$ [notice the $-$ sign]

We have, $\frac{1}{4}u^2=u^2-2\times 3\times f$

or $6f=u^2-u^2/4=\frac{3}{4}u^2$

$\therefore \quad f=\dfrac{u^2}{8}$

Now the bullet will go on penetrating till it loses all its velocity.

So, if the distance further covered be s then we have,

$0^2=\frac{1}{4}u^2-2fs$ (here the final velocity is nil)

or $2fs=\dfrac{u^2}{4}$

or $2\times\dfrac{u^2}{8}\times s=\dfrac{u^2}{4} \quad \therefore \quad s=1$ inch

$\therefore$ The bullet will penetrate 1 inch more.

6. A train travels from a station A to a station B in 45 minutes. At a point C, somewhere between A and B, it attains its maximum velocity of 45 miles per hour.

If it travels with uniform acceleration from A to C and uniform retardation from C to B, find the distance between A and B.

(C. U. 1936)

Let the distance between A and C be x miles and that between C and B be y miles. Also let the uniform acceleration between A and C be f mile/hr^2 and the time from A to C be t hrs.

Now f is given by $v=u+ft$

or, $45=0+ft \quad \therefore \; f=\dfrac{45}{t}$ (t is in hrs)

again $v^2=u^2+2fx$

or $45^2=0^2+2fx=2x\;\dfrac{45}{t}$

$\therefore \quad x=\dfrac{45t}{2}$ miles ... (1)

Let the retardation from C to B be f' in miles/hr^2. The time taken to travel from C to B is 45 mins $- t = (\frac{3}{4} - t)$ *hrs*

Velocity at station B is zero,

$\therefore \quad 0 = v - f't = 45 - f' \times (\frac{3}{4} - t)$

$\therefore \quad f' = \dfrac{45 \times 4}{3 - 4t} = \dfrac{180}{3 - 4t}$

Also v is given by

$0^2 = v^2 - 2f'y$

$2f'y = 45^2 \qquad$ or, $\dfrac{2 \times 180}{3 - 4t} y = 45^2 \qquad \ldots \qquad (2)$

$y = \frac{45}{8}\ (3 - 4t)$

from (1) and (2) the distance between A and B is given by

$x + y = \dfrac{45t}{2} + \dfrac{45}{8}(3 - 4t)$

$= \frac{45}{8} \times 3 = 16\frac{7}{8}$ miles.

7. A train stops at two stations 4 miles apart and takes 8 minutes on the journey from one station to the other.

If its motion is first that of uniform acceleration x and then that of uniform retardation y, mile and minute being the units of distance and time respectively, prove that

$$\frac{1}{x} + \frac{1}{y} = 8$$

Let the train achieve its maximum velocity v, at a distance d miles from the first station, at time t after start.

then $\quad v = xt \ldots\ldots(1)$

again, $0 = v - y(8 - t)$

or, $\quad v = y\ (8 - t) \ldots(2)$

$\therefore$ from (1) and (2)

$\dfrac{1}{x} + \dfrac{1}{y} = \dfrac{t}{v} + \dfrac{8 - t}{v} = \dfrac{8}{v} \ldots\ldots\ldots(3)$

again $v^2 = 2xd$.

also $0^2 = v^2 - 2y\ (4 - d)$ or, $v^2 = 2y(4 - d)$

$\therefore \quad \dfrac{1}{x} + \dfrac{1}{y} = \dfrac{2d}{v^2} + \dfrac{8 - 2d}{v^2} = \dfrac{8}{v^2} \ldots\ldots\ldots(4)$

$\therefore$ from (3) and (4) $\dfrac{8}{v} = \dfrac{8}{v^2} \quad \therefore \quad v = 1$

from (3) we have

$$\frac{1}{x}+\frac{1}{y}=\frac{8}{1}=8.$$

8. A train travels between two stopping stations 7 miles apart in 14 minutes. Assuming that its motion is one of uniform accleration for part of the journey and of uniform retardation for the rest, prove that the greatest speed on the journey in 60 m.p.h.

Proceeding in the same way as in example (7) we get

$$\frac{1}{x}+\frac{1}{y}=\frac{7}{v} \text{ and } \frac{1}{x}+\frac{1}{y}=\frac{7}{v^2}$$

$$\therefore \quad v=1 \text{ mile/mt.}=60 \text{ miles/hr.}$$

9. A particle starts with an initial velocity u and passes successively over the two halves of a given distance with accelerations f and f' respectively.

Show that the final velocity is the same as if the whole distance were traversed with uniform acceleration $\frac{1}{2}(f+f')$ [C. U. 1940]

Let the total distance covered by the particle be $2s$.

then for the first half, we have

$$v_1^2=u^2+2fs \quad \cdots \quad \cdots \quad \text{(i)}$$

For the 2nd half the initial velocity is v_1 and let the final velocity be v_2.

$$\therefore \; v_2^2=v_1^2+2f's$$

$$=u^2+2fs+2f's \text{ [from (i)]}$$

$$=u^2+2s(f+f') \quad \cdots \quad \cdots \text{(ii)}$$

Now if the whole distance were covered with an acceleration f_0 to get the same final velocity

$$\text{then } v_2^2=u^2+2.f_0\,(2s)$$

$$=u^2+4f_0s \quad \ldots \quad \ldots \quad \text{(iii)}$$

Comparing (ii) and (iii)

$$f_0=\tfrac{1}{2}(f+f').$$

10. If a, b, c be the spaces described in the pth, qth and rth seconds by a body starting with a given velocity u and moving with uniform acceleration f, show that

$$a(q-r)+b(r-p)+c(p-q)=0 \qquad \text{[C. U. 1962]}$$

By § 4.7 we have,

$$a=u+\tfrac{1}{2}f(2p-1)$$
$$b=u+\tfrac{1}{2}f(2q-1)$$
$$c=u+\tfrac{1}{2}f(2r-1)$$

$$\therefore\ a(q-r)+b(r-p)+c(p-q)$$
$$=u[(q-r)+(r-p)+(p-q)]$$
$$+\tfrac{1}{2}f\{(2p-1)(q-r)+(2q-1)(r-p)+(2r-1)(p-q)\}$$
$$=u\times 0+\tfrac{1}{2}f[2\{p(q-r)+q(r-p)+r(p-q)\}$$
$$-\{(q-r)+(r-p)+(p-q)\}]$$
$$=u\times 0+\tfrac{1}{2}f[2\times 0-0]=0.$$

11. A particle starting with a given velocity moves for 3 secs. with constant acceleration during which time it describes 81*ft*; the acceleration then ceases and during the next 3 secs, it describes 72 ft. Find its initial velocity and acceleration.

Let the initial velocity be u and the acceleration be f in F. P. S. units.

Then we have

$$81=3u+\tfrac{1}{2}f.3^2$$
$$\text{or,}\quad 81=3u+\tfrac{9}{2}f$$
$$\text{or,}\quad 27=u+\tfrac{3}{2}f\cdots\quad\cdots\quad\cdots(\text{i})$$

also, velocity after 3 secs

$$v=u+3f.$$

For the next 3 seconds, acceleration ceases, hence it travels with a uniform velocity v and describes 72 ft. in 3 secs.

Hence $72=v.3=(u+3f)\times 3$

or, $24=u+3f\ \cdots\ \cdots\ (\text{ii})$

From (i) and (ii)

$$27-24=-\tfrac{3}{2}f$$

or, $3=-\tfrac{3}{2}f$ $\qquad\therefore\ f=-2$ ft./sec^2,

and $24=u+3f=u-6$

$\therefore\ u=24+6=30$ ft./sec.

Ex. 12. A train starts from rest with an acceleration 2·5 m/sec². After attaining a speed of 90 km/hr. it travels with uniform velocity. Find the time required to travel a distance of 1 km. from start.

Let, 90 km/hr. speed is attained after t secs. from start.

Now 90 km/hr. $=\frac{90\times 1000}{60\times 60}=25$ m/sec.

$\therefore\ 25=2\cdot5t,\ \therefore\ t=10$ secs.

If s be the distance traveled in 10 secs,

$s=\frac{1}{2}ft^2=\frac{1}{2}\times\frac{5}{2}\times 100=125$ metres.

Now, 1 km $-$ 125 ms $=875$ ms.

The time required to travel this 875 metre $=\frac{875}{25}=35$ secs. so the total time required to travel 1 km. from start is $(10+35)$ $=45$ secs.

13. A point moving with unifrom acceleration describes distances s_1, s_2 feet in successive intervals of t_1, t_2 seconds. Prove that the acceleration is

$$2(s_2t_1-s_1t_2)/t_1t_2(t_1+t_2).$$

Let the acceleration be f, then the particle describes s_1 ft in t_1 seconds and (s_1+s_2)ft in (t_1+t_2) seconds. Let u be the initial velocity. Then $s_1=ut_1+\frac{1}{2}ft_1^2$ (i)

$$s_1+s_2=u(t_1+t_2)+\frac{1}{2}f(t_1+t_2)^2 \quad \text{(ii)}$$

From (i)

$$\frac{s_1}{t_1}=u+\frac{1}{2}ft_1 \quad \text{(iii)}$$

From (ii) $$\frac{s_1+s_2}{t_1+t_2}=u+\frac{1}{2}f(t_1+t_2) \quad \text{(iv)}$$

(iv)—(iii) gives

$$\frac{s_1+s_2}{t_1+t_2}-\frac{s_1}{t_1}=\frac{1}{2}t_2$$

$$\text{or,}\quad \frac{1}{2}ft_2=\frac{(s_1+s_2)t_1-s_1(t_1+t_2)}{t_1(t_1+t_2)}$$

$$=\frac{s_2t_1\quad s_1t_2}{t_1(t_1+t_2)}$$

$$\therefore\quad f=\frac{2(s_2t_1-s_1t_2)}{2t_1t_2(t_1+t_2)}.$$

14. If a point moving under uniform acceleration describes successive equal distances in times t_1, t_2, t_3, then

$$\frac{1}{t_1}-\frac{1}{t_2}+\frac{1}{t_3}=\frac{3}{t_1+t_2+t_3}$$

Let the point describe equal distances, each equal to d, in times t_1 t_2, t_3.

Let the velocity of the particle at the beginning of the intervals t_1, t_2, t_3 be u, v, w respectively ; also let the uniform acceleration be f.

$\therefore$ for the motion, during the interval t_1, $d = ut_1 + \frac{1}{2}ft_1^2$

or $\quad \dfrac{d}{t_1} = u + \frac{1}{2}ft_1 \qquad \cdots$(i)

Similiarly we have;

$$\frac{d}{t_2} = v + \tfrac{1}{2}ft_2 \text{ and } \frac{d}{t_3} = w + \tfrac{1}{2}ft_3$$

Now, $v = u + ft_1$ and $w = u + f(t_1 + t_2)$

So we have, $\quad \dfrac{d}{t_2} = u + ft_1 + \frac{1}{2}ft_2 \qquad \cdots$(ii)

$$\frac{d}{t_3} = u + f(t_1 + t_2) + \tfrac{1}{2}ft_3 \qquad \cdots\text{(iii)}$$

From (i), (ii) and (iii) $\quad \dfrac{d}{t_1} + \dfrac{d}{t_3} - \dfrac{d}{t_2} = u + \frac{1}{2}ft_1 + u + f(t_1 + t_2) + \frac{1}{2}ft_3$

$$-(u + ft_1 + \tfrac{1}{2}ft_2)$$

$$= u + \tfrac{1}{2}f(t_1 + t_2 + t_3) \qquad \cdots\text{(iv)}$$

Now the point travels a distance equal to $3d$ in $(t_1 + t_2 + t_3)$ seconds.

$\therefore \quad 3d = u(t_1 + t_2 + t_3) + \frac{1}{2}f(t_1 + t_2 + t_3)^2$

or $\quad \dfrac{3d}{t_1 + t_2 + t_3} = u + \frac{1}{2}f(t_1 + t_2 + t_3) \qquad \cdots$ (v)

$\therefore$ from (iv) and (v) we get

$$d\left(\frac{1}{t_1} + \frac{1}{t_3} - \frac{1}{t_2}\right) = \frac{3d}{t_1 + t_1 + t_3}$$

$$\text{or,} \quad \frac{1}{t_1} - \frac{1}{t_2} + \frac{1}{t_3} = \frac{3}{t_1 + t_2 + t_3}$$

15. A distance s is divided into n equal parts at the end of each of which the acceleration of a moving particle is increased by f/n ; show that the velocity of the particle after describing the distance is $\quad \sqrt{fs(3 - 1/n)}$

Let v_r be the velocity at the end of the rth interval, then

$$v_r^2 = v_{r-1}^2 + 2f_{r-1}s/n \quad \ldots \quad \ldots \quad \text{(i)}$$

putting, $r = 1, 2, 3 \cdots\cdots n$ in (i)

we have, (taking $v_0 = 0, f_0 = f$),

$$v_1^2 = 2f_0 s/n = 2fs/n$$

$$v_2^2 = v_1^2 + 2f_1 s/n$$

$$= v_1^2 + 2\left(f + \frac{f}{n}\right)\frac{s}{n}$$

$$v_3^2 = v_2^2 + 2f_2 s/n$$

$$= v_2^2 + 2(f + 2f/n)s/n$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$v_{n-1}^2 = v_{n-2}^2 + 2f\left\{f + (n-2)\frac{f_n}{n}\right\}\frac{s}{n}$$

$$v_n^2 = v_{n-1}^2 + 2\{f + (n-1)f/n\}s/n$$

Adding, we have

$$v_n^2 = 2s/n\,[nf + f/n(1 + 2 + \cdots + (n-1))]$$

$$= \frac{2fs}{n}\left[n + \frac{1}{n}\cdot\frac{n(n-1)}{2}\right]$$

$$= \frac{2fs}{n}\left(n + \frac{n-1}{2}\right)$$

$$= \frac{f.s}{n}\,(2n + n - 1) = \frac{fs}{n}\,(3n - 1)$$

$$= fs\left(3 - \frac{1}{n}\right)$$

$$\therefore \quad v_n = \sqrt{fs\left(3 - \frac{1}{n}\right)}.$$

16. Two particles start at the same instant from a point A and move in the same direction along a straight line AB. The first has a uniform velocity of 40 ft./sec. while the second starts with an initial velocity of 16 ft./sec. and has a uniform acceleration of 6 ft./sec^2. Find the time that elapses before the two particles meet again. (C. U. 1965)

Let them meet, t seconds after the start.

If s be the distance traversed by the particles, then for the first particle

$$s=40.t \quad \ldots \quad \text{(i)}$$

and for the second particle

$$s=16t+\tfrac{1}{2}6.t^2=16t+3t^2 \quad \ldots \quad \text{(ii)}$$

From (i) and (ii) $40t=16t+3t^2$.

or, $3t^2-24t=0$ or, $8t-t^2=0$

or, $t(8-t)=0$, $\therefore$ $t=0$ or 8

$t=0$ indicates the starting point.

$\therefore$ $t=8$ sec. is the answer,

i.e. they meet 8 secs. after they start.

17. In the above example, find when, before the particles meet again, the distance between them is maximum and what is the maximum distance.

Let in t seconds after they start s_1, s_2 be the distances traversed by the first particle and the second particle respectively.

$$\text{then,} \quad s_1=40t$$
$$s_2=16t+3t$$

the distance between them is

$$x=s_1-s_2=24t-3t^2$$
$$=-3[16-8t+t^2]+48$$
$$=48-3(4-t)^2$$

$\therefore$ x is 48 minus a positive quantity.

Hence x is maximum when

$$3(4-t)^2=0 \text{ or } 4=t$$

i.e., 4 secs. after start and then $x=48$ ft.

18. Two points P and and Q move in a straight line $\overleftrightarrow{AB}$. The point P starts from A in the direction $\overrightarrow{AB}$ with velocity u_1 and acceleration f_1 and at the same time Q starts from B in the direction $\overrightarrow{BA}$ with velocity u_2 and acceleration f_2. If they pass one another at the middle point of $\overline{AB}$ and arrive at the ends of $\overline{AB}$ with equal velocities, prove that

$$(u_1+u_2)(f_1-f_2)=4(f_1u_2-f_2u_1)$$

Let s be the distance between A and B. Then according to given conditions

$$\frac{s}{2}=u_1t+\tfrac{1}{2}f_1t^2 \text{ for P} \quad \cdots \quad \text{(i)}$$

also $$\frac{s}{2}=u_2t+\tfrac{1}{2}f_2t^2 \text{ for Q} \quad \cdots \quad \text{(ii)}$$

and since they appear at the ends of $\overline{AB}$ with equal velocity v, say

$$v^2=u_1^2+2f_1s=u_2^2+2f_2s$$

$$\therefore \quad 2s(f_1-f_2)=u_2^2-u_1^2$$

$$\therefore \quad s=\frac{u_2^2-u_1^2}{2(f_1-f_2)} \quad \cdots \quad \text{(iii)}$$

From (i) and (ii), $u_1t+\frac{1}{2}f_1t^2=u_2t+\frac{1}{2}f_2t^2$

or, $u_1+\frac{1}{2}f_1t=u_2+\frac{1}{2}f_2t$

or, $u_2-u_1=\frac{1}{2}t(f_1-f_2$

$$\therefore \quad t=\frac{2(u_2-u_1)}{f_1-f_2} \quad \cdots \quad \text{(iv)}$$

Putting the values of s and t from (iii) and (iv) in (i) we get

$$\frac{s}{2t}=u_1+\tfrac{1}{2}f_1t$$

$$\text{or,} \quad \frac{\dfrac{u_2^2-u_1^2}{2(f_1-f_2)}}{\dfrac{2(u_2-u_1)}{f_1-f_2}}=u_1+\tfrac{1}{2}f_1\times\frac{2(u_2-u_1)}{f_1-f_2}$$

$$\text{or,} \quad \frac{u_2+u_1}{4}=u_1 \quad \frac{f_1(u_2-u_1)}{f_1-f_2}$$

$$=\frac{u_2f_1-u_1f_2}{f_1-f_2}$$

or, $(u_1+u_2)(f_1-f_2)=4(u_2f_1-u_1f_2)$ proved.

Examples

1. A particle is moving along a straight line $\overrightarrow{OA}$ and its distance in cms. from a given point in the line after t seconds from start is given by

$$s=t^4-2t^3-1$$

Find its velocity at the end of 2 seconds and its acceleration at the end of 3 seconds.

2. A particle moves along a straight line $\overleftrightarrow{AB}$ and its distance from A after t secs. is s ft. If s and t satisfy the relation

$$s = \cdot 5t + \cdot 25t^2, \text{ for all } t$$

prove that the acceleration is uniform.

3. A particle is moving along a straight line $\overleftrightarrow{OA}$. If distance from O in t secs is given by

$$x = t^3 - 2t^2 - 6,$$

what is its acceleration when it is at a distance 3 cm. from O.

4. If $s = at^2 + bt + c$, where t is the time, s is the distance traversed, v is the velocity and a, b, c are constants, prove that $4a(s-c) = v^2 - b^2$. [C. U. 1958]

5. A particle moves along a straight line and at a distance x from a fixed point O on the line, its velocity is given by

$$v^2 = 4x - x^2.$$

Show that $(f+x)^2 = 4$, where f is the acceleration.

6. A particle moves from rest at distance c from a fixed point O, with an acceleration $\dfrac{\mu}{x^2}$ away from O at a distance x. Find its velocity when it is at a distance $2c$ from O.

7. Discuss the motion represented by the following

(i) $s = 2t^2,\ 0 \leqslant t \leqslant 1$
$= 4t$ for $t >$.

(ii) $f = -x - 2v$.

8. A particle is moving along a straight line on which O is a fixed point.

The acceleration of the particle at time t after start is given by $f = 5t - 10$ and initially the particle is at rest at A where OA$= 16$ cms. Discuss the motion.

9. (In the following examples f is constant)

(i) given $u = 4, f = 1, t = 3$ in C. G. S. units ; find s.

(ii) $u = 3, f = 2, t = 5$ in F P. S. units ; find v and s

(iii) $u = 15$ mt./sec., $f = -5$ cm./sec.^{2}, $t = 4$ secs. find v in km /hr.

(iv) $u = 5, v = 6, s = 5\cdot 5$ in C. G. S. units ; find f and t.

10. A point has a velocity of 15mt /sec. at a certain instant and 10 secs. after has a velocity of 45mt,/sec. If the velocity changes uniformly, find the space described.

11. A particle starts from rest and at the end of 10 seconds is moving with a velocity of 8 mt/sec. If the acceleration be uniform find the velocity 5 seconds later and also the total distance described.

12. A train is travelling at 48 miles an hour when the brakes are applied. If it comes to rest at a station in 2 minutes, find at what distance from the station were the brakes applied.

13. A bullet moving with a velocity of 1200 ft./sec. has its velocity reduced to one half after penetrating one inch into a target. Assuming the resistance to be uniform, how far will it penetrate before it is stopped

14. A particle starting from rest moves in a straight line first with uniform acceleration a, and then with uniform retardation b. If it comes to rest in time t measured from the beginning after having described a space s, prove that

(C. U. 1945)

$$t^2 = 2s\left(\frac{1}{a}+\frac{1}{b}\right)$$

15. A bullet passes through a wall 9·6 inches thick and its velocity changes from 1200 to 800 ft. thereby. Find the time required by the bullet to pass through the wall and the velocity when half the wall is penetrated. (C. U. 1942)

16. A particle, moving with uniform acceleration, describes in the last second of its motion $\frac{9}{25}$th of the whole distance. It started from rest, find how long it was in motion and through what distances did it move, if it described 6cms. in the first second.

17. A particle moving with uniform acceleration describes equal distances s in consecutive t and $t/2$ seconds. Prove that in next $3t/2$ seconds it describes a distance equal to 5 s.

18. A train stopping at two stations 2 miles apart takes 4 minutes on the journey from one of the stations to the other.

Assuming that its motion is first of uniform acceleration x and then that of uniform retardation y, prove that $\frac{1}{x}+\frac{1}{y}=4$, a mile and a minute being the unit of distance and time respectively.

(C. U. 1934)

19. The velocity of a train increases at the constant rate f_1 from 0 to v; then remains constant for an interval and finally decreases to 0 at the constant rate f_2. If x be the total distance described, prove that the total time taken is

$$\frac{x}{v}+\frac{v}{2}\left(\frac{1}{f_1}+\frac{1}{f_2}\right)$$

20. A bus starts from rest with an acceleration of 1 mt/sec^2. Show that a passenger who can run at the rate of 10 mt./sec. cannot catch the bus if he is more than 50 ft behind it.

21. A body moves from rest with uniform acceleration. Show that in any interval the space average of the velocity is 4/3 of the time average.

22. A body moving in straight line travels distances AB, BC, CD of 153 ft., 320 ft., 135 ft. respectively in three successive intervals of 3 secs., 8 secs., and 5 secs. Show that these facts are consistent with the hypothesis that the body is subjected to uniform retardation. Find the distance from A to the point where the body comes to rest.

23. If v_1, v_2, v_3, be the average velocities in three successive intervals of time, t_1, t_2, t_3 of a point moving in a straight line with uniform acceleration, show that

$$\frac{v_1-v_2}{v_2-v_3}=\frac{t_1+t_2}{t_2+t_3}.$$

24. Two trains on the same line are approaching one another with velocities u_1 and u_2 respectively. When there is a distance x between them each is seen from the other and they apply brakes to produce retardations f_1 and f_2 respectively. Prove that it is just possible to a avoid a collision if

$$u_1^2f_2+u_2^2f_1=2f_1f_2x.$$

25. An express train tries to overtake a goods train on the same line. When they are seen from each other the distance between them is x and velocities are u_1 and u_2 respectively. Prove that it is just possible to avoid a collision if $(u_1 \quad u_2)^2 = 2(f_1 + f_2)x$ where f_1 is the greatest retardation and f_2 is the greatest acceleration that can be produced in the trains

26 A particle moves along a straight line with a constant acceleration. If the distances of the moving particle from a fixed point on the line be x_1, x_2, x_3 at times t_1, t_2, t_3, prove that the acceleration is

$$2\left\{\frac{(x_2 - x_3)t_1 + (x_3 - x_1)t_2 + (x_1 - x_2)t_3}{(t_2 - t_3)(t_3 - t_1)(t_1 - t_2)}\right\}$$

27. A particle starts from rest at A and moves with uniform acceleration f in a straight line. T secs. later a second particle starts from A and moves with uniform velocity u in the same line. If $u > 2f\text{T}$, prove that the second particle will be ahead of the first for a time $\frac{2}{f}\sqrt{u(u - 2f\text{T}}$.

28. The greatest possible acceleration of a train is 75 cm./sec^2. and the greatest possible retardation is 1 mt./sec^2. Find the least time taken to run between two stations 10·56 km. apart if the maximum velocity the train can have is 59·4 km. ph.

29. For the first $\frac{1}{3}$rd of the distance between two stations a train is uniformly accelerated and for the last 1/4th of the distance it is uniformly retarded. It starts from rest at one station and comes to rest at the other. Prove that the ratio of its greatest velocity to its average velocity is 19 : 12. (C. H. 1970)

30. A particle moves in a straight line in such a manner that its velocity, t seconds after it is projected with velocity u from the point from which the distance x is measured, is $ue^{a(t+x)}$ where a is a positive constant; prove that the time taken to attain the velocity $2u$ is

$$\frac{1}{a}\log\frac{2u+2}{2u+1}$$

and the distance traversed during the interval is

$$\frac{1}{a} \log \frac{2u+1}{u+1}.$$

31. A particle starts from rest with acceleration f; at the end of time t it becomes $2f$, at the end of time $2t$, $3f$ and so on. Show that the total distance described in nt seconds is

$$\frac{n(n+1)(2n+1)}{12} ft^2.$$

32. A constable seeing a thief at a distance x ft, starts with velocity u and moves with acceleration α in order to catch him whilst the thief runs with acceleration β, starting from rest. Show that the constable will overtake the thief either if $\alpha \geqslant \beta$ or if $\alpha < \beta < \alpha + \frac{u^2}{2x}$.

CHAPTER V

LAWS OF MOTION

§ 5·1. In previous chapters we discussed kinematics, i.e., the geometry of motions without enquiring into the nature of forces which produce these motions. In the present chapter we shall discuss the relation between the forces and the motion produced by them. As mentioned earlier, the definition of force comes from Newton's laws of motion which form the basis of what is known as 'Newtonian Mechanics' In the following the laws of motion are stated.

Newton's laws of motion.

First law.

Everybody continues in its state of rest or of uniform motion in a straight line, except in so far as it be compelled by any external impressed force to change that state.

Second law.

The rate of change of momentum is proportional to the impressed force and takes place in the direction in which the force acts.

Third law.

To every action, there is an equal and opposite reaction.

These laws cannot be proved and have to be taken as basic axioms. They have been verified indirectly for bodies moving with velocities which are small compared to the velocity of light. Also Newton's laws of motion provide very precise and accurate predictions for the positions and motions of celestial bodies like the Earth, the moon and the sun and the other planets and hence these laws are assumed to be fairly true. Over two hundred years after Newton formulated them, the laws of motion were accepted as the fundamental axioms of classical mechanics and every motion was supposed to be completely explained by these laws. But it must be mentioned that they have failed to explain the apsidal motion of mercury's orbit and proved inadequate in atomic and nuclear Physics.

However, for motions involving magnitudes of ordinary dimensions these laws proved immensely successful.

§ **5·2.** In the following we discuss the implications of the three laws.

1. **First law**—This is also known as *Law of Inertia.**

This law was first discovered by Galileo. The first law states that a body has no innate tendency to change its state of rest or of uniform motion by itself. If it is initially at rest it will continue to be at rest or if it is in uniform motion along a straight line, it will continue to move uniformly along the straight line. Only an external agent can change its state of rest or of uniform motion. This external agent is referred to as *force.* Hence the first law gives a qualitative definition of force. In the following, illustrations of the principle of inertia are taken from everyday life.

1. A person in a moving car will tend to lean forward if the car suddenly comes to a halt.

This occurs because the part of his body in contact with the car comes to rest with the car while the upper part of his body tends to continue to be in motion and hence he leans forward. Similarly when the car starts from rest, the upper part of his body will tend to continue to be at rest and the person will tend to lean backward.

2. If a ball is rolled on a plane it comes to rest within a short time owing to the friction of the ground. But if the plane be made smooth, for example long smooth track on ice or a track made of looking glass, the motion continues for a long time. In this case also the ball comes to rest because of air resistance and other external forces but it shows that if the body were allowed to move freely then the motion once gained would have continued indefinitely.

3. Stack some coins or metal discs one on top of the other and then hit the lower most one with a knife.

**Inertia* is the Latin for idleness.

The stack will be found erect with one coin, the lower most missing. Since the motion was imparted suddenly to the lowermost coin the other coins in the stack did not get sufficient time to overcome their inertia.

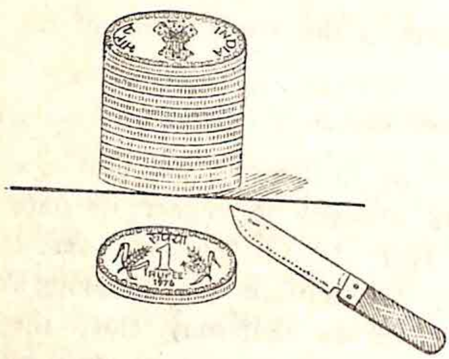


Fig. 40

From the first law we can conclude that if a body is at rest or moves with uniform velocity, no resultant force is acting on it. To change the motion of a body there must be some external forces acting on it.

§ 5·3. **Momentum :**

Before explaining *the second law* we give here the definition of momentum.

Momentum—*the momentum of a body is the product of its mass and velocity. The momentum is a vector quantity which has the same direction and sense as the velocity.*

Unit of momentum—This is defined as the momentum of unit mass moving with unit velocity In the F. P. S. system, the momentum is expressed as lb. ft /sec. and in the C. G. S. system as gm. cm./sec.

N. B. The momentum defined here is sometimes referred to as *linear momentum* to distinguish it from *angular momentum* which arises in the study of rotations of bodies.

§ 5·4. **Second law of motion :**

The first law gives a qualitative definition of force while the second law provides a method for measuring a force, i.e., it gives a quantitative definition of force. The second law states that the magnitude of a force is proportional to the rate of change of momentum produced and its direction is that of the change of momentum

The second law also implies the '*Principle of physical independence of forces.*'

Since the moment vector generated has the same direction as the force vector irrespective of the state of the body, if several forces act on a body each produces its own effect unchanged in magnitude and direction as if the other forces did not exist and as if it were the only force acting on the body. Again the force producing a change in the motion of a body depends on the mass of a body. The same force will produce different motions to different bodies, the motion generated in the heavier body will be less than that produced in the lighter body.

If mass of a body remains constant then the effect of a force is to change the velocity of the body, i.e., to generate acceleration (or retardation as the case may be) in it. The resultant effect of several forces is the resultant acceleration. For a body whose mass continually increases (for instance falling rain drop) a force will be required just to keep the velocity unchanged, for the momentum (mass × velocity) increases with mass.

§ 5·5. To deduce the formula P = mf.

Let at any instant a force P acts on a particle of mass m and let v be the velocity and f be the acceleration of the particle at that instant. From Newton's second law of motion, force is proportional to the rate of change of momentum.

$$\therefore \quad P \propto \frac{d}{dt}(mv)$$

$$\text{or,} \quad P \propto m\frac{dv}{dt} + v\frac{dm}{dt}$$

if m remains constant, then $\dfrac{dm}{dt} = 0$

$$\therefore \quad P \propto m\frac{dv}{dt} = mf$$

or, $P = k.mf$ where k is a constant whose value depends on the units used.

If we define a unit of force as that force which acting on a unit mass produces unit acceleration, then

when $m = 1, f = 1, P = 1 \qquad \therefore \; k = 1$

$\therefore \; P = mf$, where the unit of force is defined as above,

$\therefore$ magnitude of force = mass × acceleration.

N. B. Equation of motion of a particle is $P=m\dfrac{d^2x}{dt^2}$.

§ 5·6. **Units of force ; absolute units.**

Poundal is the unit of force as defined in § 5·5 in the F. P. S. system. A poundal is the amount of force which acting on a mass of one pound generates in it an acceleration of 1 ft./sec². In the C. G. S. system the unit of force is *Dyne*. A dyne is the amount of force which acting on a mass of one gramme generates in it an acceleration of 1 cm./sec².

A poundal and a dyne are absolute units of force.

Relation between the F. P. S. and the C. G. S. units of force.

We have

$$\frac{1 \text{ poundal}}{1 \text{ dyne}}=\left(\frac{1 \text{ lb.}}{1 \text{ gm.}}\right)\left(\frac{1 \text{ ft./sec.}^2}{1 \text{ cm./sec.}^2}\right)=\frac{1 \text{ lb.}}{1 \text{ gm.}}\cdot\left(\frac{1 \text{ ft.}}{1 \text{ cm.}}\right)$$
$$=453{\cdot}6\times30{\cdot}48$$
$$=13825{\cdot}7 \text{ nearly}$$

$\therefore$ 1 poundal = 13825·7 dynes nearly.

§ 5·7. **Newton's law of gravitation.**

Every particle of matter in the universe attracts every other particle in the direction of the straight line joining them with a force whose magnitude is directly proportional to the product of their masses and inversely proportional to the square of the distance between them.

Let m, m' be the masses of two particles and r the distance between them, then the force of attraction F which either particle exerts on the other is given by

$$F\propto\frac{mm'}{r^2} \quad \text{or,} \quad F=G.\frac{mm'}{r^2}$$

where G is the gravitational constant which depends on the units in which the quantities included in the formula are measured.

N. B. $G=6{\cdot}67\times10^{-8}$ in c. g. s. units.

§ 5·8. **Weight and Gravity :**

Gravity is a particular case of gravitation. It is the force with which the Earth attracts other bodies. The weight of a

body is owing to gravity. It is known (first demonstrated by Galileo) that owing to the attraction of the Earth everybody moves towards the Earth with a uniform acceleration g, the value of g being about 32 ft./sec.^{2} or 981 cm./sec^2.

Let m be the mass of a body,
then weight of the body $=$ mass $\times g = mg$ in absolute units.
weight of a mass of 1 lb $=32$ poundals nearly.
weight of a mass of 1 gm. $=981$ dynes nearly.
Also by the law of gravitation,

$$W = G.\frac{Mm}{r^2} = mg \text{ where } M = \text{mass of the Earth,}$$

$m =$ mass of the body, $r =$ distance between the centre of gravity of the Earth and that of the body;

$$\therefore \quad g = \frac{GM}{r^2}$$

$\therefore$ g is not a constant but changes with r. So g varies at different latitudes on the Earth's surface (since the Earth is not spherical) and it also varies with the altitude of the body at a fixed latitude.

§ 5·9. Gravitational unit of force :

We have just seen that g varies from place to place.

Hence to define the standard pound weight we must fix the value of g or take the value of g at a fixed place.

The standard pound weight is taken as that force which acting on a mass of 1 lb. will generate in it an acceleration of 32 ft./sec^2. *and the standard gramme weight is that force which acting on a body of mass 1 gm. will generate in it an acceleration of 981 cm/sec*^{2}.

N. B. "Absolute units" of force i.e., poundals and dynes are independent of g, while "gravitational units" of force (pound-wts. gram.-wts.) are dependent on gravity.

§ 5·10. Mass and Weight :

Though in common language we frequently use the words, mass and weight loosely, to mean the same thing, they have

separate meanings in Mechanics. The mass of a body is the quantity of matter in a body, whereas the weight of the body is the *force* with which the Earth attracts the body. From the equation $w = mg$, it is seen that the weight is proportional to the mass if g is constant. But, as we have just seen, g varies from place to place and hence the weight of a body will vary from place to place though its mass will remain the same.

Now at a given place on the Earth, the Earth's attraction on the bodies of equal masses (whatever their respective sizes may be) will be equal and hence their weights will also be equal. That is why we determine the mass of a body by weighing it with the help of a common balance. The correct way to speak of the weight of a body of total mass say 100 gm. will be to say that the weight of the body is equal to 100 gramme-weight

N. B. Mass is a scalar, weight is a force—a vector.

§ 5·11. Physical Independence of Forces :

As discussed earlier the second law implies physical independence of forces. Each force produces an acceleration in its own direction independent of the presence of other forces. For example, if an object be thrown up by a passenger inside the compartment of a moving train it falls in his hands. This shows that the vertical motion, along which the gravity acts, is independent of the horizontal motion which only produces effect in its own direction.

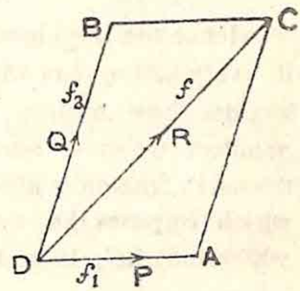
§ 5·12. Parallelogram of Forces :

With the help of the principle of physical independence of forces we can establish the famous theorem of *parallelogram of forces.* The theorem is stated below :

If a particle be acted on by two forces represented in magnitude direction and sense by two given straight lines drawn from a point, their resultant is a single force represented in magnitude, direction and sense by the diagonal drawn from the point of the parallelogram which has the given straight lines as adjacent sides.

Let the forces P and Q act on a particle of mass m. Let the accelerations generated by P and Q be f_1 and f_2 represented by $\overline{DA}$ and $\overline{DB}$ respectively.

Complete the parallelogram DACB. By the parallelogram of accelerations, the resultant acceleration is represented by $\overline{DC}$. Thus $\overline{DC}$ is the acceleration produced by the joint action of the forces P and Q, which means that the resultant R of the two forces P and Q acts along $\overline{DC}$ (Newton's second law of motion) and produces an acceleration represented by $\overline{DC}$.



$$\therefore \quad R = m\,\overline{DC}$$

In vector notation if $P = m.\overline{DA}$

$Q = m.\overline{DB}$ then $R = m.\overline{DC} = m.(\overline{DA} + \overline{DB})$

$$\text{or} \quad R = P + Q$$

Hence if P and Q are represented by m times $\overline{DA}$ and m times $\overline{DB}$, R is represented by m times $\overline{DC}$. So the resultant R is represented by $\overline{DC}$ if P and Q are represented by $\overline{DA}$ and $\overline{DB}$ respectively.

§ 5·13. Third law of motion.

Newton's second law of motion gives the relation between the acceleration of a body and the force acting on it and any problem in mechanics can be solved by it in principle. But Newton's third law is a general property of forces It states that *action equals reaction.* Suppose we have two particles, the first one exerting a force on the second one, then, at the same time, according to Newton's third law the second particle will push on the first one with an equal force, in the opposite direction ; further these forces effectively act in the same line. The third law implies one very important conservation law of nature, *i.e.*, conservation of momentum To see how it can be derived, we consider in details the case of two particles, discussed above. The forces between these two particles are equal and opposite. But according to Newton's second law, force is the rate of change of momentum.

So the rate of change of momentum say p_1 of the first particle equals to (minus) the rate of change of momentum p_2 of the second particle, then

$$\frac{d}{dt}(p_1)=-\frac{d}{dt}(p_2) \quad \text{or} \quad \frac{d}{dt}(p_1+p_2)=0$$

$\therefore$ $p_1+p_2=$ a constant independent of time.

Hence the total momentum of the particles is conserved. Now if every action has an equal and opposite reaction then one may wonder how motion is at all possible. Before we discuss this problem we must bear in mind that for motions on the surface of the earth friction is absolutely necessary though friction is the force which opposes the very motion. We give below some examples which may help to understand Newton's third law.

1. A body resting on a table: A body resting on a table exerts a pressure on the table which is equal to its own weight. The table, on the other hand, also exerts on the body an equal upward force ('called normal reaction' of the table), thus balancing the weight of the body and keeping it in position.

If the table is removed the body will fall but then the pressure on the table will also disappear.

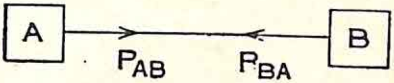
Here the action, in the form of weight of the body, equals the reaction or the upward force exerted by the table and the action is opposite to the reaction.

2. If a horse draws a huge block of stone tied by a rope, the horse is drawn back equally towards the stone. In this case the horse, of course, does not move backward but what actually happens is that the stone tends to prevent the horse from moving forward. If the rope were suddenly cut, the pull exerted by the rope will disappear and if the horse did not discontinue to exert the same effort with his feet he would start so quickly into motion that he would probably fall over.

In the above we have given some statical illustration of Newton's third law. Before we discuss some dynamical examples let us discuss different kinds of action and reaction.

(*i*) Thrust : When the action and reaction of two bodies tend to keep them apart from one another then the action and reaction are called *thrust* or a *push*.

(*ii*) Pull : When the action and reaction of two bodies tend to keep them together they constitute a *pull* or *tension*. If two bodies are connected by a string and any one of them drags or tends to drag the other then each exerts a pull on the other no matter in which direction they tend to move. In the adjacent figure P_{AB} is the pull of B on A and P_{BA} is the pull of A on B.



(*iii*) Attraction and repulsion : When two bodies act on one another at a distance, the force between them is called an attraction if it tends to bring them together or repulsion if it tends to keep them away from eaeh other, or to separate them.

(*iv*) Friction : When one body is in contact with another, the force offering resistance to relative motions between the surfaces in contact is called *friction*. That friction is essential for motion on the surface of the Earth can be seen from the following examples.

(*a*) Walking : Walking affords a very good example of the application of Newton's third law to locomotion. When we start walking, we press backwards on the ground with our feet and the reaction of the ground gives us an equal and opposite impulse which gives us the push forward. You have observed many a time how difficult it is to walk on a very smooth or wet ground. This is because smooth or wet floors have very little friction and so we tend to fall over while trying to walk on them.

(*b*) Motion of a horse and cart :

When a horse pulls a carriage forwards the carriage pulls the horse backwards, with an equal force.

To see how motion is possible we examine the system in detail.

Let A and B represent the carriage and the horse respectively. Let the mutual pull between the horse and the carriage be P.

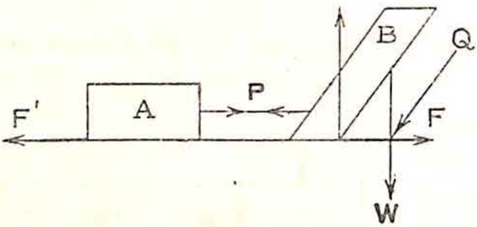


Fig. 41

The horse in attempting to drag the carriage strikes the ground with its hoofs and thus exerts an oblique force say Q.

By Newton's third law the ground offers an equal and opposite counter force.

Let the horizontal and vertical components of this counter force be F and R respectively. F acts in the direction of motion of the horse and R balances the weight W of the horse.

Let F' be the friction of the ground acting on the carriage in the direction opposite to that of its motion. Considering the motion of the horse alone, from Newton's second law of motion

$$F - P = M'f \qquad \ldots \qquad \text{(i)}$$

where M' is the mass and f is the acceleration of the horse which starts from rest.

Again considering the carriage alone,

$$P - F' = Mf \qquad \ldots \qquad \text{(ii)}$$

where M is the mass of the carriage.

From (i) and (ii) we have

$$F - F' = (M + M').\,f$$

$\therefore$ $f > 0$ if $F > F'$, i.e., motion will commence if the horizontal reaction F of the ground is greater than F', the frictional force opposing the motion of the carriage. So in starting the motion the horse has to strike the ground with such a force that $F > F'$. When motion becomes uniform $F = P = F'$. So in starting the motion, greater force is necessary than in preserving it.

§ 5·14 Units and Dimensions :

In mechanics units of time, length and mass are taken to be fundamental and other quantities are expressed in terms of them. The symbols [L], [T], and [M], stand for theoretical units of length, time and mass respectively in any system of units,

Dimensions of some quantities often discussed in mechanics are given below.

Velocity ; $\frac{L}{T} = LT^{-1}$

Acceleration : LT^{-2}

Momentum : MLT^{-1}

Force : MLT^{-2}

§ 5·15 Worked out Examples.

1. What is the momentum of a 300 pound bicycle moving at the rate of 30 miles an hour ?

$$30 \text{ miles}/hr = \frac{3\cancel{0} \times 176\cancel{0} \times 3}{6\cancel{0} \times 6\cancel{0}} \text{ ft/sec} = \frac{176}{4} = 44 \text{ ft/sec}$$

$\therefore$ the momentum = mass × velocity

$= 300 \times 44$ lb ft/sec $= 13200$ lb ft/sec

2. A force produces in a body of 100 kg. mass an acceleration of 10 mt/sec². Find the force (a) in dynes (b) in terms of gramme-weight.

Here $f = 10$ mt/sec^2 $= 1000$ cm/sec^2

$\therefore$ force $= 100 \times 1000 \times 1000$ dynes $= 10^8$ dynes.

$= \frac{10^8}{981}$ gm. wt. $= 1{\cdot}02 \times 10^5$ gm. wt. nearly

3. A uniform force of 520 dynes, acting for half a minute, changes the velocity of a body moving in a straight line from 290 cm/sec to 3·5 metres/sec. Find the mass of the body.

Change in velocity = 3·5 metres/sec—290 cm/sec

= 350 cm/sec—290 cm/sec = 60 cm/sec

This change occurs in $\frac{1}{2}$ minute or 30 seconds.

$\therefore$ acceleration $f = \frac{60}{30} = 2$ cm/sec^2.

Now $P = mf$

Here $P = 520$ dynes, $f = 2$ cm/sec^2

$$\therefore \quad m = \frac{P}{f} = \frac{520}{2} = 260 \text{ gm.}$$

$\therefore$ Mass of the body is 260 gms.

4. A train is moving on a horizontal railway at the rate of 30 miles an hour. If the steam be suddenly turned off, how far

will it run before it stops, the resistance being taken as equal to the weight of 5 lbs. per ton. [C. U. 1956]

30 miles/hr $=44$ ft/sec (see Ex. 1). Let the retardation be f

here $u=44$ ft/sec

Let the mass of the train be m tons ; then the resistance is,

$5m$ lbs wt.

$\therefore$ from $P=mf$ we have

$5m\times 32=mf\times 2240$ (1 ton $=2240$ lbs.)

$$\therefore\ f=\frac{5\times 32}{2240}\text{ ft/sec}^2=\frac{1}{.14}\text{ ft/sec}^2$$

Now from the formula, $v^2=u^2-2fs$

we have, $v=0$, $u=44$ ft/sec^2, $\therefore\ u^2=2fs$

$$\text{or,}\quad s=\frac{u^2}{2f}=\frac{44\times 44}{2\times\frac{1}{14}}=\frac{44\times 44}{2}\times 14\ ft.$$

$$=\frac{44\times 44\times 14}{2\times 1760\times 3}\text{ miles}=\frac{77}{30}=2\tfrac{17}{30}\text{ miles}$$

$\therefore$ the required distance $=2\frac{17}{30}$ miles.

5. A mass of m lbs. is acted on by a constant force of P poundals under which, in t seconds, it moves a distance x ft, from rest and acquires a velocity of v ft. per second.

Show that $x=\frac{1}{2}\ \dfrac{mv^2}{P}$.

The acceleration f generated by the force is given by, $P=mf$,

$$\therefore\ f=\frac{P}{m}\qquad\ldots\qquad(1)$$

Again from the formula $v^2=u^2+2fx$,

we have, (here $u=0$) $v^2=2fx$

$$\therefore\ x=\frac{v^2}{2f}=\frac{v^2}{2\frac{P}{m}}\quad(\text{ from (1) }),\quad\therefore\ x=\tfrac{1}{2}\ \frac{mv^2}{P}$$

6. A mass of 4 lbs falls 200 ft, from rest and is then brought to rest by penetrating 2 feet into some mud. Find the average thrust of the mud on it.

The body falls under gravity with an acceleration 32 ft/sec^2. The velocity acquired after this fall is given by

$$v^2 = 2gs \quad [\text{ See Chapter VI }]$$

here $g = 32$, $s = 200$

$$\therefore \quad v^2 = 2 \times 32 \times 200 = 12800$$

Now let the retardation due to the thrust of the mud be f, then we have,

$$v^2 = 2fx \text{ where } x = 2 \text{ ft.}$$

$$\therefore \quad 12800 = 4f, \quad \therefore \quad f = 3200 \text{ ft/sec}^2$$

Hence the resultant upward force acting on the body

$$= mf = 4 \times 3200 \text{ poundals} = 400 \text{ lbs. wt.}$$

The wt. of the body $= 4$ lbs. wt.

$\therefore$ The total thrust of the mud $= 400 + 4 = 404$ lbs. wt.

7. A shot of mass 100 lbs. moving at the rate of 1600 ft per sec. strikes a fixed target. How far will the shot penetrate the target, assuming that it offers an average resistance of the weight of 12000 tons? [C. U. 1938]

The resistance $= 12000$ tons wt.

$$= 12000 \times 2240 \text{ lbs. wt.}$$

$$= 12000 \times 2240 \times 32 \text{ poundals}$$

$\therefore$ The retardation f due to the resistance is given by

$$P = mf$$

$$\therefore \quad 12000 \times 2240 \times 32 = 100 \times f$$

$$\therefore \quad f = 2240 \times 32 \times 120 \text{ ft/sec}^2$$

Let the shot penetrate x ft. into the target, then x is given by

$$v^2 = 2fx$$

Here $v = 1600$ ft/sec

$$\therefore \quad 1600 \times 1600 = 2 \times 2240 \times 32 \times 120 \times x$$

$$\therefore \quad x = \frac{1600 \times \cancel{1600}}{2 \times \cancel{2240} \times \underset{2}{\cancel{32}} \times \cancel{120}} \text{ ft.}$$

$$= \frac{100}{4 \times 14 \times 12} \text{ ft.} = \frac{25}{14} \text{ inch} = 1\frac{11}{14} \text{ inch.}$$

8. A man weighing 12 stones is descending in a lift with acceleration 8 ft/sec^2. Find the thrust of his feet on the lift. Calculate the same when he is ascending with the same acceleration.

What would happen to this thrust if the chain of the lift broke (i) during descent. (ii) during ascent ? [C. U. 1943]

When the man is descending with an acceleration f, the resultant force on it is downwards. Let R be the thrust of his feet on the lift. Then the equation of motion is, $mg-R=mf$.

$$\therefore \quad R=mg-mf=m\,(g-f)=m\times(32-8)=m.\,24$$

$$=\overset{3}{\cancel{12}}\times\frac{\overset{3}{\cancel{24}}}{\underset{4}{\cancel{32}}}\text{ st. wt.}=9\text{ stone wt.}$$

When the man is ascending, the resultant upward force on him is $R-mg$ and the equation of motion is $R-mg=mf$.

$$\therefore \quad R=m(g+f)=12\times\frac{(32+8)}{32}\text{ st. wt.}$$

$$=12\times\frac{40}{32}\text{ st. wt.}=15\text{ stone wt.}$$

When the chain of the lift breaks the thrust becomes nil in both the cases.

9. A particle, whose mass is m, is acted upon by a force $m\left(x+\frac{a^4}{x^3}\right)$ towards the origin. If it starts from rest at a distance a; show that it will arrive at the origin in time $\pi/4$.

From Newton's second law, $P=mf$

$$\text{or} \quad -m\left(x+\frac{a^4}{x^3}\right)=mf. \quad \therefore \quad f=-\left(x+\frac{a^4}{x^3}\right) \qquad \ldots(1)$$

($-ve$ sign because the force is towards the origin)

We can write f as $v\dfrac{dv}{dx}$.

$$\therefore \quad \text{from (1),} \quad v\frac{dv}{dx}=-\left(x+\frac{a^4}{x^3}\right) \quad \text{or,} \quad vdv=-\left(x+\frac{a^4}{x^3}\right)dx$$

Integrating both sides we have

$$\frac{v^2}{2}=-\frac{x^2}{2}+\frac{a^4}{2x^2}+C, \text{ where } C \text{ is a constant.}$$

Since the particle starts from rest at a distance a from the origin, we have, $v=0$, when $x=a$

$$\therefore \quad 0=-\frac{a^2}{2}+\frac{a^4}{2a^2}+C=-\frac{a^2}{2}+\frac{a^2}{2}+C=0+C$$

$$\therefore \quad C=0.$$

$$\frac{v^2}{2}=-\frac{x^2}{2}+\frac{a^4}{2x^2}=\frac{a^4-x^4}{2x^2}, \quad \therefore \quad v^2=\frac{a^4-x^4}{x^2}.$$

$\therefore \quad v=-\dfrac{\sqrt{a^4-x^4}}{x}$ ($-ve$ sign because x decreases with time)

or, $\dfrac{dx}{dt}=-\dfrac{\sqrt{a^4-x^4}}{x}$, or, $-\dfrac{xdx}{\sqrt{a^4-x^4}}=dt$

Integrating both sides

$$-\int\frac{xdx}{\sqrt{a^4-x^4}}=t+C' \quad \ldots \quad (1)$$

Putting $x^2=a^2 \sin z$, $2xdx=a^2 \cos z\, dz$

$a^4-x^4=a^4-a^4 \sin^2 z=a^4(1-\sin^2 z)=a^4 \cos^2 z$

$$\therefore \quad \text{L. H. S.}=\tfrac{1}{2}\int\frac{a^2 \cos z dz}{a^2 \cos z}$$

$$=\tfrac{1}{2}\int dz=\frac{z}{2}=\frac{1}{2}\sin^{-1}\frac{x^2}{a^2}$$

$\therefore$ from (1) $-\tfrac{1}{2}\sin^{-1}\dfrac{x^2}{a^2}=t+C'$

at $t=0$, $x=a$, $\therefore \quad -\dfrac{1}{2}\sin^{-1} 1=0+C'$

$$\therefore \quad C'=-\tfrac{1}{2}\pi/2=-\pi/4$$

$\therefore \quad -\tfrac{1}{2}\sin^{-1} x^2/a^2=t-\pi/4 \quad \ldots \quad (2)$

The particle reaches the origin in a time t given by (2) when $x=0$.

$$\therefore \quad 0=t-\pi/4, \quad \therefore \quad t=\pi/4.$$

10. A thin glass plate can just support a weight of 27 lbs. A body is placed on it and the plate is raised with the body on it with a gradually increasing acceleration. It is found that the plate breaks when the acceleration is 4 ft/sec^2. Find the mass of the body.

Let the mass of the body be m lbs. Since the body is ascending the resultant force on it is upwards. If R be the thrust on the plate when the body is ascending with an acceleration of 4 ft/sec^2, R is given by

$$R-mg=mf, \quad \text{or,} \quad R=m(g+f)=m(32+4)=m\times 36,$$

Since the plate can just support a weight of 27 lbs and it breaks down when $f=4$ ft/sec^2,

so, we have $27\times32=m\times36$,

$$m=\frac{27\times32}{36}\text{lbs.}=24\text{ lbs.}$$

11. A train runs from rest 1 mile down an incline of 1 in 100. If the resistance be equal to 8 lbs. wt. per ton, how far the train will be carried along the horizontal level at the foot of the incline ?

Here the slope α of the incline is given by $\sin\alpha=\frac{1}{100}$.

Let m be the mass of the train in lbs. ; then the component of wt. down the plane is

$$mg\sin\alpha=\frac{m\times32}{100}\text{ poundals.}$$

The total resistance at the rate of 8 lbs per ton

is $\frac{8\times32\times m}{2240}$ poundals.

$\therefore$ The resultant force acting on the train is, $\frac{32m}{100}-\frac{256m}{2240}$

$$=\frac{32m}{10}\left(\frac{1}{10}-\frac{8}{224}\right)\text{ poundals}=\frac{144\times32m}{10\times2240}\text{ poundals}$$

So, the acceleration is given by $f=\frac{144\times32}{10\times2240}$ ft/sec^2.

The velocity acquired after run of 1 mile down the plane is given by

$$v^2=2fs=\frac{2\times144\times32}{10\times2240}\times1760\times3\text{ ft.}$$

Along the horizontal level, the only force acting on the train is the resistance which is, $\frac{32\times8m}{2240}$ poundals

$\therefore$ the retardation f' is given by

$$\frac{32\times8m}{2240}=mf',\quad\therefore\quad f'=\frac{32\times8}{2240}$$

$\therefore$ the required distance is given by

$$v^2=2f'x\text{ or }x=\frac{v^2}{2f'}$$

$$=\frac{2\times144\times32\times1760\times3\times2240}{10\times2240\times2\times32\times8}\text{ ft}$$

$$=\frac{2\times144}{10\times2\times8}\text{ miles}=\tfrac{9}{5}\text{ miles}=1\tfrac{4}{5}\text{ miles.}$$

12. A load W is raised by a rope, from rest to rest through a height h ; the greatest tension which the rope can safely bear is nW. Show that the least time in which the ascent can be made is $\left\{\frac{2nh}{(n-1)g}\right\}^{\frac{1}{2}}$.

Since the load is raised from rest to rest it achieves its maximum velocity after ascending a certain distance. Let this distance be x, then the maximum velocity achieved is given by $v^2=2fx \cdots\cdots (1)$

Also for the rest $(h-x)ft$ the load is raised against gravity. so we have, $v^2=2g(h-x) \quad \cdots \quad (2)$

Now the time is minimum when f is maximum.

The maximum f is given by

$\left(\text{the mass of the load being} \frac{W}{g}\right)$, $T_{max}-\left(\frac{W}{g}\right)f=W$

$$\text{but } T_{max}=nW \quad \therefore \quad nW=W\left(\frac{f+g}{g}\right)$$

$$\text{or} \quad ng=f+g \quad \cdot \quad f=(n-1)g$$

$$\text{from (1) and (2), } 2fx=2g(h-x)$$

$$\text{or} \quad 2(n-1)\,gx=2g\,(h \quad x)$$

$$\text{or} \quad (n-1)\,x=h-x, \quad \text{or} \quad nx=h \quad \therefore \quad x=h/n$$

$$\therefore \quad v^2=2fx=2\frac{(n-1)gh}{n} \quad \cdots \quad (3)$$

Now the time t_1 taken to ascend the first x feet is given by

$$v=ft_1 \text{ or } t_1=v/f=\frac{v}{(n-1)g}$$

and the time t_2 to ascend the rest of the distance is given by

$$v=gt_2 \text{ or } t_2=\frac{v}{g}$$

$\therefore$ the least total time is

$$t=t_1+t_2=\frac{v}{(n-1)g}+\frac{v}{g}=\frac{v}{g}\left(\frac{1}{n-1}+1\right)=\frac{nv}{(n-1)g}$$

$$=\frac{n}{(n-1)g}\sqrt{\frac{2(n-1)}{n}gh} \text{ (From (3))}=\sqrt{\frac{2nh}{(n-1)g}}$$

Exercise on chapter V

1. Find the momentum of a car of mass 3 tons moving at the rate of 40 miles/hr.

2. A certain force acting on a mass of 200 lbs. for five minutes gives it a velocity of 5 yards per minute.

Find the force (i) in poundal (ii) in lb-wt.

3. A constant force of 10773 dynes acts along a straight line on a mass of 9 lbs. at rest, for 2 minutes. How many feet does the mass move during the interval ? (given 1 lb. = 453·6 gms. 1 ft = 30·4 cm) [C. U. 1963]

4. A uniform force of 200 dynes changes the velocity of a body of mass 24 gms. moving in a straight line from 200 to 300 metres per sec. Find the time for which the force acts.

5. A mass of 4 lbs. falls through 100 ft. from rest and is then brought to rest by penetrating 2 ft. into some sand. Find the thrust of the sand on it, supposing it to be uniform. [C. U. 1950]

6. A bullet weighing half an ounce leaves the muzzle of a rifle-barrel 2 ft. long, with a velocity of 2000 ft. per sec. Find the force acting on the bullet in the barrel, assuming it to be uniform ; and also the time taken by the bullet to traverse the barrel. [C U. 1938]

7. A rope can just support a mass of 20 lbs, when at rest. Show that the rope will break if it raises a mass of 16 lbs with an acceleration greater than 8 ft/sec^2.

8. A force of 100 dynes acts for 4 secs. on a particle of mass 40 gms. initially at rest, and then ceases to act. What is the displacement of the particle in 10 secs ? What is its velocity then ?

9. A railway train exclusive of engine weighs 435 tons and starting along a level line from rest attains a speed of 40 miles per hour in 7 minutes. Calculate the average pull between the

engine and the train taking the resistance to be 15 lbs. wt. per ton. [C. U. 1935]

10. Find the velocity of a 4 lbs. shot that will just penetrate through a wall 10 inches thick, the resistance being 42 tons wt.

11. A train runs down an incline of 1 in 200 with the steam shut off; the resistance is 15 lbs. per ton and its velocity at the top is 30 miles per hour. Show that its velocity after it has run down 1000 yards is 27·4 miles per hour.

12. A bullet of mass 3 oz. moving horizontally with a velocity of 1280 ft/sec strikes a fixed target. If the resistance to the bullet is 1600 lbs. wt. find the depth of penetration and the time that elapses before the bullet comes to rest.

13. A bullet weighing 30 gms. is fired into a fixed block of wood with a velocity of 244 metres per second and is brought to rest in $\frac{1}{150}$ sec. Find in dynes, and in grammes-weight, the resistance exerted by the wood, supposing it to be uniform.

14. A particle of mass m moves from rest at a distance c from a fixed point O, under a force $\frac{m\mu}{x^2}$ away from O at a distance x.

Find its velocity when it is at distance $2c$ from O.

15. A particle of mass m, moving along the x-axis, is acted on by an attractive force which is given by the formula $\frac{mk^2a^3}{x^2}$ for $x \geqslant a$ and by the formula $\frac{2mk^2x}{a}$ for $x < a$. If the particle starts from rest at distance $2a$, prove that it will reach the origin with velocity $2k\sqrt{a}$,

16. A body whose true weight was 13 ounces appeared to weigh 12 ounces when weighed by means of a spring balance in a moving lift. What was the acceleration of the lift at the time of weighing?

17. Weight of a body by means of a spring balance is found to be 2 *lbs* at the equator. In Calcutta, the weight of the same body by a spring balance is $\frac{1}{5}$ ounce more. A ball can be thrown 16 *ft.* vertically upwards in Calcutta. To what height it can be thrown at the equator?

CHAPTER VI

Vertical Motion Under Gravity

§ **6·1.** In the previous chapter we discussed briefly about gravity. If a heavy particle be dropped from any height, it falls vertically downwards with a constant acceleration which is due to the attraction exerted by the Earth on the particle. Towards the end of the 16th century, Galileo first established the laws of falling bodies ; the laws are stated below.

1. At any place the acceleration of a falling body due to gravity is constant.

2. At any place the acceleration due to gravity is the same for all bodies.

Galileo found, by experimenting with different bodies, that the distance described by any body in any time was proportional to the square of the time and all bodies describe the same distance in the same interval of time. Some times it is found that light bodies, for example, a feather or a piece of paper, fall more slowly than heavy bodies like pieces of lead or iron. But this difference is due to different resistance of the air on them. In vacuo all bodies fall with same velocities. This was shown by Newton in his famous 'Guinea and feather experiment'. Newton put a guinea and a feather in a long glass tube, closed at one end and the other end with an air tight cap and a stop cock. On inverting the tube it was found that the feather fell more slowly than the guinea, but when air was pumped out from the tube by means of an air-pump and the experiment was repeated, the guinea and the feather fell together side by side throughout the entire length of the tube. Galileo also arrived at the same conclusion before Newton by dropping bodies of different weights from the top of the leaning tower of Pisa in Italy.

As mentioned before, the acceleration due to gravity, denoted by g, varies from place to place. But while working out different

problems we shall take the value of g, unless otherwise stated, to be 32 ft/sec^2 (in the F. P. S system) or 981 cm/sec^2 (in the C. G. S system).

§ 6.2. **A body falling freely under gravity.**

Let a body of mass m fall freely from O.

Free fall means the particle is just let go from O or in other words its initial velocity is nil. For a frame of reference, take O as origin and the vertical through O as the X-axis, the downward direction being taken as the positive direction. Now the force acting on the body due to gravity is mg. From 2nd law of motion;

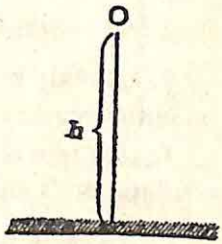


Fig. 42

$$m\frac{d^2x}{dt^2}=P=mg \quad \cdots \quad \text{(i)}$$

or $$\frac{d^2x}{dt^2}=g \quad \cdots \quad \text{(ii)}$$

Integrating (ii), $\dfrac{dx}{dt}=gt+c \quad \cdots \quad$ (iii)

Now the initial condition being,

at $t=0$, $\dfrac{dx}{dt}=0$ we get $c=0$ (from (iii))

$\therefore$ (iii) becomes $\dfrac{dx}{dt}=gt \quad \cdots \quad$ (iv)

Integrating again we have $x=\frac{1}{2}gt^2+C' \quad \cdots \quad$ (v)

Again if we measure the distance from O, then at $t=0$, $x=0$

$\therefore$ from (v), $C'=0$ $\therefore$ (v) becomes $x=\frac{1}{2}gt^2$

$\therefore$ If the body be dropped from a height h above the ground, *the time to reach the ground* is given by

$$h=\tfrac{1}{2}gt^2 \quad \text{or} \quad t=\sqrt{\frac{2h}{g}}. \quad \cdots \quad \text{(vi)}$$

$\therefore$ the velocity on reaching the ground is given by

$$\frac{dx}{dt}=gt=g\sqrt{\frac{2h}{g}}=\sqrt{2gh}; \text{ (i.e., } v=\sqrt{2gh}) \text{ [From (vi)]}$$

§ 6·3. **Motion of a body projected vertically downwards.**

Let the particle be projected vertically downwards from a point O with a velocity u. Here the problem is similar to that

discussed in §6·2 ; only thing to note here is that the initial velocity is not nil but u.

Since the acceleration is constant, we have from the formula $s=ut+\frac{1}{2}ft^2$, if h be the distance described by the particle in t seconds, then.

$$h=ut+\frac{1}{2}gt^2,\ v_t^2=u^2+2gh$$

also $h_t=u+\frac{1}{2}g(2t-1)$ where h_t is the distance described in the t-th second (see Ch. IV)

§ 6·4. Motion of a body projected vertically upwards

Let a body be projected vertically upwards from a point O with an initial velocity u

In this case if we take the upward direction as positive then the acceleration is along the negative direction, i.e, $f=-g$.

∴ The motion is given by the following equations

$$v=u-gt,\ h=ut-\frac{1}{2}gt^2$$
$$v^2=u^2-2gh;\ h_t=u-\frac{1}{2}g(2t-1).$$

N. B. Some times downward direction is taken as the positive direction, then necessary changes in the sign of the vector quantities like, u, g, h appearing in the above equations will have to be made.

§ 6·5. Greatest height

A particle, projected vertically upwards, ceases to rise after sometime, i. e. its velocity at that point is zero and it rises no further. The height at this instant is referred to as the *greatest height* attained by the particle.

From the formula, $v^2=u^2-2gh$,

putting $v=0$; $h=H$, where H is the greatest height

$$\text{we have } H=\frac{u^2}{2g}.$$

Hence to reach a maximum height H, the velocity of projection will have to be $\sqrt{2gH}$.

Again, in the formula $v=u-gt$,

putting, $u=0$, $t=T$, we get

$T=\frac{u}{g}$, the *time to reach the greatest height* or the time of ascent.

§ 6.6. Time to attain any given height.

From $h = ut - \frac{1}{2} gt^2$, if h is given, we get, $gt^2 - 2ut + 2h = 0$...(i)

This equation is quadratic in t and consequently has two roots. The roots are real if $u^2 \geqslant 2gh$. Also, if $u^2 \geqslant 2gh$, the roots are both real and positive. The reason for double roots will be clear from the follow.ng. Let a paιticle be projected vertically upwards from a point O Let P be a point on its path such that OP $= h$. Now the particle passes through P twice, once on its way up and the second time on its way down while descending. If t_1 and t_2 be the roots of (*i*) and $t_2 > t_1$, then t_1 gives the time to reach P on its way up and t_2 gives the time when the particle is at the same point on its way down

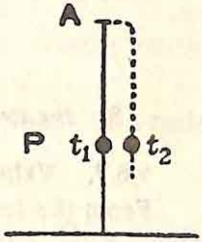


Fig. 43

If $u^2 \leqslant 2gh$, the roots are imaginary. It means that when $u^2 < \sqrt{2gh}$, it is not possible for the particle to attain a height h, for $u = \sqrt{2gh}$, h is the greatest height attained by the particle as discussed in § 6 5.

Now, from (i) we have $t = \dfrac{u \pm \sqrt{u^2 - 2gh}}{g}$

$$\therefore \quad t_1 = \frac{u - \sqrt{u^2 - 2gh}}{g} \text{ and } t_2 = \frac{u + \sqrt{u^2 - 2gh}}{g}$$

If OA be the greatest height attained by the particle, then the time to reach A is given by $T = \dfrac{u}{g}$.

$\therefore$ The time to reach A from P on the way up $= T - t_1$

$$= \frac{\sqrt{u^2 - 2gh}}{g}$$

and the time to reach P from A on the way down $= t_2 - T$

$$= \frac{\sqrt{u^2 - 2gh}}{g}$$

Hence the time of rise to the greatest height A *from any point* P *is always equal to the time of fall from the greatest height to* P.

Again when P coincides with O i e., when $h = 0$, we have from (i) $gt^2 - 2ut = 0$ or, $t\,(u - \frac{1}{2} gt) = 0$, here $t_1 = 0$, $t_2 = \dfrac{2u}{g}$.

The first root gives the time when the particle starts (the initial position) and the second root gives the time when the particle comes back to O, its starting point. Hence t_2 gives the whole time of flight. So, *the time of flight* $=\frac{2u}{g}$

Also the time of descent = time of flight—time of ascent

$$=\frac{2u}{g}-\frac{u}{g}=\frac{u}{g}.$$

So, *the time of ascent = the time of descent* $=\frac{u}{g}$.

§ 6.7. **Velocity at any height h.**

From the formula $v^2=u^2-2gh$,

v is known if u, g, and h are given.

$\therefore$ $v=\pm\sqrt{u^2-2gh}$, the positive value gives the velocity at a point P on the way up and the negative value gives the velocity at a point P on the the same point on the way down.

Hence *velocities at any point when moving up and down are equal in magnitude but opposite in direction*

Again if H be the greatest height attained, then from $v^2=u^2-2gh$, we have,

$v^2=2gH-2gh\left[\because\ u^2=2gH\right]=2g(H-h)$.

But $H-h$ is the distance of the point P at a height h from the greatest height.

Hence the velocity at any point is equal to the velocity due to a fall from rest from maximum height to the same point.

N. B. velocity on return to the starting point is equal in magnitude but opposite in sign to the velocity of projection.

§ 6 8. **Motion of a particle dropped from an ascending or a descending object.**

If a particle be dropped from a body moving vertically, for example a balloon or a lift, the initial velocity of the particle *relative to the moving object* is zero but is not zero relative to a frame of reference in which the Earth is supposed to be stationary. In fact initial velocity of the particle will be same as that of the moving object. If the particle be dropped from an object ascending with a velocity, say v, the motion of the particle

will be the same as that of a particle projected upwards with a velocity v as discussed in § 6·4, while the motion of a particle dropped from a descending body will be that of a particle projected vertically downwards as discussed in § 6.3.

§ 6·9. **Worked out Examples.**

1. A stone is projected vertically upwards with a velocity sufficient to carry it to a height of 50 ft. ; find its velocity when it is half way up.

Let the initial velocity of projection be u, then the greatest height attained is given by $h=\dfrac{u^2}{2g}$ ... (1)

Let its velocity be v when it is half way up. Then we have

$$v^2=u^2-2g.\frac{h}{2}=u^2-gh \quad \cdots \quad (2)$$

From (1) and (2) we have $u^2=2gh-gh=gh$

Here, $h=50$, $g=32\ ft/\text{sec}^2$,

$\therefore\ v^2=50\times 32=1600,\ \therefore\ v=40\ ft/\text{sec}.$

2. A ball is projected vertically upwards with a velocity of 30 ft/sec. Find the greatest height attained and the total time of flight.

The greatest height is given by,

$$h=\frac{u^2}{2g}=\frac{30\times 30}{2\times 32}\text{ ft.}=\frac{225}{16}\text{ ft.}=14{\cdot}06\text{ ft.}$$

The total time of flight is

$$t=\frac{2u}{g}=\frac{2\times 30}{32}\text{ sec.}=1{\cdot}87\text{ sec.}$$

3. A ball is thrown vertically upwards. Prove that it will be at half the greatest height after times whose ratio is $3+2\sqrt{2}:1$.

Let the velocity of projection be u.

Then the greatest height attained is given by $H=\dfrac{u^2}{2g}$.

At any time t, the height attained by the ball is given by

$h=ut-\frac{1}{2}gt^2$.

When $h=H/2$ we have $H/2=ut-\frac{1}{2}gt^2$,

or, $\dfrac{u^2}{4g}=ut-\frac{1}{2}gt^2$, or, $gt^2-2ut+\dfrac{u^2}{2g}=0$.

Solving we get $t=\frac{u}{g}\pm\frac{\sqrt{u^2-u^2/2}}{g}=\frac{u}{g}\pm\frac{u}{\sqrt{2}g}$

So, the roots are $t_1=\frac{u}{g}+\frac{u}{\sqrt{2}g}=u\frac{(\sqrt{2}+1)}{\sqrt{2}g}$

and $t_2=\frac{u}{g}-\frac{u}{\sqrt{2}g}=\frac{u(\sqrt{2}-1)}{\sqrt{2}g}$.

$\therefore\ t_1:t_2=(\sqrt{2}+1):(\sqrt{2}-1)=3+2\sqrt{2}:1.$

4. A body falls freely from the top of a tower and during the last second it falls $\frac{9}{25}$ th of the whole distance. Calculate the height of the tower. (C. U. 1966)

Let h be the height of the tower and t secs. be the whole time of fall.

Using the formula $h_t=\frac{1}{2}g(2t-1)$ we get

$$\tfrac{9}{25}h=\tfrac{1}{2}g(2t-1)\quad\cdots\quad(1)$$

Again in t secs. it falls the whole distance h

$$\therefore\ h=\tfrac{1}{2}gt^2\quad\cdots\quad(2)$$

Eliminating h from (1) and (2) we get

$$\tfrac{1}{2}gt^2=\frac{25}{9}\times\tfrac{1}{2}g(2t-1),\quad\text{or,}\quad t^2=\frac{25}{9}(2t-1),$$

$$\text{or,}\quad 9t^2-50t+25=0;\quad\text{or,}\quad(9t-5)\times(t-5)=0,$$

$$\therefore\ t=5/9,\quad\text{or,}\quad 5$$

but from the problem $t>1$

$$\therefore\ t=5\text{ secs,}\quad\therefore\ h=\tfrac{1}{2}gt^2=\tfrac{1}{2}\times32\times25\text{ ft.}=400\text{ ft.}$$

5. A particle after falling freely for some time under the action of gravity is observed to pass through 768 *ft.* in 4 secs ; how far will it fall in the next 4 secs ?

Let the velocity of the particle just before it starts to pass through 768 ft. be u ft/sec.

Then we have (taking $g=32$ ft/sec^2)

$$768=4u+\tfrac{1}{2}g\times4^2=4u+256$$

$$\therefore\ 4u=512\quad\therefore\ u=128\text{ ft/sec.}$$

Now in 8 secs. the particle passes through a distance given by, $h=8u+\frac{1}{2}g\times8^2=8u+64\times16$

$$=8(u+128)\text{ ft.}=8\times256\text{ ft,}=2048\text{ ft.}$$

In the first 4 secs. it passes through 768 ft. Hence in the next 4 secs. it will pass through $(2048-768)$ ft $=1280$ ft.

6. A body is thrown vertically upwards; if t be the time that it takes to reach a certain height and t' the subsequent time taken to reach the ground again, show that the greatest height attained is $\frac{g}{8}(t+t')^2$

Here total time of flight is $T=t+t'$

Now if the initial velocity of projection be u then we have,

$T=\frac{2u}{g}$ .. (1). Now the greatest height attained is given by

$$H=\frac{u^2}{2g} \quad \cdots \quad (2)$$

From (1) and (2) (eliminating u) we have

$$H=\left(\frac{gT}{2}\right)^2 \frac{1}{2g}=\frac{T^2g}{8}=\frac{g(t+t')^2}{8}.$$

7. A stone is dropped into a well and the sound of its striking the water is heard in $2\frac{33}{56}$ secs. If the velocity of sound be 1120 ft/sec, find the depth of the well. [C. U. 1932]

Let the depth of the well be h; then the time taken by the stone to strike the water is given by,

$h=\frac{1}{2}gt^2=16t^2\cdots(1)$ .taking $g=32$ ft/sec^2)

Again the time t', taken by sound to travel the whole depth is given by, $h=1120\,t'$ ..(2).

Since $t+t'=2\frac{33}{56}$, we have [from (1) and (2)]

$$\tfrac{1}{2}gt^2=1120(2\tfrac{33}{56}-t)$$

$$=1120\times\frac{(145-56t)}{56}$$

or, $16t^2=20\times(145-56t)$,

or, $4t^2+280t-725=0$, or, $(2t+145)(2t-5)=0$

$\therefore$ $t=-\frac{145}{2}$ secs, or, $5/2$ secs,

Neglecting the negative value we have,

$t=5/2$ secs.

$\therefore$ $h=16t^2=16\times\frac{25}{4}$ ft. $=100$ ft.

8. If a stone dropped freely strikes the water in a well with a velocity of 112 ft/sec. and the splash is heard in $3\frac{5}{8}$ secs., find the velocity of sound.

The time taken by the stone to strike the water is given by,

$$y = gt.$$

$$\therefore \text{ here } 112 = 32t$$

$$\therefore t = \frac{112}{32} \text{ sees.} = 3{\cdot}5 \text{ secs.} = 3\tfrac{1}{2} \text{ secs.}$$

The splash is heard after $3\frac{2}{3}$ secs.

Hence the time taken by sound to travel the depth of the well is $(3\frac{2}{3} - 3\frac{1}{2})$ sec. $= \frac{1}{6}$ secs.

The depth of the well is given by

$$h = \tfrac{1}{2}gt^2 = 16 \times (\tfrac{7}{2})^2 \text{ ft.} = 196 \text{ ft.}$$

$\therefore$ the velocity of sound is given by $u \times \frac{1}{6} = 196$.

$$\therefore u = 1176 \text{ ft./sec.}$$

9. If a bomb dropped from an aeroplane rising vertically with uniform velocity reaches the ground in 5 secs., find the height of the aeroplane when the bomb reaches the ground.

Let the velocity of the aeroplane be v ft./sec., then the initial velocity of the bomb is v ft/sec. upwards If h be the height of the plane at that instant, then

$$h = -v5 + \tfrac{1}{2}g5^2 = -5v + 400$$

Now when the bomb reaches the ground the aeroplane gains another $5v$. ft. Hence the height of the plane when the bomb reaches the ground is,

$$h + 5v = 16 \times 25 \text{ft} = 400 \text{ ft.}$$

10. A balloon has been ascending vertically from the ground with a uniform velocity for 6 seconds, when a stone is dropped from ft. It is found to reach the ground in 10 secs Find the velocity of the balloon and its height when the stone is dropped.

Let the velocity of the balloon be v ft./seç. When the stone was dropped its initial velocity was v ft./sec. upwards.

The time to reach the ground is given by,

$$h = -vt + \tfrac{1}{2} gt^2 \qquad \ldots \qquad (1)$$

where h is the height of the balloon at the instant when the stone was dropped.

Also the balloon had been ascending from the ground for 6 secs.

Hence $h = v \times 6 = 6v$ ... (2)

$\because$ the stone takes 10 secs to reach the ground

from (1) putting $t = 10$ we have

$h = -10v + 16 \times 10^2 = -10v + 1600$ ... (3)

From (2) and (3) we have

$6v = -10v + 1600$, or, $16v = 1600$

$\therefore$ $v = 100$ and $h = 6 \times 100 = 600$ ft.

Hence the velocity of the balloon was 100 ft./sec and its height was 600 ft.

11. A man is standing on a platform which descends with a uniform acceleratation 5ft/sec^2. After having descended for 2 secs. he drops a ball ; what will be the velocity of the ball after 2 more seconds.

After 2 seconds the velocity of the platform is

$$u = ft = 5 \times 2 = 10 \text{ ft/sec.}$$

$\therefore$ the ball falls with an initial velocity of 10ft/sec downwards. After 2 more seconds the velocity of the ball is given by,

$$v = u + gt = 10 + 32 \times 2 = 74\text{ft/sec.}$$

12. From an aeroplane rising vertically with uniform acceleration f, a ball is dropped ; 4 secs. after this another ball is dropped from the aeroplane. Show that the distance between the two balls 2 secs. after the second ball is dropped is $16\,(g+f)$,

Let the initial velocity of the first ball be u (upwards). Then after 4 secs the velocity of the aeroplane will be $u + 4f$, hence the initial velocity of the 2nd ball is $u + 4f$ upwards.

In 6 secs. the distance traversed by the first ball in given by,

$$h = -6u + \tfrac{1}{2}g \times 6^2 = -6u + 18g.$$

In 2 secs. the distance traversed by the 2nd, ball is given by

$$h' = -2(u + 4f) + \tfrac{1}{2}g \times 2^2 = -2u - 8f + 2g.$$

But the 2nd. ball is dropped from a height higher than that of the first ball by an amount given by (it is the distance through which the aeroplane rises in 4 secs.)

$$h_0 = u \times 4 + \tfrac{1}{2}f \times 4^2 = 4u + 8f$$

$\therefore$ the distance between the two balls 2 secs. after the second ball is dropped is,

$$h + h_0 - h' = -6u + 18g + 4u + 8f - (-2u - 8f + 2g)$$
$$= 16g + 16f = 16(g+f).$$

13. A particle is projected upwards with a velocity u; n seconds afterwards another particle is projected upwards with a velocity v from the same point; if the particles meet at the highest point reached by the first, show that $v-u=g^2n^2/2(u-ng)$.

the greatest height attained by the first particle is given by,

$$h=\frac{u^2}{2g} \text{ and the time is } t=\frac{u}{g}.$$

The 2nd. particle is thrown after n secs. and hence, by the problem, it attains a height h in $(t-n)$ secs.

$$\therefore \quad h=v.(t-n)-\tfrac{1}{2}g(t-n)^2$$

$$or, \quad \frac{u^2}{2g}=v.\left(\frac{u}{g}-n\right)-\tfrac{1}{2}g\left(\frac{u}{g}-n\right)^2$$

$$or, \quad u^2=2v(u-ng)-(u-ng)^2=(u-ng)(2v-u+ng)$$

$$=(u-ng)(2v-2u)+(u-ng)(u+ng)$$

$$or, \quad u^2=2(v-u)(u-ng)+u^2-g^2n^2$$

$$or, \quad 2(v-u)(u-ng)=g^2n^2$$

$$\therefore \quad v-u=\frac{g^2n^2}{2(u-ng)}$$

14. If a particle takes t seconds less time and acquires a velociety v ft/sec more at one place on the earth's surface than at another in falling freely through the same height, show that the geometric mean of the numerical values of g at the two places is v/t.

Let the values of g at the two places be g_1 and g_2 respectively. Let the particle take t_1 sceonds and acquire a velocity v_1 ft/sec. In falling freely through a height h in the first place and take t_2 secs. and acquire a velocity v_2 ft/sec in falling through the same height in the second place.

By the problem, $v_1-v_2=v$ and $t_1=t_2-t$

Now we have

$$v_1=g_1t_1 \quad \ldots \quad (1)$$

$$v_2=g_2t_2=g_2(t_1+t) \quad \ldots \quad (2)$$

$$\therefore \quad v_1-v_2=t_1(g_1-g_2)-g_2t$$

$$or, \quad v=t_1(g_1-g_2)-g_2t \quad \ldots \quad (3)$$

$$\text{Also, } h=\tfrac{1}{2}g_1t_1^2=\tfrac{1}{2}g_2t_2^2$$

$\therefore \quad g_1 t_1^2 = g_2 t_2^2 = g_2(t+t_1)^2 = g_2 t_1^2 + 2g_2 t_1 t + g_2 t^2$

$or, \quad t_1^2 (g_1 \quad g_2) = g_2 t t_1 + g_2 t^2 \cdots (4)$

From (3) we get $v t_1 = t_1^2 (g_1 - g_2) - g_2 t t_1$

$= 2g_2 t t_1 + g_2 t^2 - g_2 t t_1 = g_2 t t_1 + g_2 t^2$ [by (4)]

$or, \quad t_1(v - g_2 t) = g_2 t^2$

$$\therefore \quad t_1 = \frac{g_2 t^2}{v - g_2 t}$$

Putting this value of t_1 in (3) we get,

$$v = \frac{(g_1 - g_2) g_2 t^2}{(v - g_2 t)} - g_2 t = \frac{g_1 g_2 t^2 - v g_2 t}{(v - g_2 t)}.$$

$or, \quad v^2 - v g_2 t = g_1 g_2 t^2 - v g_2 t, \quad or, \quad v^2 = g_1 g_2 t^2$

$\therefore \quad g_1 g_2 = v^2 / t^2.$

Hence the geometric mean of the numerical values of g is given by, $\sqrt{g_1 g_2} = \dfrac{v}{t}$.

Exercises on Chapter VI

1. A ball is projected vertically upwards with a velocity sufficient to attain a height of 200 ft. Find its initial velocity.

2. A ball is dropped from a height of 50 ft. Find its velocity when it is half way down.

3. A stone is projected vertically upwards from the top of a tower with a velocity of 64 ft./sec and reaches the foot of the tower in 6 secs. ; find the height of the tower.

4. A body falling freely under gravity describes 24·5 metres in a certain second. Find the total distance described by the body.

5. A particle falling from rest under gravity passes a cartain point with a velocity of 120 ft./sec From what height above the point did the body begin to fall ?

6. A cricket ball is thrown vertically upwards ; find through what distance it goes in the last half second of its ascent.

7. A body falling freely from the top of a tower is observed to pass through $\frac{8}{9}$ths of the height of the tower in the last second of its motion. Find the height of the tower.

8 A ball thrown up is caught by the thrower 7 secs. afterwards. How high did it go ? Find the velocity with which it was thrown.

9. A stone is dropped from a height of 200 ft above the ground. At the same time a ball is projected upwards from the ground with a velocity of 80 ft./sec. in the same vertical line. Show that they will meet midway and find the time of meeting.

10. A particle thrown vertically upwards takes t secs. to rise to a height h and t' secs. is the subsequent time to reach the ground again. Show that, $h=\frac{1}{2}gtt'$.

[C. U. 1963]

11. A body is dropped from a height of 384 ft. and after 4 secs another body is projected vertically upwards from the ground with a velocity of 128 ft./sec. Find when and where they will meet.

12. A stone falling from the top of a vertical tower has descended x ft. when another is let fall from a point y ft. below the top. If they fall from rest and reach the ground together, show that the height of the tower is

$$\frac{(x+y)^2}{4x} \text{ ft}$$

[C. U. 1935]

13. A stone dropped into a well and the sound of the splash is heard in $3\frac{9}{70}$ secs. ; if the velocity of sound is 1120 ft./sec., find the depth of the well

14. A stone dropped into a well reaches the water with a velocity of 80 ft./sec and the sound of its striking the water is heard in $2\frac{7}{12}$ secs after it is let fall. Find the velocity of sound.

15. If a body after having fallen for 3 secs. breaks a piece of glass and thereby loses one-third of its velocity, find the entire distance through which it will have fallen in 4 secs.

16. A stone dropped into an empty pit of depth h is heard to strike the bottom after time t. Prove that $2h(1+gt/v)=gt^2$, where v is the speed of sound supposed so large compared with h that $(h/v)^2$ may be ignored.

[C. U. 1967]

17. The distance described by a freely falling particle in the last second of its motion is to that described in the last but one second as 3 : 2. Find the height from which the particle drops.

18. A particle is projected vertically upwards with a velocity of u ft./sec, and after t secs, another particle is projected upwards from the same point with the same initial velocity.

Prove that they will meet at a height of

$$\frac{4u^2-g^2t^2}{8g} \text{ ft. after } \left(\frac{t}{2}+\frac{u}{g}\right) \text{ secs}$$

19. From a balloon which a ascending with a velocity of 32 ft./sec. a stone is let fall. It reaches the ground in 17 secs. How high was the balloon when the stone was dropped ?

20. A ballon is ascending vertically and at a height of 1500 ft. a stone is released. If the stone reaches the ground in 10 seconds, find the height through which the stone rise immediately after the release.

21. A man in a lift, ascending with an acceleration f ft./sec^2 throws a ball vertically upwards with a velocity v ft./sec, relatively to the lift, and catches it again in t secs. ; show that $f+g=2v/t$. [C. U. 1964]

22. Three bodies are simultaneously projected vertically downwards from heights h_1, h_2, h_3 with velocities v_1, v_2, v_3 respectively, and they all reach the ground at the same moment.

Show that $\dfrac{h_1-h_2}{v_1-v_2}=\dfrac{h_2-h_3}{v_2-v_3}=\dfrac{h_3-h_1}{v_3-v_1}$

CHAPTER VII

PROJECTILE

§ 7·1. In the previous chapter we discussed rectilinear motion of a particle under gravity. There we investigated the motion of a body projected vertically upwards or downwards. In this chapter we shall consider free motion of a particle projected in any direction in space. It will be assumed that the only force acting on the particle is gravity and we shall neglect air resistance etc. and consider the motion in vacuo. The motion will take place in a *vertical plane* containing the vertical through the point of projection and the initial direction of projection.

A particle projected in any direction in space is called a *projectile.* The path described by a projectile is called the *trajectory.*

The velocity with which a projectile is projected is defined to be the *velocity of projection* and the angle made by the direction of the velocity of projection with the *horizontal* is called the *angle of projection.*

N. B. Horizontal direction lies along the line lying in the plane of horizon through the point of projection.

§ 7·2 Motion of a projectile in vacuo.

Let a particle be projected from O with a velocity u at an angle α with the horizon.

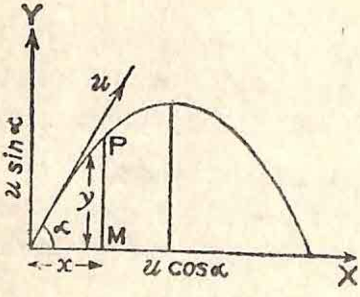


Fig. 44

Draw axes $\overrightarrow{OX}$, $\overrightarrow{OY}$ through O (take $\overrightarrow{OX}$ along the horizontal direction)

Let $P(x, y)$ be the position of the particle at any time t referred to the axes $\overrightarrow{OX}$, $\overrightarrow{OY}$.

Draw $\overline{PM}$ perpendicular to $\overrightarrow{OX}$, then $OM = x$, $PM = y$.

Initial velocity of projection is $u \cos \alpha$ along $\overrightarrow{OX}$ and $u \sin \alpha$ along $\overrightarrow{OY}$.

Now gravity acts vertically downwards and it has no component along the horizontal direction.

Hence the horizontal velocity is constant and we have

$$\frac{dx}{dt}=u\cos\alpha \quad \cdots \quad (1)$$

Integrating both sides we get, $x=u\cos\alpha\, t+c$ $\cdots(2)$

$\because$ $x=0$, at $t=0$; $\therefore$ $c=0$

$$\therefore \quad x=u\cos\alpha . t \quad \cdots \quad (3)$$

Now the acceleration due to gravity is g downwards $or -g$ upwards)

$\therefore$ for vertical upward motion,

$$\frac{d^2y}{dt^2}=-g \quad \cdots \quad (4)$$

Integrating both sides we get

$$\frac{dy}{dt}=-gt+c_1 \quad \cdots \quad (5)$$

Now, at $t=0$; $\frac{dy}{dt}$ = velocity along the vertical

$=u\sin\alpha$, $\quad\therefore \quad c_1=u\sin\alpha$.

$\therefore$ from (5) we get

$$\frac{dy}{dt}=u\sin\alpha-gt \quad \cdots \quad (6)$$

Integrating again, $y=u\sin\alpha\, t-\frac{1}{2}gt^2+c_2$ $\cdots$ (7)

Using the initial condition, at $t=o$, y is o we have, $c_2=o$.

$\therefore$ Equation (7) reduces to $y=u\sin\alpha . t-\frac{1}{2}gt^2$ $\cdots$ (8)

Now, the *greatest height* is obtained at A say, when the particle ceases to move upwards, i.e., when $\frac{dy}{dt}=o$

$\therefore$ from (6) we get $o=u\sin\alpha-gT_1$

or, $\quad T_1=\frac{u\sin\alpha}{g}\cdots(9)$ which gives

the *time to reach the greatest height.*

Putting the value of T_1 from (9) in (7) we get, the greatest height say H.

$$\therefore \quad H=u\sin\alpha.\frac{u\sin\alpha}{g}-\frac{1}{2}g\left(\frac{u\sin\alpha}{g}\right)^2$$

$$\text{or,} \quad H=\frac{u^2\sin^2\alpha}{2g} \quad \cdots \quad (10)$$

The particle will come back to the ground after a time which is obtained from (7) by putting $y=0$,

i.e., $0=u\sin\alpha.t-\frac{1}{2}gt^2$, or, $t(u\sin\alpha-\frac{1}{2}gt)=0$

solving it we have $t=0$ or, $t=\frac{2u\sin\alpha}{g}$.

Here $t=0$ gives the starting time and

$$t=\frac{2u\sin\alpha}{g}=T \text{ (say)} \quad \cdots \quad (11)$$

gives the total time of flight. Comparing (9) and (11)

we have $T=2\times T_1$

i.e., The total time of flight is twice the time to reach the greatest height. Hence the time to reaeh the greatest height is equal to the time to reach the ground from the greatest height (or maximum height).

Again putting $y=0$ ih equation (8) we get

$$0=u\sin\alpha.t-\tfrac{1}{2}gt^2$$

$$\therefore\quad t=0, \quad \text{or,} \quad t=\frac{2u\sin\alpha}{g}.$$

Here $t=0$ gives the initial time and $t=\frac{2u\sin\alpha}{g}$ is the time when the particle strikes the ground.

Putting this value of t in equation (3) we get

$$x=u\cos\alpha\times\frac{2u\sin\alpha}{g}$$

$$=\frac{u^2\sin 2\alpha}{g}.$$

Hence the *range* is given by

$$R=\frac{u^2\sin 2\alpha}{g}.$$

N. B. The range is maximum when $\sin 2\alpha$ is maximum, i.e. $2\alpha=90^\circ$, or, $\alpha=45^\circ$.

§ 7·3. To find the velocity of the particle at any point.

Let u be the velocity at any point $P(x, y)$ making an angle θ with the horizon. Then $v\cos\theta$ = horizontal component of

the velocity(=const)$=u\cos\alpha$ and the vertical component of the velocity is $v\sin\theta$ which is given by

$$v^2\sin^2\theta=u^2\sin^2\alpha-2gh \text{ (here } h=y).$$

Hence $v^2=v^2\cos^2\theta+v^2\sin^2\theta$

$$=u^2\cos^2\alpha+u^2\sin^2\alpha-2gh=u^2(\sin^2\alpha+\cos^2\alpha)-2gh$$

$=u^2-2gh$ and the angle θ is given by $\tan\theta=\dfrac{v\sin\theta}{v\cos\theta}$

$$=\pm\frac{\sqrt{u^2\sin^2\alpha-2gh}}{u\cos\alpha}$$

the + sign giving the velocity while ascending and the —sign the velocity while descending

§ 7·4. The path of a projectile in vacuo is a parabola.

From equations (3) and (7) of § 7·2 we get

$$x=u\cos\alpha.t \qquad \dots \qquad (1)$$

$$y=u\sin\alpha.t-\tfrac{1}{2}gt^2 \qquad \dots \qquad (2)$$

Eliminating t from (1) and (2)

we get $y=u\sin\alpha.\dfrac{x}{u\cos\alpha}-\dfrac{1}{2}g\left(\dfrac{x}{u\cos\alpha}\right)^2$

$$=x\tan\alpha-\frac{1}{2}.\frac{gx^2}{u^2\cos^2\alpha} \qquad \dots \qquad (3)$$

Equation (3) can be written in the form

$$y=Ax+Bx^2 \qquad \dots \qquad (4)$$

where $A=\tan\alpha$, $B=-\dfrac{g}{2u^2\cos^2\alpha}$.

Equation (4) represents a parabola.

Hence the path of a projectile in vacuo in a parabola.

Alternative proof :

Let a particle be projected from a point O with a velocity u at an angle α with the horizon. Assume that the particle reaches a maximum height and let A be the highest point of the path described by the particle. Let P be the position of the particle at any instant t from A.

Let $\overrightarrow{XL}$ be the vertical drawn through A. Draw perpendicular $\overrightarrow{\overline{PN}}$ from P upon AL.

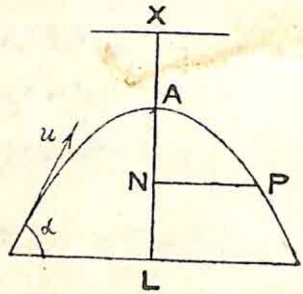


Fig. 45

PN = horizontal displacement of the particle

Since the horizontal velocity is constant and equal to $u \cos \alpha$,

$$\therefore \quad PN = u \cos \alpha \; t$$

AN = vertical displacement from A in time $t = \frac{1}{2}gt^2$

$$\therefore \quad \frac{PN^2}{AN} = \frac{(u \cos \alpha \; t)^2}{\frac{1}{2}gt^2} = \frac{2u^2 \cos^2 \alpha}{g}$$

= a constant independent of t.

Hence from the definition of a parabola, the locus of P is a parabola whose vertex is at A and whose axis is $\overrightarrow{AL}$ and latus rectum is given by $4a = \frac{2u^2 \cos^2 \alpha}{g}$.

§ 75. **Motion of a particle projected horizontally from a point at any height above the ground.**

Let a particle be projected horizontally with a velocity u, from a point O at a height h above the ground

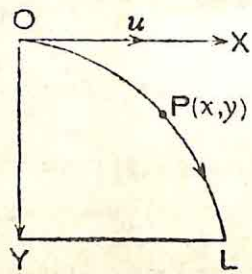


Fig. 46

Let $\overrightarrow{OY}$ be the vertical through O meeting the gronnd at Y. Here the horizontal velocity $= u =$ constant.

Initial vertical velocity $= 0$.

Hence if we take $\overrightarrow{OY}$ (downwards) as the Y-axis and the horizontal through O as the X-axis, then the position of P at any time is given by

$$x = u.t \qquad \ldots \qquad \text{(i)}$$

$$y = 0.t + \tfrac{1}{2}gt^2 = \tfrac{1}{2}gt^2 \qquad \ldots \qquad \text{(ii)}$$

Eliminating t from (i) and (ii), we get

$$y = \tfrac{1}{2}g.\frac{x^2}{u^2} = \frac{gx^2}{2u^2}.$$

The above equation represents a parabola whose latus rectum is given by $4a=\frac{g}{2u^2}$.

Also the time for the fall is given by $h=\frac{1}{2}gT^2$

$$\therefore\ T=\sqrt{\frac{2h}{g}}.$$

The particle falls on the ground at a point L whose distance from the ground is given by $YL=u.\ T=u\sqrt{\frac{2h}{g}}$.

§ 7·6. Motion on an inclined plane.

Let OP be an inclined plane inclined at an angle β to the horizontal.

Let a particle be projected with a velo city u at angle α to the horizon in the vertical plane through a line of greatest slope. Let the path of the projectile meet the plane at P.

OP=R (say) is the range on the plane.

Resolving along and perpendicular to the line of greatest slope, the initial velocity along the plane is $u\cos(\alpha-\beta)$.

and along the perpendicular to the plane is $u\sin(\alpha-\beta)$. Also the components of g along and perpendicular to the plane are $-g\sin\beta$ and $-g\cos\beta$ respectively.

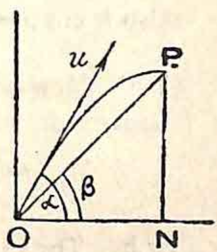


Fig. 46

∴ the displacement perpendicular to the plane is

$$y=u\sin(\alpha-\beta)\,t-\tfrac{1}{2}g\cos\beta\,t^2 \quad \ldots \quad (1)$$

(use $s=ut+\frac{1}{2}ft^2$)

When the particle strikes the plane its displacement perpendicular to the plane is zero and hence from (1), the time of flight T is given by

$$0=u\sin(\alpha-\beta)T-\tfrac{1}{2}g\cos\beta T^2 \quad \ldots \quad (2)$$

$$\text{or,}\quad T=\frac{2u\sin(\alpha-\beta)}{g\cos\beta} \quad \ldots \quad (3)$$

Similarly the displacement along the plane is given by

$$x = u \cos(\alpha - \beta)t - \tfrac{1}{2} g \sin \beta t^2 \quad \cdots \quad (4)$$

[$\because$ the component of g along the plane (upwards) is $-g \sin \beta$]

$\therefore$ the range R on the plane is obtained by putting

$$t = T \text{ in } (4)$$

$$\therefore \quad R = u \cos(\alpha - \beta) \cdot \frac{2u \sin(\alpha - \beta)}{g \cos \beta} - \frac{1}{2} g \sin \beta \frac{4u^2 \sin^2(\alpha - \beta)}{g^2 \cos^2 \beta}$$

$$= \frac{2u^2 \sin(\alpha - \beta)}{g \cos^2 \beta} [\cos(\alpha - \beta) \cos \beta - \sin(\alpha - \beta) \sin \beta]$$

$$= \frac{2u^2 \sin(\alpha - \beta) \cos \alpha}{g \cos^2 \beta} = \frac{u^2}{g \cos^2 \beta} [\sin(2\alpha - \beta) - \sin \beta]$$

(Alt. method) :

$\because$ the horizontal velocity = constant = $u \cos \alpha$,

$\therefore$ we have ON $= u \cos \alpha . T$

Also R $\cos \beta =$ ON $= u \cos \alpha . T$

$$\therefore \quad R = \frac{u \cos \alpha}{\cos \beta} \cdot T = \frac{2u^2 \cos \alpha \sin(\alpha - \beta)}{g \cos^2 \beta}$$

$$= \frac{u^2}{g \cos^2 \beta} [\sin(2\alpha - \beta) - \sin \beta]$$

N. B. The range is maximum,

when $\sin(2\alpha - \beta)$ is maximum, i.e., when

$$\sin(2\alpha - \beta) = 1 = \sin \frac{\pi}{2} \quad \text{or,} \quad 2\alpha - \beta = \frac{\pi}{2}; \text{ or,} \quad \alpha = \tfrac{1}{2}\left(\frac{\pi}{2} + \beta\right).$$

Maximum value of $R = \dfrac{u^2}{g \cos^2 \beta}(1 - \sin \beta) = \dfrac{u^2}{g(1 + \sin \beta)}$

§ 7. Motion down an inclined plane.

Let a particle be projected from the point P down an inclined plane and also let the velocity of projection be u at angle α to the horizontal. Here the component of g along the plane (downwards) is $g \sin \beta$ and the components of velocity of

projection along and perpendicular to the plane are $u \cos(\alpha+\beta)$. and $u \sin(\alpha+\beta)$ respectively. Following the same procedure as in § 7·6, we get for T′, the time after which the particle is again on the plane,

$$0 = u \sin(\alpha+\beta)T' - \tfrac{1}{2} g \cos\beta \, T'^2$$

$$\text{or,} \quad T' = \frac{2u \sin(\alpha+\beta)}{g \cos\beta}.$$

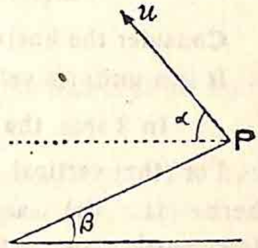


Fig. 41

Similarly the range is given by

$$R' = \frac{2u^2}{g \cos^2\beta} \cos\alpha \sin(\alpha+\beta)$$

$$= \frac{u^2}{g \cos^2\beta}\left[\sin(2\alpha+\beta) + \sin\beta\right]$$

N. B. To get the formulas for T′ and R′ from T and R respectively (see § 7·6) replace β by $-\beta$. Maximum value of

$$R' = \frac{u^2}{g \cos^2\beta}(1+\sin\beta) = \frac{u^2}{g(1-\sin\beta)} \text{ when } \alpha = 45^\circ - \frac{\beta}{2}.$$

Worked out Examples.

1. A particle is projected from the ground at an angle of 45° with a velocity of 60 ft /sec. Find the range and the greatest height reached by the particle.

Here $u = 60$ ft./sec.; $\alpha = 45^\circ$ Taking $g = 32$ ft./sec^2

we have $R = \dfrac{u^2 \sin 2\alpha}{g} = \dfrac{60 \times 60 \times \sin 90^\circ}{32}$

$$= \frac{3600}{32} \text{ ft.} = 112{\cdot}5 \text{ ft.}$$

and the greatest height is given by

$$H = \frac{u^2 \sin^2\alpha}{2g} = \frac{60 \times 60 \times (\sin 45^\circ)^2}{2 \times 32} \text{ ft.}$$

$$= \frac{3600 \times \frac{1}{2}}{64} \text{ft.} = 28{\cdot}125 \text{ ft}$$

2. A ball is projected from the ground at an angle of $\sin^{-1}\frac{4}{5}$ to the horizon with a velocity of 30m/sec. Find its position and its velocity at the end of 2 secs.

Here $\alpha = \sin^{-1}\frac{4}{5}$, $\quad \therefore \sin\alpha = \frac{4}{5}$

$\therefore$ initial horizontal velocity of the ball

$$=30\times\cos\alpha=30\sqrt{1-\sin^2\alpha}=30.\tfrac{3}{5}$$
$$=18 \text{ m/sec.}$$

Initial vertical velocity of the ball

$$=30\times\sin\alpha=30\times\tfrac{4}{5}=24 \text{ m/sec.}$$

Consider the horizontal motion.

It is a uniform velocity of 18m/sec.

$\therefore$ In 2 secs. the horizontal displacement is 18m. $\times 2=36$ m.

For the vertical motion, the initial velocity is 24m/sec and there is the acceleration due to gravity $g=980\text{cm/sec}^2$ downwards, or, $-9\cdot80\text{m/sec}^2$ upwards.

$\therefore$ the vertical displacement in 2 secs. is given by

$$y=24\times2-\tfrac{1}{2}\times9\cdot8\times2^2$$
$$=(48-19\cdot6)\text{m.}=28\cdot4\text{m.}$$

(use the formula $s=ut+\frac{1}{2}ft^2$)

also its vertical velocity is then given by

$$v_y=24-9\cdot8\times2 \quad =24-19\cdot6=4\cdot4\text{m/sec}$$

$\therefore$ the magnitude of its velocity is,

$$v=\sqrt{(4\cdot4)^2+18^2}=18\cdot53\text{m/sec.}$$

(since the horizontal velocity is always 18m/sec.)

and the velocity then makes an angle θ with the horizontal where θ is given by

$$\tan\theta=\frac{v_v}{u\cos\alpha}=\frac{4\cdot4}{18}=\frac{11}{45}, \quad \therefore \quad \theta=\tan^{-1}\frac{11}{45}$$

N. B. Here $v\cos\theta=u\cos\alpha$ and $v\sin\theta=v_y$.

3. A stone is dropped from a balloon moving horizontally with a velocity 96 ft./sec. and reaches the ground in 4 seconds. Find the height of the balloon and the velocity of the stone on striking the ground.

Here the initial velocity of the stone is the velocity of the balloon, which is 96 ft./sec. horizontally.

$\therefore$ here $\alpha=0$, $\therefore$ initial vertical velocity is nil.

Hence for vertical motion the distance covered in 4 secs. is

$h=\frac{1}{2}gt^2=\frac{1}{2}\times32\times4^2$ ft. $=256$ ft. which is the height of

the balloon. The vertical velocity on striking the ground is given by

$$v_v = gt = 32 \times 4 = 128 \text{ ft /sec.}$$

the horizontal velocity = 96 ft./sec.

$$\therefore \quad v = \sqrt{v_v^2 + 96^2} = \sqrt{128^2 + 96^2} = 160 \text{ ft./sec.}$$

$$\text{and } \tan\theta = \frac{v_v}{96} = \frac{128}{96} = \frac{4}{3}$$

$\therefore$ its velocity on striking the ground is 160 ft/sec. at an angle of $\tan^{-1}\frac{4}{3}$ with the ground

4. A cricket ball, thrown by a man from a height of 7 ft. at an angle of 30° with the horizon, with a speed of 60 ft/sec., is caught by another fieldsman at a height of 3 ft. from the ground. How far apart were the two men ?

The horizontal velocity of the ball is $60 \cos 30^\circ = 30\sqrt{3}$ ft/sec.

The initial vertical velocity of the ball is

$$60 \sin 30^\circ = \frac{60}{2} \text{ or } 30 \text{ ft/sec upwards.}$$

Take the point of projection as origin and the vertical through the point of projection as the y-axis, the positive direction being downwards. The vertical displacement during the motion of the ball is $(7-3)$ ft. = 4ft. downwards. Since the initial vertical velocity is 30ft/sec upwards, so we have

$4 = -30.\ t + \frac{1}{2} gt^2 = -30.\ t + 16t^2$ (use $s = ut + \frac{1}{2} ft^2$ and $g = 32$ ft/sec^2),

or, $16t^2 - 30t - 4 = 0$.

or, $8t^2 - 15t - 2 = 0$,

or, $(t-2)(8t+1) = 0$ or $t = 2$ or $t = -\frac{1}{8}$

Neglecting the negative value, we get $t = 2$

$\therefore$ the motion takes place for 2 secs.

During these 2 secs. the horizontal displacement of the ball is $30\sqrt{3} \times 2$ ft. $= 60\sqrt{3}$ ft.

Hence the men were $60\sqrt{3}$ ft. apart.

5. A gun is fired at an elevation $\tan^{-1}\frac{1}{20}$ towards a person on the same horizontal plane as the gun. If the shot and the sound of the gun reach him at the same instant, find the range, the velocity of sound being 1120 ft/sec.

Here $\alpha=\tan^{-1}\frac{1}{20}$, $\therefore$ $\tan\alpha=\frac{1}{20}$.

Now if T be the time taken by the shot to reach the man, then $T=\dfrac{2u\sin\alpha}{g}$ ...(1) where u is the velocity of projection.

Also the range is given by

$R=1120T$ ...(2) and $R=u\cos\alpha.T$ ... (3)

From (2) and (3) we have

$1120T=u\cos\alpha.T$. $\therefore$ $u\cos\alpha=1120$ ... (4)

Again,

$$R=1120T=1120\times\frac{2u\sin\alpha}{g}\text{[from (1)]}$$

$$=\frac{1120\times2}{32}\times u\cos\alpha\tan\alpha$$

$$=\frac{1120\times2\times1120}{32}\times\tan\alpha\text{ [from (4)]}$$

$$=\frac{1120\times2\times1120}{32}\text{ ft.}\times\frac{1}{20}=3920\text{ ft.}$$

6. A particle, thrown horizontally from a height of 19·6 metres from the ground, reaches the ground at a horizontal distance of 100 metres Find the velocity of projection ($g=980$ cm/sec^2).

[C. U. 1957]

Let u be the horizontal velocity of projection. The initial vertical velocity is zero. Considering the vertical motion, we have $19{\cdot}6=v_yt+\frac{1}{2}gt^2$, where t is the time to reach the ground,

or, $19{\cdot}6\text{m.}=\frac{1}{2}\times9{\cdot}80\text{m.}\times t^2$ ($\because$ $v_y=0$)

or, $t^2=\dfrac{2\times19{\cdot}6}{9{\cdot}80}=\dfrac{2\times196}{98}=4$, $\therefore$ $t=2$ secs.

Since the horizontal velocity is constant.

$\therefore$ the horizontal displacement is $2u=100$

$\therefore$ $u=50$ metres/sec.

7. A particle is projected with a velocity u at an angle of elevation α from a point on a horizontal line. If R be the range, T be the time of flight, and H the maximum height attained by the particle, prove that

$g^2T^4-4T^2u^2+4R^2=0$ and $16gH^2-8Hu^2+gR^2=0$. [C. U. '45]

We have, $H = u^2 \sin^2\alpha/2g$ ... (i)

$R = u^2 \sin 2\alpha/g$ ... (ii)

$T = 2u \sin\alpha/g$ ... (iii)

From (ii) we get

$$R^2 = (u^2.2 \sin\alpha \cos\alpha/g)^2$$

$$= \frac{4u^4 \sin^2\alpha \cos^2\alpha}{g^2} = \frac{4u^4 \sin^2\alpha(1-\sin^2\alpha)}{g^2} \quad \text{...(iv)}$$

From (iii) we have, $T^2 = 4u^2 \sin^2\alpha/g^2$

$$\sin^2\alpha = \frac{T^2 g^2}{4u^2} \quad \text{... (v)}$$

Eliminating $\sin^2\alpha$ from (iv) and (v) we have

$$R^2 = \frac{4u^4.T^2 g^2}{g^2\, 4u^2} \cdot \left(1 - \frac{T^2 g^2}{4u^2}\right)$$

$$= u^2 T^2 \left(\frac{4u^2 - T^2 g^2}{4u^2}\right) = T^2 . \frac{(4u^2 - T^2 g^2)}{4}$$

or, $g^2 T^4 - 4u^2 T^2 + 4R^2 = 0.$

To get the second relation, we eliminate α from (i) and (ii)

Since (i) gives $\sin^2\alpha = \dfrac{2Hg}{u^2}$,

we get from (iv)

$$R^2 = \frac{4u^4}{g^2}\,\frac{2Hg}{u^2}\left(1 - \frac{2Hg}{u^2}\right) = \frac{8u^2\,H}{g}\left(\frac{u^2 - 2Hg}{u^2}\right)$$

$$= \frac{8H}{g}(u^2 - 2Hg)$$

or, $16gH^2 - 8Hu^2 + gR^2 = 0.$

8. A body is projected so that on its upward path it passes through a point x ft. horizontally and y ft. vertically from the point of projection. If R ft. be the range on the horizontal plane through the point of projection, show that the angle of projection is $\tan^{-1}\left(\dfrac{y}{x} . \dfrac{R}{R-x}\right)$. [C. U. 1944]

The path of the parabola (see § 7·4), is given by

$$y = x \tan \alpha - \frac{gx^2}{2u^2 \cos^2 \alpha} \quad \cdots \quad \text{(i)}$$

Also we have $R = \dfrac{2u^2 \cos \alpha \sin \alpha}{g} \cdots$ (ii)

Eliminating u from (i) and (ii), we have,

$$y = x \tan \alpha - \frac{gx^2}{R\, g \cot \alpha}$$

$$(\because \ 2u^2 \cos^2 \alpha = 2u^2 \cos \alpha . \sin \alpha \cot \alpha = \text{etc})$$

or, $y = x \tan \alpha - \dfrac{x^2 \tan \alpha}{R} = x \tan \alpha \left(1 - \dfrac{x}{R}\right)$

$$\therefore \ \tan \alpha = \frac{y}{x\left(1 - \frac{x}{R}\right)} = \frac{yR}{x(R - x)}.$$

$$\therefore \ \alpha = \tan^{-1}\left\{\frac{yR}{x(R - x)}\right\}.$$

9. The maximum range of a rifle bullet is 1200 yds. If the rifle is fired with the same elevation from a truck running at 15 m. p. h towards a target, prove that the range is increased by 110 yds. [C. U. 1963]

Let u be the velocity of projection of the bullet making an angle α with the horizontal.

Then the horizontal range is given by, $R = \dfrac{u^2 \sin 2\alpha}{g}$.

R is maximum when $\sin 2\alpha = 1$ or $\alpha = 45°$,

then $R = \dfrac{u^2}{g} = 1200 \times 3$ ft. (by the problem)

$$\therefore \ u^2 = 1200 \times 3 \times 32$$

$$\therefore \ u = \sqrt{96 \times 1200} = 240\sqrt{2} \text{ ft./sec.}$$

$\therefore$ When the bullet is fired from an elevation of 45° its horizontal velocity is

$$240\sqrt{2} \times \cos 45° = 240\sqrt{2} \times \frac{1}{\sqrt{2}} = 240 \text{ ft./sec.}$$

If it is fired from a truck moving towards the target with a velocity of 15 m p. h. or 22 ft./sec, then its horizontal velocity

is increased to (240+22) or 262 ft /sec., while its vertical velocity remains the same i.e., $240\sqrt{2}\sin 45^\circ = 240$ ft /sec. upwards.

The time of flight is

$$T = \frac{2\text{ (initial vertical velocity)}}{g} \qquad \left[\text{ use } T = \frac{2u\sin\alpha}{g}\right]$$

$$= \frac{2 \times 240}{32}\text{ secs.} = 15\text{ sec.}$$

$\therefore$ the new range is given by

$$R' = 262 \times 15\text{ ft.} = 3930\text{ ft.} = 1310\text{ yds.}$$

$\therefore$ the horizontal range is increased by

$$R' - R = (1310 - 1200)\text{ yds.} = 110\text{ yds.}$$

10. A particle is projected with velocity 64 ft./sec. at an angle of 45° with the horizontal. Find its range on a plane inclined at 30° to the horizontal and its time of flight when projected (i) up, (ii) down, the plane. Find also the greatest range on the inclined plane in the two cases.

For motion up the plane,

$$R = \frac{u^2}{g}\left[\frac{\sin(2\alpha - \beta) - \sin\beta}{\cos^2\beta}\right]$$

Here $\alpha = 45^\circ$, $\beta = 30^\circ$; $u = 64$ ft./sec.

$$\therefore \quad R = \frac{u^2}{g} \cdot \frac{\sin(90^\circ - 30^\circ) - \sin 30^\circ}{\cos^2 30^\circ}$$

$$= \frac{64 \times 64}{32} \cdot \frac{\left[\frac{\sqrt{3}}{2} - \frac{1}{2}\right]}{\left(\frac{\sqrt{3}}{2}\right)^2}$$

$$= \frac{128 \times (\sqrt{3} - 1)}{2 \times \frac{3}{4}} = \frac{256}{3}(\sqrt{3} - 1) = 62{\cdot}63\text{ ft.}$$

R is maximum when $\sin(2\alpha - \beta)$ is maximum

(assuming u and β to be fixed)

i.e , when $2\alpha - \beta = 90^\circ$ or $2\alpha = 90^\circ + \beta$ (when $\sin(2\alpha - \beta) = 1$)

Here $\beta = 30^\circ$, $\therefore$ R to be maximum $2\alpha = 90^\circ + 30^\circ = 120^\circ$

$\therefore$ $\alpha = 60^\circ$,

then $R_{max}=\frac{u^2}{g}\left[\frac{1-\sin 30^\circ}{\cos^2 30^\circ}\right]=\frac{64\times 64}{32}\times\frac{1}{2}\times\frac{4}{3}$ ft. $=85{\cdot}33.$ ft.

For motion down the plane,

$$R=\frac{u^2}{g}\,\frac{[\sin(2\alpha+\beta)+\sin\beta]}{\cos^2\beta}$$

Putting $\alpha=45^\circ$, $\beta=30^\circ$, $u=64$ and $g=32$

we get $R=\frac{256(\sqrt{3}+1)}{3}=233{\cdot}30$ ft. (nearly)

and $R_{max}=\frac{u^2}{g}\cdot\frac{(1+\sin\beta)}{\cos^2\beta}=256$ ft.

11. A ball is projected with a velocity of 28 ft./sec. up an inclined plane which passes through the point of projection and which is of elevation 30°. The ball strikes the plane at right angles. Find the range on the plane.

Let the angle of projection of the ball be α.

Now resolving along and perpendicular to the plane, the initial component of velocity of projection along the plane is $u\cos(\alpha-\beta)$ and that perpendicular to the plane is $u\sin(\alpha-\beta)$. The components of g are $-g\sin\beta$ and $-g\cos\beta$ along and perpendicular to the plane respectively.

For motion along the plane

$v_x=u\cos(\alpha-\beta)-g\sin\beta.t$ ... (i)

$x=u\cos(\alpha-\beta).\,t-\frac{1}{2}g\sin\beta.\,t^2$ ... (ii)

For motion perpendicular to the plane

$v_y=u\sin(\alpha-\beta)-g\cos\beta.\,t$ ... (iii)

$y=u\sin(\alpha-\beta)\;t-\frac{1}{2}g\cos\beta\,t^2$ ... (iv)

By the problem, $v_x=0$ when $y=0$

$\therefore$ from (i) and (iv) we respectively have

$0=u\cos(\alpha-\beta)-g\sin\beta.t$ ... (v)

$0=u\sin(\alpha-\beta)t-\frac{1}{2}g\cos\beta\,t^2$ ... (vi)

Now from (v), $t=\frac{u\cos(\alpha-\beta)}{g\sin\beta}$ ... (vii)

from (vi), $t=\frac{2u\sin(\alpha-\beta)}{g\cos\beta}$ ... (viii)

$$\frac{u\cos(\alpha-\beta)}{g\sin\beta}=\frac{2u\sin(\alpha-\beta)}{g\cos\beta}$$

or, $2\tan(\alpha-\beta)=\cot\beta=\cot 30^\circ=\sqrt{3}$

or, $\tan(\alpha-\beta)=\frac{\sqrt{3}}{2}$ ... (ix)

Now from (ii) and (vii) we have

$$x=u\cos(\alpha-\beta).\frac{u\cos(\alpha-\beta)}{g\sin\beta}-\frac{1}{2}g\sin\beta\frac{u^2\cos^2(\alpha-\beta)}{g^2\sin^2\beta}$$

$$=\frac{u^2\cos^2(\alpha-\beta)}{2g\sin\beta}=\frac{u^2\cos^2(\alpha-\beta)}{g}\qquad\left[\because\ \sin\beta=\tfrac{1}{2}\right]$$

But from (ix), $\tan(\alpha-\beta)=\frac{\sqrt{3}}{2}$ $\therefore$ $\sec^2(\alpha-\beta)=1+\frac{3}{4}=\frac{7}{4}$

$\therefore$ $\cos^2(\alpha-\beta)=\frac{4}{7}$. $\therefore$ $x=\frac{u^2}{g}\times\frac{4}{7}=\frac{28\times 28\times 4}{32\times 7}$

or, $x=14$ ft Hence the range on the plane is 14 ft.

12. The angular elevation of an enemy's position on a hill h feet high is β; show that, in order to shell it, the initial velocity of the projectile must not be less than $\sqrt{gh(1+\operatorname{cosec}\beta)}$. [C U. 1946]

Let u be the vel city of the projectile and α the angle of projection required to hit the enemy's position.

Range $AB=h\operatorname{cosec}\beta=\frac{2u^2\sin(\alpha-\beta)\cos\alpha}{g\cos^2\beta}$

$$\therefore\quad u^2=\frac{gh\operatorname{cosec}\beta.\cos^2\beta}{2\sin(\alpha-\beta)\cos\alpha}=\frac{gh\operatorname{cosec}\beta\cos^2\beta}{\sin(2\alpha-\beta)-\sin\beta}$$

Now u is minimum when $\sin(2\alpha-\beta)-\sin\beta$ is maximum, i.e., when $\sin(2\alpha-\beta)$ is maximum and $=1$

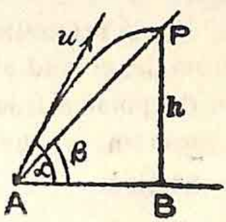


Fig. 47

$$\therefore\quad u^2_{min}=\frac{gh\operatorname{cosec}\beta\cos^2\beta}{1-\sin\beta}$$

$$=gh\operatorname{cosec}\beta\,(1+\sin\beta)$$

$$\left[\because\ \frac{\cos^2\beta}{1-\sin\beta}=\frac{1-\sin^2\beta}{1-\sin\beta}=1+\sin\beta\right]$$

$$\therefore\quad u_{min}=\sqrt{gh\operatorname{cosec}\beta\,(1+\sin\beta)}$$

$$=\sqrt{gh(1+\operatorname{cosec}\beta)}.$$

Exercise on Chapter VII

1. A ball is projected from a point with a velocity 100 ft/sec at an angle of 45° to the horizon. Find the greatest height and the total time of flight.

2. A boy can throw a ball 40 yds. vertically upwards. Prove that the greatest horizontal distance he can throw it is 240 ft.

3. A ball is projected from the ground at an elevation of $\cos^{-1}\frac{3}{5}$ with the horizontal with a velocity of 100 ft/sec. Find the distance of the ball from the point of projection after 2 seconds.

4. A stone is projected with a velocity 50ft/sec. at an angle of 30° to the horizontal. Find the equation of its trajectory. Also find out how long it will be in air.

5. Show that when the maximum horizontal range is 100 kilometres, the time of flight is about 2 minutes 23 seconds.

6. A football is kicked and just passes over a bar 12 ft high and 20 feet away. Find the direction in which the ball is kicked, if the velocity generated by the kick is 40 ft/sec.

7. A projectile thrown from a point in a horizontal plane comes back to the plane in 4 secs. at a distance of 64 yds, from the point of projection. Find the velocity and angle of projection.

8. Prove that if the time of flight of a bullet over a horizontal range R is T seconds, the inclination of the direction of projection to the horizontal is given by

$$\tan^{-1}\left(\frac{gT^2}{2R}\right)$$

9. A projectile fired horizontally from a height of 19·6 metres from the ground reaches the ground at a point A. If the distance of the point A from the foot of the perpendicular from the point of projection on the ground is 100 metres, find the velocity of projection.

10. A stone is dropped from a balloon moving horizontally with a velocity of 100·8 km. *p. h.* and reaches the ground in $5\frac{5}{7}$ secs. Find the height of the balloon, and the velocity of the stone on reaching the ground.

11. A ball is thrown from the top of a house 96 ft. high with a velocity of 80 ft./sec. at an elevation of 30°. Find when, where, and with what velocity it will strike the ground.

12. A particle is projected from the top of a tower with a velocity of 30 ft. sec. at an elevation of 30° and reaches the ground in 4 secs. Find the height of the tower.

13. A cricket ball thrown from a height of 6 feet at an angle of 30° with the horizon with a speed of 60 feet/sec, is caught by another fieldsman, at a height of 2 feet from the ground. How far apart were the two men ?

14. A bullet shot from the top of a tower 272 ft. high strikes the ground in 17 secs. at a distance of 4352 ft. from the foot of the tower. Find the velocity and angle of projection.

15. Prove that the equation to the path of a projectile may be written in the form $y = x \tan \alpha \, (R - x)/R$,
where R is the range on the horizontal plane through the point of projection and α is the angle of elevation of the projection.

[C. U. '62]

16. If h and h' be the greatest heights in the two paths of a projectile with a given velocity for a given range R, prove that

$$R = 4\sqrt{hh'}.$$

17. A body is projected at an angle α to the horizontal so as just to clear two walls of equal height a, at a distance $2a$ from each other. Show that the range is equal to $2a \cot \frac{}{2}$

18. A particle is projected with velocity 32 ft./sec. at an angle of 60° with the horizontal. Find its range on a plane inclined at 30° to the horizontal and its time of flight when projected (i) up, (ii) down the plane. Find also the greatest range on the inclined plane in the two cases.

19. A gun is fired from the sea level. It is then taken to a height h feet above the sea level and fired making the same angle α with the horizon. Show that the range is increased by

$$\frac{1}{2}\left[\left(1 + \frac{2gh}{u^2 \sin^2 \alpha}\right)^{\frac{1}{2}} - 1\right]$$

20. A ball is projected at an angle α to the horizontal up a plane which passes through the point of projection and is of elevation β. Show that it strikes the plane

(i) horizontally, if $\tan\alpha = 2\tan\beta$

(ii) normally, if $\tan\alpha = 2\tan\beta + \cot\beta$.

21. A particle is projected with an initial velocity u. If the greatest height attained by the particle be H, prove that the range R on the horizontal plane through the point of projection is

$$R = 4\sqrt{\left\{H\left(\frac{u^2}{2g} - H\right)\right\}}.$$ (C. U. 1940)

22. A fort is on the top of a hill of height h above sea level. Prove that the greatest horizontal distance at which a gun in a ship can hit the fort is $2\sqrt{k(k-h)}$, where $\sqrt{2gk}$ is the muzzle velocity of the shot. (C. U. 1964)

23. A particle is projected from the foot of a plane inclined at an angle of 30° to the horizon, with a velocity of 3·27 metres per sec at an angle of 60° with horizon. Find the velocity with which the particle hits the plane.

24. A particle projected with velocity V, strikes at right angles a plane through the point of projection inclined at angle β to the horizon. Show that the height of the point struck above the horizontal plane through the point of projection is $\dfrac{2V^2 \sin^2\beta}{g(1+3\sin^2\beta)}$ and the time of flight is $\dfrac{2V}{g\sqrt{1+3\sin^2\beta}}$

CHAPTER VIII

SIMPLE HARMONIC MOTION

§ 8·1. Simple harmonic motion is the simplest type of oscillatory motion and is of great importance in physical and mechanical problems.

Motion of the bob of a simple pendulum, the oscillatory motion of a particle attached at the extremity of an elastic string stretched along its length, are examples of simple harmonic motion.

Definition. If a point moves along a straight line in such a way that its acceleration is always directed towards a fixed point in the line and is proportional to its distance from that point the particle is said to execute a simple harmonic motion.

§ 8·2. Analytical treatment of Simple Harmonic Motion.

Let $\overleftrightarrow{XX'}$ be a straight line along which a particle moves in such a way that its acceleration is always directed towards O, a fixed point in the line and is proportional to its distance from O.

Let P be the position of the point at any instant t after start. Let $OP = x$. We take $x > 0$ when P is on the right side of O and $x < 0$ when P is on the left side of O. Now the acceleration f is proportional to OP and is directed towards O.

Fig. 48

Now, the distance of P from 0 is $|x|$. For, if the point P is on the right side of O, then $x > 0$ and so the distance is $|x|$ and if P is on the left side of O, then $x < 0$ and so the distance is $-x = |x|$.

$\therefore$ the magnitude of the acceleration varies as $|x|$ and is directed towards O.

or, the magnitude of $f = \mu^2 |x|$ and is towards O, where μ^2 is a constant.

If $x>0$, then $f=\mu^2 \mid x \mid$, towards O

$$= +\mu^2 x, \text{ along } \overrightarrow{PO} = -\mu^2 x, \text{ along } \overrightarrow{OX}.$$

If $x<0$, then $f=\mu^2 \mid x \mid$, towards O

$$= \mu^2 \mid x \mid \text{ along } \overrightarrow{P'O} \text{ (see fig.)}$$

$$= -\mu^2 x \text{ along } \overrightarrow{OX}. \ (\because \ \mid x \mid = -x \text{ when } x<0)$$

So in any case $f = -\mu^2 x$ ... (1)

But we know $f = \dfrac{d^2 x}{dt^2}$,

$\therefore$ Equation of motion of the particle is

$$\frac{d^2 x}{dt^2} = -\mu^2 x \quad \cdots \quad (2)$$

multiplying both sides of (2) by $2\dfrac{dx}{dt}$

we get

$$2\frac{dx}{dt}\frac{d^2 x}{d^2 t} = -2\mu^2 x\frac{dx}{dt}$$

$$\text{or, } \frac{d}{dt}\left(\frac{dx}{dt}\right)^2 = -\mu^2 \frac{d}{dt}(x^2) \quad \cdots \quad (3)$$

Integrating both sides of (3) with respect to t

$$\left(\frac{dx}{dt}\right)^2 = -\mu^2 x^2 + C \quad \cdots \quad (4)$$

Suppose the particle starts from rest from a position A at a distance a from O, then at $t=0, x=a, \dfrac{dx}{dt}=0$ $\therefore$ from (4) we have,

$$0 = -\mu^2 a^2 + C, \quad \therefore \quad C = \mu^2 a^2 \quad \ldots \quad (5)$$

$\therefore$ we have

$$\left(\frac{dx}{dt}\right)^2 = \mu^2(a^2 - x^2), \quad \text{or,} \quad \frac{dx}{dt} = \pm\mu\sqrt{a^2 - x^2} \quad \ldots \quad (6)$$

which gives the velocity when the particle is at a distance x from O. When the particle is at A $(x=a)$ its velocity is zero. When it approaches the fixed point O, its velocity increases till it reaches O.

At O the acceleration is zero, but the velocity is maximum. Then the particle continues to move in the negative side of O,

until it reaches A′ $(x=-a)$ when v is again zero, (see equation 6) then the particle starts to come back towards O, gains its maximum velocity at O and then comes to rest at A. The whole process is then repeated, and the particle oscillates to and fro about O between the points $x=a$ and $x=-a$.

Now from equation-(6) we see when the particle is moving towards O we have, $\frac{dx}{dt}=-\mu\sqrt{a^2-x^2}$ ... (7) (the negative sign, because x decreases with time).

$$\text{or,}\quad -\frac{dx}{\sqrt{a^2-x^2}}=\mu dt \quad \dots \quad (8)$$

Integrating, $\cos^{-1}\frac{x}{a}=\mu t+C'$ .... (9) where C' is a constant.

Since at $t=0$, $x=a$, $\therefore$ $\cos^{-1}\frac{a}{a}=0+C'$ or $0=0+C'$ $\quad C'=0$

$\therefore$ from (9) we have $x=a\cos\mu t$ ... (10)

Equation (10) is the equation of the path of a particle having simple harmonic motion.

the maximum value of x is a and that of $\frac{dx}{dt}$ is μa

when $x=0$ i. e. at O, $\cos\mu t=0$, or, $\mu t=\pi/2$

$$\therefore\quad t=\frac{\pi}{2\mu}.$$

$\therefore$ the particle takes time $\frac{\pi}{2\mu}$ to come from A to O. Similarly for coming from O to A′, or, from A′ to O or from O to A the particle takes the same time $\frac{\pi}{2\mu}$. Hence the time required for returning to A after starting from A i. e, the complete period of oscillation is $4.\frac{\pi}{2\mu}=\frac{2\pi}{\mu}$.

N. B. The most general solution of the equation (2) is

$$x=A\cos(\mu t+\epsilon)$$

It is clear that as t changes, $\cos(\mu t+\epsilon)$ changes periodic between -1 and $+1$ and so x varies between $+a$ and $-a$.

the motion is oscillatory.

A is the greatest distance to which the particle moves on either side and is called the *amplitude* of the oscillation.

Again as t increases by $\frac{2\pi}{\mu}$, x becomes

$$x' = \text{A} \cos\left\{\mu\left(t+\frac{2\pi}{\mu}\right)+\epsilon\right\}$$

$$= \text{A} \cos(\mu t + 2\pi + \epsilon) = \text{A} \cos(\mu t + \epsilon)$$

$\therefore$ value of x remains unchanged

$$\text{also } \frac{dx}{dt} = -\mu\text{A} \sin(\mu t+\epsilon)$$

remains unchanged when t increases by $\frac{2\pi}{\mu}$,

the interval $\frac{2\pi}{\mu}$ is called the *periodic time* or *period* of the oscillation.

ϵ is called the *Epoch* and the angle $\mu t+\epsilon$ is called the *argument*, (sometimes it is also called the *phase*; usually *phase* is $\frac{\mu t+\epsilon}{2\pi}$ times the time period.)

If the particle starts from rest $\epsilon=0$.

§ 8·3. Simple harmonic motion and uniform circular motion.

If a point P describes a circle with centre O and radius a with a uniform angular velocity ω and if Q be the orthogonal projection of P on a fixed diameter of the circle, then the motion of Q along the diameter is a Simple Harmonic Motion.

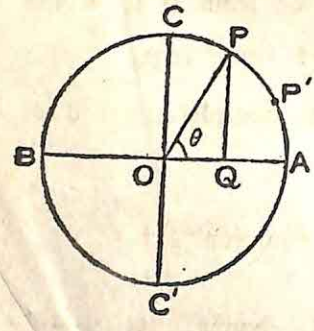


Fig. 49

Let A be the position of P at $t=0$ and let θ be the angle AOP where P is the position of the point at any instant t.

Since the angular velocity ω is a constant, we have $\theta=\omega t$

...t O be the origin and $\overleftrightarrow{\text{OA}}$ be the axis of x. Let OQ $= x$.

$$\therefore \quad x = OQ = OP \cos\theta = a \cos\omega t \qquad \cdots \qquad (1)$$

$$\therefore \quad \frac{dx}{dt} = -a\omega \sin\omega t \qquad \cdots \qquad (2)$$

$$\text{and} \quad \frac{d^2x}{dt^2} = -a\omega^2 \cos\omega t \qquad \cdots \qquad (3)$$

$$= -\omega^2 x \text{ (from (1))}$$

But $\frac{d^2x}{dt^2}$ is the acceleration of Q in the direction $\overrightarrow{OA}$

$\therefore$ acceleration of Q is always directed towards O and varies as $|x|$, the magnitude of OQ.

$\therefore$ the motion of Q is a S. H. M. The *time period* of the motion is $\frac{2\pi}{\omega}$ and the *amplitude* is a.

Note : If P′ be the starting point of P and $m\angle AOP' = \theta'$, then θ' is called the Epoch. In this case $\theta = \omega t + \theta'$.

§ 8·4. The Simple Pendulum.

A *simple pendulum* is a heavy particle suspended from a fixed point by means of a light inextensible string and oscillating in a vertical plane.

Let O be the fixed point and P the position of the particle of mass m at time t. Let $m\angle AOP = \theta$.

Let $OA = OP = l$ (P moves in a circular arc) = length of the pendulum, $s = \text{arc } AP = l\theta$.

Now the forces acting on the particle are the tension T acting along $\overrightarrow{PO}$ and the weight mg vertically downwards. Since there is no motion of P along $\overrightarrow{PO}$, the forces along $\overleftrightarrow{OP}$ i. e. the component $mg \cos\theta$ of the weight mg along $\overrightarrow{OP}$ and the tension T along $\overrightarrow{PO}$ balance each other. The resultant force acting on P is the component $mg \sin\theta$ acting along the tangent $\overrightarrow{PA}$ at P to the arc AP.

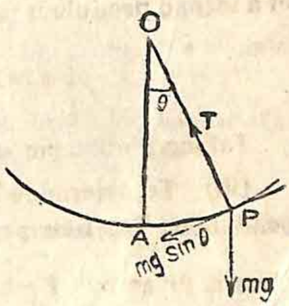


Fig. 50

$\therefore$ Equation of motion of the particle is $m\frac{d^2s}{dt^2} = -mg$

But $s=l\theta$, $\therefore$ $\frac{d^2s}{dt^2}=l\frac{d^2\theta}{dt^2}$

and for small θ, $\sin\theta=\theta$ (θ is measured in radians)

$\therefore$ From (1) $ml\frac{d^2\theta}{dt^2}=-mg\theta$

$$\text{or,}\quad l\frac{d^2\theta}{dt^2}=-g\theta,\quad \text{or,}\quad \frac{d^2\theta}{dt^2}=\frac{-g}{l}\theta$$

this can be compared to the equation (2) of § 8·1 viz,

$$\frac{d^2x}{dt^2}=-\mu^2x$$

Here $\mu^2=\frac{g}{l}$, or, $\mu=\sqrt{\frac{g}{l}}$.

$\therefore$ *the motion of* a simple pendulum is S. H. M. and the time period is $\frac{2\pi}{\mu}=2\pi\sqrt{\frac{l}{g}}$.

N. B. (i) The period of a simple pendulum varies *directly as* the square root of l and *inversely* as the square root of g. Hence a clock will become fast or slow according as the length of the pendulum is shortened or increased or according as g increases or decreases.

(ii) *A second's pendulum is a pendulum whose time period is 2 seconds, i.e.* it swings from rest to rest in *one second.* The length of a second pendulum is given by

$$2=2\pi\sqrt{\frac{l}{g}},\quad \text{or,}\quad l=\frac{g}{\pi^2}.$$

Taking $g=980$ cm/sec^2, $l=99{\cdot}39$ cm. or $39{\cdot}12$ inches nearly.

(iii) To determine 'g' at any place with the help of a simple pendulum: The time period of a simple pendulum of length l at any place is given by $T=2\pi\sqrt{\frac{l}{g}}$.

$$\therefore\ \frac{l}{g}=\left(\frac{T}{2\pi}\right)^2,\ \therefore\ g=\frac{4\pi^2l}{T^2}\ \ \dots\ \ \text{(i)}$$

Since l is known and T can be found out from experiment and we can calculate g from (i).

Worked out Examples

1. A particle executing an S.H.M starts from a point 5 cm. from the centre of motion with a speed of 1 cm/sec. ; the time period is 11 secs. Find the maximum speed and acceleration.

The general solution of an S. H. M is

$$x=a\cos(\mu t+\epsilon) \quad .... \quad (1)$$

$$\frac{dx}{dt}=-a\mu\sin(\mu t+\epsilon) \quad .... \quad (2)$$

Since the period is 11 sec $\therefore \quad \frac{2\pi}{\mu}=11$ or $\mu=\frac{2\pi}{11}$ ... (3)

Now at $t=0,\ x=5,\ \frac{dx}{dt}=1$

$\therefore$ from (1) and (2)

$$5=a\cos\epsilon$$

and $1=-a\mu\sin\epsilon$

Eliminating ϵ, we have

$$(a\cos\epsilon)^2+(a\sin\epsilon)^2=5^2+\left\{\frac{11}{\mu}\right\}^2$$

or, $a^2=25+\frac{121}{4\pi^2}=28\cdot09 \quad \therefore \quad a=5\cdot3$ cms.

The maximum speed is $\mu a=\frac{2\pi}{11}\times5\cdot3=3\cdot03$ cm/sec.

and the maximum acceleration is

$$\mu^2=\frac{4\pi^2}{121}\times5\cdot3=1\cdot73\text{ cm/sec}^2.$$

2. The speed v (in cm/sec) of a point moving along the x-axis is given by $v^2=12-2x-2x^2$ (x is in cm.)

Prove that the motion is S. H. M and find the amplitude and the period.

We have, $v^2=12-2x-2x^2$ .... (1)

differentiating both sides with respect to x we have

$$2v\frac{dv}{dx}=-2-4x$$

or, $2\frac{d^2x}{dt^2}=-2-4x \quad \left(f=\frac{d^2x}{dt^2}=v\frac{dv}{dx}\right)$

or, $\frac{d^2x}{dt^2}=-2x-1=-2(x+1/2)$ .... (2)

Putting $x=z-1/2$ we have $\frac{d^2x}{dt^2}=\frac{d^2z}{dt^2}$

and (2) becomes $\frac{d^2z}{dt^2}=-2z.$

$\therefore$ The motion is simple harmonic about the point $z=0$ i.e. $x=-1/2$ i.e. the point $\frac{1}{2}$ cm. to the left of o (the origin).

Here $\mu^2=2$, $\therefore$ $\mu=\sqrt{2}$

$\therefore$ the period is $\frac{2\pi}{\sqrt{2}}=\pi\sqrt{2}$ sec.

Now $v^2=12-2x-2x^2$,

$\therefore$ $v=0$ when $12-2x-2x^2=0$

or, $x^2+x-6=0$, or, $(x-2)(x+3)=0$, i.e. when $x=2$ or, -3

$\therefore$ the amplitude is $\frac{2+3}{2}=2{\cdot}5$ cm.

3. A particle is projected with velocity v directly away from a fixed point at a distance a from the point of projection. If the acceleration be attractive and of magnitude $n^2 \times$ (distance from the fixed point), find the amplitude of the S. H. M.

The general solution of an S. H. M. is $x=A\cos(\mu t+\epsilon)$.

By the problem $\frac{dx}{dt}=v$ when $x=a$ and $t=0$, also $\mu=n$.

Now $\frac{dx}{dt}=-\mu A\sin(\mu t+\epsilon)$

$\therefore$ $v=-\mu A\sin\epsilon$ and $a=A\cos\epsilon$

Eliminating ϵ we get $\left(\frac{v}{\mu}\right)^2+a^2=A^2$, or, $A^2=a^2+\frac{v^2}{n^2}$

$\therefore$ the amplitude is given by $A=\sqrt{a^2+\frac{v^2}{n^2}}$

4. A point moving with S H.M. has an accleration of 4 ft/sec^2, when at its greatest distance from the centre and a maximum velocity of 8 ft/sec. Find the amplitude and the velocity at distance 3 ft. from the centre.

The acceleration is given by $f=-\mu^2 x$

$\therefore$ If the greatest distance of the particle be a then the amplitude is a and the magitude of acceleration is $\mu^2 a$.

$\because$ the maximum velocity is μa

$\therefore$ By the problem $\mu^2 a=4$ and $\mu a=8$.

Eliminating a we get $\mu = 4/8 = 1/2$, $\therefore$ $a = 8/\mu = 8 \times 2 = 16$

$\therefore$ the amplitude is 16 ft.

the velocity at any point is given by $v = \mu \sqrt{a^2 - x^2}$

$\therefore$ at $x = 8$

$$v = \tfrac{1}{2}\sqrt{16^2 - 8^2} = \tfrac{1}{2}\sqrt{256 - 64} = \tfrac{1}{2}\sqrt{192}$$
$$= 4\sqrt{3} \text{ ft/sec.}$$

5. A particle moving with S. H. M. in a straight line has velocities v_1, v_2 at distances x_1, x_2 from the centre of its path. Show that if T be the period of its motion,

$$T = 2\pi\sqrt{\frac{x_1^2 - x_2^2}{v_2^2 - v_1^2}}.$$ [C. U. 1969]

The velocity at any point x is given by $v = \mu\sqrt{a^2 - x^2}$.

By the problem, $v_1 = \mu\sqrt{a^2 - x_1^2}$, $v_2 = \mu\sqrt{a^2 - x_2^2}$

Eliminating a, we get

$$\frac{v_2^2}{\mu^2} - \frac{v_1^2}{\mu^2} = (a^2 - x_2^2) - (a^2 - x_1^2)$$

$$\frac{v_2 - v_1^2}{} = x_1^2 - x_2^2 \quad \text{or,} \quad \mu^2 = \frac{v_2^2 - v_1^2}{x_1^2 - x_2^2}$$

But $T = \dfrac{2\pi}{\mu}$, $\therefore$ $T = 2\pi\sqrt{\dfrac{x_1^2 - x_2^2}{v_2^2 - v_1^2}}$.

6. If a seconds pendulum be lengthened by $\frac{1}{100}$th of its length, how many seconds will it lose in a day ?

Let the original length of the pendulum be l,

then $T = 2\pi\sqrt{\dfrac{l}{g}}$;

when its length is l', the time period changes to T', where

$$T' = 2\pi\sqrt{\frac{l'}{g}}$$

$$\therefore \frac{T'}{T} = \sqrt{\frac{l'}{l}} = \sqrt{\frac{l + l/100}{l}} = \sqrt{1 + \frac{1}{100}} = 1{\cdot}005$$

$$\left[\because \sqrt{1 + \frac{1}{100}} = \left[1 + \frac{1}{100}\right]^{/2} = 1 + \frac{1}{200} = 1{\cdot}005\right]$$

[neglecting $(\frac{1}{100})^2$ etc.]

$\therefore$ $T' = 1{\cdot}005T$

$\because$ it losses ·005 second in a second.

In a day there are 86400 secs.

$\therefore$ it loses $86400 \times \cdot 005$ secs $= 432$ sec; in a day.

7. A seconds pendulum gains 36 secs. a day; how high must it be raised above the sea level in order that it may keep correct time (radius of earth $= 4000$ miles.)

Since the seconds pendulum gains 36 secs. a day, it gains $\frac{36}{86400}$ secs. in a sec. $\therefore$ half of its time period T' at sea level is given by $T' = \frac{86400}{86436}$

Let it keep correct time at a height h above the sea level i.e. half of its time period $T = 1$ sec.

$$\therefore \quad \frac{T}{T'} = \frac{1 \text{ second}}{\frac{86400 \text{ seconds}}{86436 \text{ secs}}} = \frac{86436}{86400}$$

Now g at a place is inversely proportional to the square of the distance of the place from the earth.

So if g' be the value of g at sea level and g be the value of g at a height h miles above sea level,

$$\text{then } \frac{g'}{g} = \frac{(4000+h)^2}{(4000)^2} = \left(1 + \frac{h}{4000}\right)^2 = \left(1 + \frac{2h}{4000}\right)$$

$$\left[\text{neglecting } \left(\frac{h}{4000}\right)^2\right]$$

$$\text{But } \frac{T}{T'} = \sqrt{\frac{g'}{g}}. \quad \therefore \quad \frac{T^2}{T'^2} = \frac{g'}{g} = \left(1 + \frac{2h}{4000}\right)$$

$$\therefore \quad 1 + \frac{2h}{4000} = \left(\frac{86436}{86400}\right)^2 = 1 + \frac{2 \times 36}{86400}, \text{ or, } \quad \frac{2h}{4000} = \frac{72}{86400}$$

$$\therefore \quad h = \frac{5}{3} = 1\tfrac{2}{3} \text{ miles.}$$

8. Show that the number of oscillations lost by a pendulum of length l which makes n oscillations in a given time, if the length is increased to $l+x$ is $\frac{nx}{2l}$.

Let T, T' be the time periods of the pendulum when its lengths are l, $l+x$ respectively.

$$\text{Now } T = 2\pi\sqrt{\frac{l}{g}}$$

$$T' = 2\pi\sqrt{\frac{l'}{g}} = 2\pi\sqrt{\frac{l+x}{g}}.$$

$$\therefore \quad \frac{T'^2}{T^2} = \left(\frac{l+x}{l}\right) = 1 + \frac{x}{l} \quad \therefore \quad \frac{T'}{T} = \left(1+\frac{x}{l}\right)^{1/2} = 1 + \frac{x}{2l} \text{ (nearly)}$$

$$\therefore \quad T' = T\left(1+\frac{x}{2}\right).$$

Now it makes n oscillations in t_0 seconds with length l and n' oscillations t_0 secs with length $l+x$

$$\text{then } t_0 = nT$$

$$t_0 = n'T'$$

$$\therefore \quad \frac{n'}{n} = \frac{T}{T'} = \left(1+\frac{x}{2l}\right)^{-1}$$

$$\therefore \quad n' = n\left(1+\frac{x}{2l}\right)^{-1} = n\left(1-\frac{x}{2l}\right)\left[\text{ neglecting } \left(\frac{x}{l}\right)^2 \text{ etc.}\right]$$

$$\therefore \quad n - n' = \frac{nx}{2l}.$$

Hence the number of oscillations lost by the pendulum is $\frac{nx}{2l}$

Exercises On the Chapter VIII

1. A particle is executing an S. H. M. of time period 11 seconds. Find its acceleration when it is at a distance of 7 cm. from the centre of motion.

2. A point executing an S. H. M. has velocities 5 ft,/sec., 12 ft./sec. when at distances 6 ft, $2\frac{1}{2}$ ft. respectively from the centre O. Find its greatest velocity. the period and its acceleration when at its greatest distance from O.

3. A particle performing S. H. M. in a straight line starts at a point 14 ft, from the centre of its path and has a maximum velocity of 22 ft./sec.. Find its periodic time.

4. If the period of a S. H. M. is 8 secs. and the amplitude 4 ft, find the maximum velocity and also the velocity when the particle is 2 ft. from the central position.

5. The speed v, (in ft./sec.) of a point moving along the axis of x is given by

$$v^2 = -9x^2 + 18x + 27,$$

where x is in feet Prove that the motion is simple harmonic and find the centre of motion, the amplitude and the period.

6. A particle oscillating harmonically in a straight line has velocities v_1, v_2 and accelerations f_1, f_2 in two positions on the path.

If d be the distance between the two positions, show that

$$d=\frac{v_1^2-v_2^2}{f_1+f_2}$$

7. If the distance x of a moving point at any time t is given by $$x=\alpha \cos nt+\beta \sin nt$$
where α, β are constants, show that the motion of the point is simple harmonic.

8. Show that the general solution of an S. H. M. can be written in the form

$$x=\text{A} \cos \mu t+\text{B} \sin \mu t.$$

9. A particle executing simple harmonic motion in a straight line is observed to be at distances x, y, z from the centre of its path at the t-th, $(t+1)$-th and $(t+2)$-th seconds respectively, show that the time of complete oscillation is

$$\frac{2\pi}{\cos^{-1}\left(\dfrac{x+z}{2y}\right)}.$$ [C. U. '73]

10. A particle executing simple harmonic motion in a straight line has velocities v_1, v_1, v_3 respectively at distances x_1, x_2, x_3 from the centre of the path.

Prove that $x_1^2(v_2^2-v_3^2)+x_2^2(v_3^2-v_1^2)+x_3^2(v_1^2-v_2^2)=0.$

[C. U '72]

11. A particle is performing S. H. M. of period T about a centre O, and it passes through a point P with a velocity v in the direction OP. If OP be equal to x, and if the particle returns to P in time t, the show that $t=\frac{\text{T}}{\pi}\tan^{-1}\frac{v\text{T}}{2\pi x}$. [B. U.]

12. A seconds pendulum loses 15 seconds per day. Find the alteration to be made in its length so that it may keep correct time.

13. The length of a seconds pendulum is increased by ·05 inch. How many seconds it will *loose* in a day.

14. A seconds pendulum keeps correct time at sea level. It is taken to a height 2 miles above sea level. What alteration should be made so that it will still keep the correct time.

(radius of earth = 4000 miles).

15. A pendulum which beats seconds at equator gains 5 minutes a day when carried to the pole. Compare the values of g at the two places.

16. If L and l be the lengths of a second's pendulum at the surface of the earth and at a height h, show that the earth's radius is

$$\frac{\sqrt{l}}{\sqrt{L}-\sqrt{l}}h.$$

SHORT ANSWER TYPE QUESTIONS

In Higher Secondary Examinations *Short Type Questions* on Mechanics are set every year. In the next few pages a set of such questions on Dynamics is given ; we have not solved them. Most of them are in the main body of the text in some form or other. So, in many cases we have referred to the relevant article, worked out example or exercise. In some cases suitable hints have also been given.

Correct or Justify the following statements (1—75).

1. For a given displacement along every trajectory the average speed and average velocity of a particle are equal.

[**Ans.** Incorrect. See Note. § 2·2]

2. A particle undergoes displacements 9 cms, 13·5 cms and 22·5 cms in times 2 seconds, 3 seconds and 5 seconds respectively. The particle may be taken to move with uniform velocity.

[**Ans.** Correct.]

3. A cyclist goes from a place A to a place B with a velocity of 30 miles per hour and from B returns to A with a velocity of 20 miles per hour. The average speed of the cyclist during his journey is 25 miles per hour.

[**Ans.** Incorrect. Average speed $=\frac{2}{\frac{1}{30}+\frac{1}{20}}=24$ miles perhour.]

4. A particle possesses simultaneously two uniform velocities u and v. The greatest magnitude of the resultant velocity is $\sqrt{u^2+v^2}$.

[**Ans.** Incorrect. The greatest magnitude of the resultant velocity is $u+v$. See § 2·6 Note 1.

5. In order to cross a river without current a man began to swim following in a direction perpendicular to the length of the river. When he was at half-way to the opposite bank, the tide came. If the man swims following the same course, the time taken by him to swim the last half is equal to the time taken to swim the first half.

[**Ans.** Correct].

6. The shortest time taken by swimmer to cross a flowing river is independent of the velocity of the current.

[**Ans.** Correct statement].

7. A swimmer will have to swim perpendicularly to the direction of the current in a flowing river in order to reach the other bank in shortest time. [W. B. H. S. 1981]

[**Ans.** Correct statement]

8. The velocity required to cross a flowing river along the shortest route is greater than the velocity required to cross the river in the shortest time.

[**Ans.** Correct statement. See Ex. 10 Exercise 1.]

9. A swimmer wants to cross a river flowing along a straight course at the rate of 2 km. perhour so as to reach the directly opposite point on the other bank. If he can swim at the rate of 4 miles perhour in stillwater, he should attempt to swim in a direction making an angle 120° with the direction of the current.

[**Ans.** Correct statement. See Ex. 11, w. out in chapter II].

10. In case of uniform circular motion velocity and acceleration of a particle remain constant.

[**Ans.** Incorrect statement. Hints. See § 2·1]

11. It is possible for a body possessing simultaneusly three velocities of 20, 10 and 8 units to remain at rest.

[**Ans.** Incorrect statement. See Triangle of velocities]

12. The relative velocity of a particle A with respect to another particle B is equal to the relative velocity of B with respect to A.

[**Ans.** Incorrect statement. Correct statement will be "The relative velocity of a particle A······equal in magnitude but opposite in direction to the relative velocity of B with respect to A.

13. When a man rushes towards a plane mirror with a velocity u, his image appears to rush towards him with velocity 2u.

[**Ans.** Correct statement.

Let at time t after start A and A′ be the positions of the man and his image and the straight line AA′ intersect the mirror at O Then $AO = A'O = x$ (say).

$$\therefore \quad \frac{dx}{dt} = -u \quad \text{(As the man rushes towards O)}$$

Also let $AA' = y$. $\therefore$ $y = 2x$.

$$\therefore \quad \frac{dy}{dt} = \frac{d}{dt}(2x) = 2\,\frac{dx}{dt} = -2u$$

The negative sign indicates that y diminshes i. e., the image approaches the man]

14. Two particles are moving along the circumference of a circle in opposite directions with the same velocity u. The least relative velocity of the one of them with respect to the other is u.

[Incorrect statement. Corrrect Ans : relative velocity$=u-u=0$. See Ex, 10 worked out, chapter III]

15. During the rains, the rain drops falling vertically appear to come down obliquely to a person sitting in a running train.

[Correct statement. See Discussion of § 3·1. and example 5 worked out in chapter III.]

16. A man is running with an umbrella in rain with a velocity equal to that of rain falling vertically. The best angle at which he should hold the umbrella so that the rain drops do not strike his face is 45° with the vertical.

[Ans. Correct statement.]

17. The relative velocity of C with respect to A is the resultant of the relative velocity of A with respect B and that of B with respect to C.

[Ans. Incorrect statement. See Ex. 11. worked out chapter III]

18. If on each of two moving bodies a common velocity be imposed, their relative velocity remains the same as before.

[Ans Correct statement]

19. A passenger in a moving train drops a stone from the window. The stone will move in a parabolic path (*i*) relative to the train and (*ii*) relative to the ground.

[Ans. (*i*) Incorrect (*ii*) Correct]

20. A stone is thrown outside by a passenger in a moving train. (i) A man on the ground and (ii) A man in the train will see the stone to move in a parabolic path.

Ans. (*i*) Correct (*ii*) Incorrect]

21. A boy sitting in a compartment of a train moving with uniform velocity throws a ball inside the compartment. The ball will fall into the hands of the boy.

[Ans. Correct statement]

22. A boy sitting in a compartment of a train throws a ball inside the compartment and the train accelerates when the ball is in the air. The ball will fall in the hands of the boy.

[Ans Incorrect statement]

23. A particle moves in a straight line and the distance of the particle from a fixed point O of the straight line at time t after start is given by $s=t^3-4t+4$. The particle moves with constant-acceleration.

[Ans. Incorrect statement. See Ex. 1. w out chapter IV]

24. For a body moving along a straight line the law of motion is $x=\frac{1}{2}vt$, where v is the velocity of the body at a distance x from a fixed point O of the straight line. The particle moves with uniform acceleration.

[Ans. Correct statement. See Ex. 5. w. out ch. IV]

25. The velocity of a particle moving along a straight line is given by the relation $v^2=as^2+2bs+c$. Hence the acceleration is uniform throughout the motion.

[Ans. Incorrect statement]

26. The velocity of a particle moving along a straight line at time t after start is given by the relation $x=a+bt+ct^2$. Hence the acceleration is constant. [Ans Correct statement.]

27. It is impossible for a particle to move in a straight line so that its velocity varies as the distance described from the commence ment of the motion.

28. The two extremities of a train running with uniform acceleration overtakes a person standing on the platform with velocities u and v. The middlepoint of the train will over take the person with velocity $\dfrac{u+v}{2}$.

[Ans. Incorrect statement. The middle point will over take the person with velocity $\sqrt{\dfrac{u^2+v^2}{2}}$.]

29. A particle moving with uniform acceleration traverses respectively distances $a+bt$ and $a+b(t+1)$ during the t th second and the next second, where a and b are constants. The acceleration of the particle is b.

[Ans. Correct statement]

30. (a) For a particle moving with uniform acceleration, the average velocity during an interval is the arithmetic mean of the velocities of the particle at the begining and end of the interval.

[Ans. Correct statement. See cor 2. § 4·4.]

(b) For a particle moving with uniform acceleration the velocity at the middle of the interval is the arithmetic mean of the velocities of the particle at the begining and end of the interval.

[**Ans.** Correct statement]

31. The velocities of a particle moving with a uniform acceleration at the begining and end of an interval t, are u and v respectively. The distances described by the particle during this interval is given by $s=\frac{u+v}{2}\,t$.

[Correct statement. See cor 2. § 4·4.]

32. For a body moving along a straight line with uniform acceleration, velocity at halftime is less than velocity at half distance.

[**Ans.** Correct statement.]

Hints, velocity $v_{\frac{s}{2}}$ at half distance

$$=\sqrt{u^2+2f.\tfrac{s}{2}}=\sqrt{u^2+fs}=\sqrt{\frac{2u^2+2fs}{2}}=\sqrt{\frac{u^2+u^2+2fs}{2}}$$

$$=\sqrt{\frac{u^2+v^2}{2}}.$$

Velocity at halftime $=v_{\frac{t}{2}}=\frac{u+v}{2}$.

Now $v^2_{\frac{s}{2}}-v^2_{\frac{t}{2}}=\frac{u^2+v^2}{2}-\left(\frac{u+v}{2}\right)^2=\frac{u^2+v^2-2uv}{2}=\left(\frac{u-v}{2}\right)^2>0$

$\therefore\ v^2_{\frac{s}{2}}>v^2_{\frac{t}{2}}$ i.e, $v_{\frac{s}{2}}>v_{\frac{t}{2}}$ i.e., $v_{\frac{t}{2}}<v_{\frac{s}{2}}$

33. If the velocity at time t after start of a particle moving along a straight line with uniform acceleration f be v, then the distance described by the particle during this time t is given by $s=vt-\frac{1}{2}ft^2$.

[**Ans.** Correct statement]

34. The velocity v of a particle at any instant is given by $v=\sqrt{\mu\left(\frac{x}{a}-1\right)}$, where x is the distance from a fixed point. The acceleration of the particle is towards the fixed point and varies inversely as the square of the distance from the point.

[Ans. Incorrect statement. Correct statement. :
$v=\sqrt{\mu\left(\frac{a}{x}-1\right)}$.

Hints : $v=\sqrt{\mu\left(\frac{x}{a}-1\right)}$ or, $v^2=\frac{\mu x}{a}-\mu$ $\therefore$ $2v\frac{dv}{dx}=\frac{\mu}{a}=$ cons.

If $v^2=\frac{\mu}{ax}$ then $2v\frac{dv}{dx}=-\frac{\mu}{ax^2}$]

35. A person is standing on a bus moving along a straight line with uniform velocity.

(a) If the bus accelerates forward, the man will fall backward.

(b) If the bus retards, the man will fall forward.

(c) If the bus moves round a curved path the man will fall backward.

[Ans. (a) Correct statement (b) Correct statement (c) Correct statement.]

36. To every action there is an equal and opposite reaction and so the action and reaction balance each other.

[Ans. Incorrect statement.]

37. To every force applied on a body there is an equal and opposite reaction and so the body cannot undergo any displacement. [Incorrect stntement]

38. A thief jumps off the terrace of a building with a heavy suitcase on his head, and falls vertically. He experiences no pressure on his head while he is in the air. [H. S. 1978]

[Ans. Correct statement.]

Hints. Let the pressure be P, m be the mass of the suitcase and R be the reaction of the head on the suitcase.

Then $mg-\mathrm{R}=mg$ [Here $f=g$, since the suitcase is falling with acceleration g.]

39. A man jumps from a height with a bucket in his hand. He experiences no pressure on his hand while he is falling.

[Ans. Correct statement. Same as Ex. 38 above]

40. A man is ascending in a lift with a heavy load in his hand. The lift is ascending with (i) an acceleration or, (ii) with a retardation f. In case (i) the load will appear lighter to the man and in case (ii) the load will appear heavier.

Ans. Incorrect statement.]

41. A lift is descending with (i) uniform acceleration f (ii) uniform retardation or (iii) with uniform velocity. In case (i), to a man in a lift a load in his hand will appear lighter; in case (ii) the load will appear heavier and in case (iii) the man will feel the original weight of the load. [Ans. Correct statement].

42. A Balloon is ascending with a uniform acceleration 245 cms per sec^2. A body is found to weigh 25 kg. by means of a spring balance attached to the balloon. The true weight of the body is 20 kg. [Ans. Correct statement.]

43. A spring balance is attached to the bottom of a lift and a load of 10lbs is suspended from the balance. The lift is ascending with a uniform velocity of 32 ft/sec^2. So, the load will appear heavier and the reading of the spring balance will be increased. [Ans. Incorrect statement.]

44. When two men pull at two ends of a rope each with a force of 150 kg, the tension of the string is still 150 kg.

[Ans. Correct statement]

45. It is easy to drag a roller than to push it

[Ans. Correct statement]

46. When a man pulls a tree by a rope with a force of 50 kg, the tension in the string is 100 kg.

[Ans. Incorrect statement]

47. A rope can just support a mass of $20 \times 453{\cdot}6$ gms when at rest. It will break if it raises a mass of $16 \times 453{\cdot}6$ gms with an acceleration greater than $8 \times 30{\cdot}4$ cm./sec^2.

46. A man is ascending in a lift. If the chain of the lift breaks, the thrust of the feet of the man on the lift is twice the weight of the man. [Incorrect statement]

49. A shot loses half its velocity in penetrating 3 cms of a target. It will penertrate 3 cms more before coming to rest.

[c. f. H. S. 1981]

[Ans. Incorrect statement. It will penetrate 1 cm. more]

50. When a train moves with uniform velocity, the pull of the engine balances the resisting force. [H. S. 1978]

[Ans. The statement is true.]

51. A particle is thrown vertically upwards with a velocity u. The greatest height attained by the particle is $\frac{u^2}{g}$.

[Ans. Incorrect statement. Correct Ans $\frac{u^2}{2g}$. See § 6·3]

52. A particle is thrown vertically upwards with a velocity u. It will reach the point of projection again after a time $\frac{u}{g}$.

Ans. [Incorrect statement Correct Ans $\frac{2u}{g}$ See §6·5 and 6·6]

53. A particle is projected vertically upward with a given velocity from the surface of the earth. The time of rise of the particle is equal to the time of fall to the earths surface. [H. S. 1981]

(Correct statement : See § 6·6)

54. The height traversed by a particle thrown vertically upwards in the last second of its ascent is independent of the velocity of projection [Ans. Correct statement.]

Hints. Time of ascent = Time of descent, so distance traversed in the last second of ascend = distance traversed in the first second of descent = $\frac{1}{2}$ g1^2)

55. A particle projected vertically upwards with a velocity of 1000 ft/sec will ascend 4ft in the last half second of its motion.

[Ans. Correct statement.]

Hints. Height ascended in the last half-second of its ascent = $\frac{1}{2}$ g. $(\frac{1}{2})^2$ = 4ft. (See Ex. 54 above)

56. A particle falling freely from a height falls through 224ft in the last second of its fall. It was falling from a height of 900ft.

(Ans. Correct statement)

57. A passenger in a train throws a ball vertically upwards inside a compartment. The point at which the ball will reach the floor of the compartment is the same if the train were at rest. The time taken by the ball to reach the floor of the compartment in the two cases are also the same.

[Ans. Correct statement.]

Hints. As the ball was thrown vertically upwards, the train's

velocity whose vertical component is zero does not affect the motion of the ball.

58. From the top of a tower a particle is let fall and at the same time from the same plaee another particle is thrown horizontally. The two particles will reach thc gronnd at the same time.

[Ans. Correct statement]

59. From an aeroplane moving horizontally with a uniform velocity, a particle is let fall. The particle will always remain vertically below the plane, till it reaches the ground.

[Ans. Correct statement]

60, From the top of a tower two bodies of masses 5 kgs and 10 kgs are let fall. As the force of attraction of the earth on the second body is greater, so it will reach the surface of the earth earlier. [Ans. Incorrect statement, See § 6·3]

61. Two masses of 10 kg and 20 kg are simultaneonsly let fall from the the top of a tower and they reach the earth's surface at the same time. So the force of attraction of the earth on the two bodies are equal in magnitude. [Ans. Incorrect]

62. A body is thrown vertically upwards to a certain height at Calcutta. The same body is thrown to a greater height at the equator with the same vertically upward velocity. The weight of the body at the equator is greater than that at Calcutta

[Ans. Incorrect statement.

Let m be the mass of the body. If g and g' be the acceleration due to gravity at Calcutta and the equator and h and h' be the two corresponding heights,

$$\text{then } h=\frac{u^2}{2g} \text{ and } h'=\frac{u^2}{2g'} \quad \text{Also, } h'>h.$$

$$\therefore \frac{u^2}{2g'}>\frac{u^2}{2g} \quad \text{or,} \quad g>g' \quad \therefore \quad mg>mg'.]$$

63. If the time taken by a body to fall freely from rest from a given height at Paris be less than that at Calcutta, then the weight of the body at Paris is less than that at Calcutta.

[Ans. Incorrect statement]

64. If a body falls from a height h with uniform velocity, then the resistance of air is equal to the weight of the body.

[Ans. Correct statement.]

65. If a waggon without engine moves down an inclined plane of inclination 1 in 100 with uniform velocity, then the resistance of the plane is equal to the weight of the waggon.

[Ans. Incorrect statement.

Resistance $=\frac{1}{100}\times$ weight of the waggon].

66. A stone is let fall from the roof of a building. It reaches the ground in 2 seconds. The height of the building is not less than 20 metres.

67. A body is thrown vertically upwards with a velocity 64 ft/sec. It will return to the point of projection after 4 seconds.

[Ans. Correct statement.]

68. A ball cannot be projected horizontally beyond 128 ft if the velocity of projection be 64 ft/sec². [Ans. Correct statement,

Hints. Maximum horizontal range of a projectile is $\frac{u^2}{g}$]

69. In a ball throwing competition, a competitor should throw the ball at an angle 60° to the horizontal direction to get the best result. [H. S. 1979]

[Ans. Incorrect statement. Required angle is 45°].

70. For a given horizontal range with a given velocity of projection, there are in general two directions of projection equally inclined to the direction of maximum range.

[Ans. Correct statement. See Note (iii) § 7·2]

71. A boy can throw a stone 50 ft. vertically upwards. The greatest horizontal distance to which he can throw the ball is 100 ft.

[Ans. Correct statement].

72. Particles are projected simultaneously with velocities of magnitude v from a given point in different directions. After t secs. they all lie on a circle. [C. U.]

[Ans. Correct statement

73. A particle executing simple harmonic motion has maximum velocity at the centre of motion but no acceleration there.

[Ans. Correct statement. See equations (2) and (6) § 8·2].

74. A paaticle executing simple harmonic moticn has maximum acceleration at the ends but has zero velocity there.

[Ans. Correct statement. See equations (2) and (6) § 8·2]

75. The tension in the string of a simple pendulum remains constant as the pendulum oscillates.

[Ans. Correct statement. See § 8·4 $T = 2\pi\sqrt{\frac{l}{g}}$]

Justify the correct Answer [76—82]

76. A particle moving along a straight line goes through a distance d with uniform velocity u and returns back with a uniform velocity v. The average speed of the particle is

(a) $\frac{u+v}{2}$ (b) $\sqrt{uv}$ (c) $\frac{2uv}{u+v}$ [Ans (c)]

77. A block slides down a smooth inclined plane from its top while another falls freely from the same point.

The second body reaches the bottom of the plane (a) earlier (b) at the same time or (c) later than the first one. [Ans. (c)]

78. The law of motion of a particle moving along a straight line is given by $x = t^3 - 6t^2 - 15t$. The velocity will be negative and the acceleration will be positive if,

(i) $2 < t < 5$ (ii) $1 < t < 3$ (iii) $2 < t < 6$ (iv) $1 < t < 5$. [Ans. (i)]

79. A body projected vertically upwards with the same velocity can reach a greater height when thrown at Calcutta than when thrown at London.

The weight of the body at London (i) less than (ii) equal to or (iii) more than that at Calcutta. [Ans. (iii)]

80. The horizontal range of a projectile for a given velocity of projection is maximum when the angle of projection is

(a) 30° (b) 45° or (c) 60°. [Ans. (b)]

81. A man is descending in a lift with a load of weight 10 kg. The lift is descending with an acceleration of 1m/sec². The pressure of the load in the hand of the man is one of the following

(a) more than 10 kg. (b) less than 10 kg. (c) equal to 10 kg. (d) zero. [Ans. (b)]

82. A man is standing in a lift and the lift is asending with an acceleration k. The man will appear to be heavier by :—

(a) (k+g) of his actual weight (b) (k−g) of his actual weight (c) (k×g) of his aetual weight (d) (k÷g) of his actual weight.

[Ans. (d)]

ANSWERS

Chapter II

1. Average speed $=44$ mt./min, average vel $=0$. 2. 17 metres.
3. (i) 0, 24 m./sec. (ii) 50 km./hr. along ox; $50\sqrt{3}$ km./hr.
   (iii) $5\sqrt{2}$ cm./sec., $5\sqrt{2}$ cm./sec.
4. $5\sqrt{7}$ m./sec. making angle $\tan^{-1}\sqrt{3}/2$ with the first.
5. (i) 25 km./hr. (ii) 7 cm./sec.
   (iii) $\frac{3}{\sqrt{2}}(\sqrt{3}+1)$ m./min. (iv) 60°.
6. No. 7. $135^\circ, 105^\circ, 120^\circ$.
8. 25m/sec, $\cos^{-1}(-\frac{3}{5})$, with the first comp.
11. $25\sqrt{2}$ m.; 135°. 12. $3\sqrt{5}, \tan^{-1}\frac{1}{2}$ N of W. 14. 150m.
16. $\tan^{-1}\frac{9}{41}$ E, 210 metre. 17. 34·6 m/sec.
18. $s\sqrt{t_1^2-t^2}/tt_1$.
19. 2 km./sec. along the first comp. in the opp direction
20. 3·7 hr, 28°35′S of W.
22. at an angle $90^\circ+\alpha/2$ with the direction of motion; 60°.

Chapter III

1. (i) 22 ft/sec (ii) 110 ft/sec.
2. 36 km/hr 3. 36 sec. 4. $8\sqrt{3}$ km/hr. vertical.
5. 10 km hr South-East 6. 5 km/hr; 25 km
7. $6\sqrt{2}$ km/hr, at an angle of 45° to the vertical.
8. 1 hr 36 mins; 150 km. 9. Towards East.
10. $\frac{10}{3}\sqrt{34}$ min/sec, from a direction making an angle $\tan^{-1}\frac{3}{5}$ with the motion of the train.
11. $17\frac{1}{2}$ m/hr, 202 yds, 36·7 yds. 12. $u\sqrt{5-4\cos\alpha}$.
13. $\sin^{-1}\frac{\sqrt{2}}{5}$, West of North.
14. at an angle $\sin^{-1}\left[\frac{V}{v}\sin\theta\right]$ with the direction of the aeroplane.
15. (i) from N 40° 30′E (ii) from S 36° 42′E
16. (i) Due N (ii) ·15 hr
18. 34° 49′E of N, 3 h. 5 min. 45 sec. 31·44 m. N.
19. 12 5 m/sec, $\tan^{-1}(-\frac{3}{4})$ 20 From 30° N of E
21. 39 km/hr. $\tan^{-1}(\frac{5}{12})$ East of North 22. 6 miles/hr.

Chapter IV

24 cm/sec, 104 cm/sec^2 3. 14 cm/sec^2 6. $\sqrt{\frac{\mu}{c}}$

7. f is const. in $0\leqslant t\leqslant 1$, $v=$const. in $t\geqslant 1$ 8. 4 secs.
9. (i) 16·5 cm. (ii) 40 ft, 13 ft/sec (iii) 53·28 km/hr.
   (iv) 1 cm/sec^2 1 sec

10. 300 metres 11. 12 m/sec, 90 m 12 $\frac{4}{5}$ miles.
13. $\frac{1}{3}$ inch more 15. ·0008 sec, 1020 ft/sec (nearly)
16. 5 sec, 150 cm. nearly 23. 729 ft. 28. 10 min 59·25 sec.

Chapter V

1. $1{\cdot}419 \times 10^9$ lb ft/sec. 2. 10 pounds $= \frac{5}{16}$ lb wt.
3. 625 ft. 4. 20 m. 5. 1600 ft/sec^2.
6. $3{\cdot}125 \times 10^4$ poundals ; ·002 secs.
8. 80 cm., 10 cm./sec. 9. $10778\frac{1}{3}$ lb. wt.
10. 1120 ft/sec. 12. 3 ft., $\frac{2}{640}$ sec.
13. $1{\cdot}12 \times 10^5$ gm. wt $= 1{\cdot}098 \times 10^8$ dynes.
14. $\sqrt{\frac{\mu}{c}}$ 16. $\frac{g}{13}$ 17. 16·1 ft.

Chapter VI

1. $80\sqrt{2}$ ft./sec. 2. 40 ft./sec. 3. 192 ft.
4. 44·1 m. 5. 225 ft. 6. 4 ft.
7. 36 ft. 8. 196 ft., 112 ft/sec. 9. 2·5 sec.
11. 60 ft. above ground 1/2 sec after the start.
13. 144 ft. 14. 1200 ft./sec. 15. 224 ft./sec.
17. 196 ft. 19. 4080 ft. 20. $1\frac{9}{16}$ ft.

Chapter VII

1. $78\frac{1}{8}$ ft, $\frac{25}{8}\sqrt{2}$ sec 3. 153·6 ft.

4. $y = \frac{\sqrt{3}}{3}x - \frac{32}{7500}x^3$ $t = 1\frac{9}{16}$ sec.

6. 45° or $\tan^{-1} 4$ with the horizon. 7. 80ft/sec, $\alpha = \tan^{-1} 4/3$
9. 50m/sec. 10. 160 m $8\sqrt{5}$m/sec ; $\alpha = \tan^{-1} 2$
11. 4 sec, $160\sqrt{3}$ ft, from base, 112 ft/sec. at an angle $\tan^{-1} \frac{11}{5\sqrt{3}}$ with the ground. 12. 196 ft. 13. $60\sqrt{3}$ ft.

14. $256\sqrt{2}$ ft/sec, $\alpha = 45°$ 18. (i) $\frac{64}{3}$ ft., $\frac{2\sqrt{3}}{3}$ sec., $\frac{64}{3}$ ft.

(ii) 64 ft, $\frac{4\sqrt{3}}{3}$, 64 ft. 23. $109\sqrt{3}$ cm/sec.

Chapter VIII

1. $2\frac{2}{7}$ cm/sec^2. 2. 13 ft/sec, π, 26 ft/sec^2. 3. 4 sec.

4. π ft/sec; $\frac{\pi\sqrt{3}}{2}$ ft/sec. 5. $x = 1, 2$ ft, $\frac{2\pi}{3}$.

12. must be shortened by about ·013 inch. 13. 55.
14. must be shortened by ·099 cm. 15. 143 : 144.

STATICS

CHAPTER ONE

FORCE

§ 1·1. **Definition.** In the Dynamics Portion of this book we have discussed the states of rest and motion of an object, force etc. In order to give the discussion of statics a complete shape we are recapitulating those ideas in this chapter.

Rest. If a body does not change its position, then it is said to be at rest. But in this universe there is no absolute rest. The earth itself is rotating round the sun about its axis. Then, what is meant by rest ? If a body does not change its position with respect to its surroundings, then it is said to be at **rest,** or at **relative rest.** In this book, a body at rest, will really mean a body at *relative rest*. Henceforth the epithet relative will not be used.

Force. From Newton's First Law of Motion we get the following definition of Force.

A *Force* is an external agent which acting on a body changes or tends to change the state of rest or the state of uniform motion in a straight line of the body.

Equilibrium. If the state of rest or the state of motion of a body remains unchanged after the application of a system of forces on the body, then the force-system is said to be in equilibrium. If a body be at rest, then the body as well as the forces acting on it are in equilibrium.

Statics. The branch of mechanics which deals with the equilibrium of bodies under the action of forces is known as *Statics*.

§ 1·2. **Measurement of Force.** In Dynamics units for measurement of forces were defined while discussing Newton's Second Law of Motion. But in Dynamics we discuss bodies in motion while in statics bodies at rest are discussed. So in statics unit force must be defined in terms of the state of equilibrium of a body. In statics force is generally measured in terms of the weight of a definite mass. In the *C. G. S.* system the unit force

is known as *one gramme-weight*. One gramme-weight is that amount of upward force which is to be applied in order to keep a mass of one gramme at rest. Though the value of the acceleration due to gravity varies from place to place, it is constant for the same place. So, in statics, since bodies in motion are not considered, gramme-weight provides a convenient unit for measurement of force. A force one gramme-weight, is in general referred to as one gramme. In the *F. P. S.* system the unit of force used in statics is one **pound weight** or one pound force. In most of the countries in the world, including **India,** now a days a new system of measurement known as ***M. K. S.*** (Metre-Kilogramme—Second) system is in use. In the ***M. K. S.*** system the unit of force is one kilogramme weight or one Kg.-weight or one Kg. The upward force required to keep a mass of 15 Kg. in equilibrium is said to be a force of 15 Kg. weight. If two forces are equal in measure, the forces are said to be equal.

§ 1·3. **Representation of a force by a directed line-segment.** To know a force completely one must know (i) point of application of the force, (ii) its magnitude, (iii) its direction and (iv) its sense.

(i) The point of a body at which a force is applied is called the point of application of the force.

(ii) The magnitude of forces has been discussed in the previous article.

(iii) The direction of a force is the direction in which it is applied. If a force is applied in a direction parallel to the straight line $\overleftrightarrow{AB}$, then the direction of $\overleftarrow{AB}$ is the direction of the force.

(iv) The direction of a force is the direction of the straight line $\overleftrightarrow{AB}$. But one has to ascertain whether the force acts in a *sense* from A towards the point B or in the opposite sense from B towards the point A. Hence forces having the same direction can have two different senses.

A line segment is made *directed* by fixing one of its end points as the *initial point* and the other as the *terminal point*.

Directed line segments will be expressed according to the following conventions. The directed line segment having the point A as its initial point and the point B as the terminal point will be expressed as $\overline{AB}$. The initial and the terminal points of the directed line segment $\overline{BA}$ are B and A respectively.

The length of a line segment represents its magnitude and the line segment indicates a definite direction. The sense of a directed line segment is from its initial point towards its terminal point. Hence the directed line segments $\overline{AB}$ and $\overline{BA}$ are of the same magnitude and direction but of opposite senses. Hence a directed line segment possesses all the attributes of a force. Hence forces can be represented by directed line segments. In order to represent forces by directed line segments one must first of all fix the *scale* of representation.

Scale. If different forces are to be represented simultaneously, then the magnitudes of the forces and the lengths of the directed line segments, which will represent them, must be in the same ratio. The equal ratio is the **scale** of representation of forces. Let 1 cm. = 1 kg. weight force be our scale. Hence forces of magnitudes 5 kg. and 12 kg. are to be represented by directed line segments of lengths 5 cms. and 12 cms. respectively.

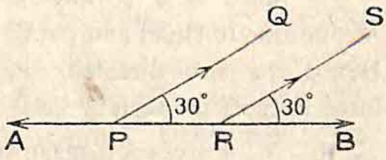


Fig. 1

Let AB be a known st. line *i.e.*, its position and hence its direction are known. In the figure the length of the line segment $\overline{PQ}$ is 4 cms. and $m\angle QPB = 30°$. Let the scale of representation be 1 cm. = 5 kg. wt. Hence the directed line segment $\overline{PQ}$ will represent a force of magnitude 4×5 kg. = 20 kg. Its direction is the direction of $\overrightarrow{PQ}$ and that its sense is from P to Q has been indicated in the figure by an arrowhead. Hence the directed line segment $\overline{PQ}$ represents the force 20 kg. completely. The directed line segment $\overline{QP}$ will represent an equal force having the same direction but opposite sense i.e., the sense from Q to P. Again, in the figure, the directed line segments $\overline{PQ}$ and $\overline{RS}$ are

of the same magnitude, direction and sense. Hence the two directed line segments $\overline{RS}$ and $\overline{PQ}$ will represent the same force in the same scale and this is true for all equal and parallel directed line segments having the same sense. Hence we obtain the following important result.

Result. All equal and parallel directed line segments having the same sense, represent the same force in the same scale.

Example. 1. The lengths of the sides containing the right-angle of a right-angled triangle are 12 cms. and 5 cms. The side of length 12 cms. represent a force of magnitude 60 gms. Find the magnitude of the force that will be represented by the hypotenuse of the triangle in the same scale.

The length of the hypotenuse $= \sqrt{12^2+5^2}$ cms. $=13$ cms. As, the side of length 12 cms. represents a force of magnitude 60 gms. hence the scale is 1 cm. $=\frac{60}{12}=5$ gms. force,

Hence a line segment of length 13 cms. will represent a force of magnitude 13×5 gms. $=65$ gms. i.e., the hypotenuse of the right-angled triangle will represent a force of magnitude 65 gms.

Ex. 2. The sides $\overline{AB}$ and $\overline{DC}$ of the parallelogram ABCD will represent the same force in the same scale.

As ABCD is a parallelogram so the directed line segments $\overline{AB}$ and $\overline{DC}$ are equal and parallel and they have the same sense. Hence the two directed line segments will represent the same force in the same scale.

Ex. 3. Forces of magnitudes 3 gms., 4 gms. and 8 gms. cannot be represented by the three sides of a triangle.

Let the scale be 1 gm. wt. force $=l$ cm. Hence the given forces will be represented by directed line segments of lengths $3l$, $4l$ and $8l$ respectively. Now, as $8l>3l+4l$, and as the sum of the lengths of any two sides of a triangle cannot be less than the length of the third side, so a triangle with sides of lengths $3l$, $4l$ and $8l$ cannot be constructed.

Hence the given forces cannot be represented by the three sides of a triangle.

§ 1·4. Principle of Transmissibility of Forces.

Before discussing the principle of transmissibility of forces one must know the following axiom.

Axiom. If two equal and opposite forces be introduced on a rigid body, then the state of the body will not be altered.

The Principle of transmissibility of Forces :

If the point of application of a force acting on a rigid body be shifted to any other point on its line of action, then the effect of the force on the body will not be altered, if the two points are rigidly connected with each other.

Proof : Let O and $\overrightarrow{OX}$ be respectively the point of application and the line of action of a force F acting on a rigid body and O′ be any other point on $\overleftrightarrow{OX}$, so that O and O′ are rigidly connected with each other. Now, introduce two equal and opposite forces F, F on the body acting at O′ along $\overrightarrow{OX}$ and $\overrightarrow{XO}$. By the axiom just stated, these forces will have no effect on the body. Now, the forces F and F acting at O and O′ respectively along $\overrightarrow{OX}$ and $\overrightarrow{XO}$ will cancel each other (according to the same axiom). Hence instead of the three forces we are left with only one force F acting at O′ along $\overrightarrow{OX}$. *i.e.*, a force acting at O′ having the same magnitude, direction and sense and acting along the same line as the given force. Hence the point of applicatioh of the force acting at O can be shifted to O′ on the line of action of the given force.

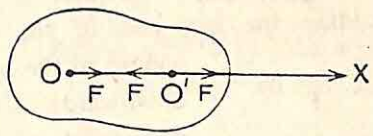


Fig. 2

§ 1·5. Classification of Forces.

Forces can be classified in three main classes, (i) Attraction and Repulsion, (ii) Thrust and tension, (iii) Reaction and Friction.

(i) **Attraction and Repulsion :** If two objects be not in contact with each other and if one of them without the help of any visible medium tries to bring the other towards it by the

application of a force, then the force is said to be a *force of attraction*. If the body tries to move the second body away from it, then the force is said to be a force of *repulsion*.

That the earth attracts every object towards its centre, and that this force of attraction is known as the weight of the object has already been discussed in Dynamics.

(ii) **Thrust and Tension :** If a force is applied on a body by pushing it, then the applied force is said to be a *push* or *thrust*. When a football is kicked or a door is opened by a push then the applied forces are thrusts.

In statics the idea of Tension is very important. The idea of tension is explained below with the help of the following experiment.

Experiment : Suspend a small heavy body by a string holding the free end of the string between the fingers. The weight of the body will try to move the body downwards. Now in order to prevent the downward motion and to keep the body at rest, you will exert by means of the muscles of your fingers and arm an upward force. Let this force be T. It will be seen that,

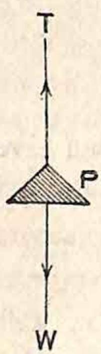


Fig. 3

if $T > w$, the body will move upwards,
if $T = w$, the body will be at rest.
and if $T < w$, the body will move downwards.

Actually, according to the principle of transmissibility, the muscular force T exerted by your fingers is transmitted by the string to the body. Again, the weight of the body is transmitted to your fingers by the string. Thus at every point of the string two equal and opposite forces come into play and the string is said to be under *tension*. This tension is equal at every point of the string.

Hence the value of T will increase with the weight of the body. But the tension transmitted in every string has an upper limit ; *i.e.*, the string will not break until the tension is not greater than what is called the force of **cohesion.** The magnitude of this force of cohesion depends on (i) the material

of the string, (ii) the cross-section of the string but not on the length of the string.

Note 1. In most cases it is found convenient to exert force if the cord is long.

2. If a cord is made of different cords knotted at different points one by one, then the tension in all parts of the long string thus formed may not be equal.

(iii) **Reaction and Friction :** Reaction and friction have already been discussed in the Dynamics portion of this book.

Exercise 1

1. A force of 15 kg. is represented by a directed line segment of length 3 cms. Determine the scale.

2. Represent two forces of magnitudes 50 kg. and 40 kg. acting at right angles to each other according to the scale determined in question 1.

3. $\overline{AB}$ and $\overline{DC}$ are two opposite sides of a parallelogram. Which of the following is true ?

In the same scale the two directed line segments $\overline{AB}$ and $\overline{CD}$.

(i) represent the same force.

(ii) represent two equal and opposite forces.

(iii) None of (i) and (ii) is true.

4. Represent by directed line segments two forces of magnitudes 50 kg. and 75 kg. acting along two straight lines inclined at angles 30° and 45° with a given straight line.

[Scale : 1 cm. = 5 kg.]

5. Can you represent three forces of magnitudes 100 gms. 200 gms. and 350 gms. by the three sides of a triangle ? Justify your answer.

COMPOSITION AND RESOLUTION OF CONCURRENT FORCES

§ 2·1. **Resultant :** If a number of forces be applied on a body at one or more points and if a single force R acting on the body be such that the action of R on the body is equivalent to the combined actions of the number of forces on the body, then the single force R is said to be the **Resultant** of the number of forces.

If the magnitude of the resultant of several forces acting on a body at one or more points of it be zero, then the total effect of the several forces on the body is nil and the body is said to be in **equilibrium**. The several forces are said to be the *component forces* of the resultant force.

In this chapter we shall discuss forces acting at a point. Forces acting at a point are said to be *concurrent forces*.

§ 2·2. **Parallelogram of Forces.**

If the magnitude, direction and sense of two forces acting at a point be known, then from the parallelogram law of forces one can know the magnitude, direction and sense of the resultant of the two forces. We state below the parallelogram law of forces.

Parallelogram law of forces : *If two forces acting at a point can be represented in magnitude, direction and sense by two adjacent sides (diverging from their point of intersection) of a parallelogram, then the resultant of the forces will be represented in magnitude, direction and sense by the diagonal of the parallelogram drawn from the point.*

Let two forces be represented in magnitude, direction and sense by the two adjacent sides $\overline{AB}$ and $\overline{AD}$ of the parallelogram ABCD. Then their resultant will be represented in magnitude, direction and sense by the diagonal $\overline{AC}$ of the parallelogram (see fig. 4).

Note 1. The parallelogram law can be verified in the laboratory.

2. A proof of the law has been given in chapter V of the Dynamics portion of the book.

3. $\overline{AB}$, $\overline{AC}$, $\overline{AD}$ all represent directed line segments.

§ 2·3. Two forces of magnitudes P and Q acting at a point are inclined with each other at an angle α. To express the resultant force of the two forces in terms of P, Q and α.

Two forcess of magnitudes P and Q are represented by the two adjacent sides $\overline{AB}$ and $\overline{AD}$ of the parallelogram ABCD and $m\angle BAD=\alpha$. Then by the parallelogram law of forces the diagonal $\overline{AC}$ of the parallelogram represents magnitude, direction and sense of the resultant R of the forces P and Q. Since the

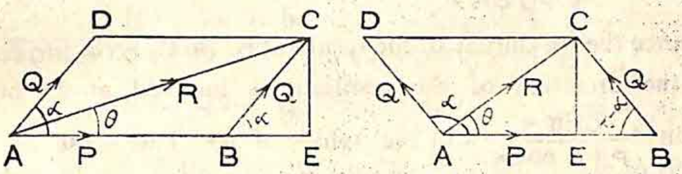


Fig. 4

two directed line segments $\overline{AD}$ and $\overline{BC}$ are equal and parallel, so both of them represent the same force and thus $\overline{BC}$ represents the force Q. From C draw a perpendicular $\overline{CE}$ on $\overline{AB}$.

CE intersects AB produced in fig. (i) and AB in fig. (ii) at E.

Now, in figure (i) ;

$$m\angle CBE=\text{corresponding } \angle DAB=\alpha.$$

In fig. (ii), $m\angle CBE+m\angle DAB=\pi$.

$\therefore \quad m\angle CBE=\pi-m\angle DAB=\pi-\alpha$.

Again, in fig. (i) $CE=BC.\dfrac{CE}{BC}=Q\sin\alpha$

and $BE=BC.\dfrac{BE}{BC}=Q\cos\alpha$.

In fig. (ii) $CE=BC.\dfrac{CE}{BC}=Q\sin(\pi-\alpha)=Q\sin\alpha$.

and $BE=BC.\dfrac{BE}{BC}=Q\cos(\pi-\alpha)=-Q\cos\alpha$.

Hence in fig. (i), $AE=AB+BE=P+Q\cos\alpha$.

and in fig. (ii), $AE=AB-BE=P-(-Q\cos\alpha)$

$$=P+Q\cos\alpha.$$

Now in both figures we obtain from the right-angled $\triangle CAE$,

$$AC^2 = AE^2 + CE^2$$

or, $R^2 = (P + Q\cos\alpha)^2 + (Q\sin\alpha)^2$

$$= P^2 + 2PQ\cos\alpha + Q^2\cos^2\alpha + Q^2\sin^2\alpha$$

$$= P^2 + 2PQ\cos\alpha + Q^2(\cos^2\alpha + \sin^2\alpha)$$

$$= P^2 + Q^2 + 2PQ\cos\alpha.$$

$$\therefore \quad R = \sqrt{P^2 + Q^2 + 2QP\cos\alpha}.$$

Naw, if the resultant force makes an angle θ with the sense of the force P, then in both the figures,

$$\tan\theta = \frac{CE}{AE} = \frac{Q\sin\alpha}{P + Q\cos\alpha},$$

$$\theta = \tan^{-1}\frac{Q\sin\alpha}{P + Q\cos\alpha}.$$

Hence the magnitude of the resultant is $\sqrt{P^2 + Q^2 + 2PQ\cos\alpha}$ and the direction of the resultant is inclined at an angle $\theta = \tan^{-1}\frac{Q\sin\alpha}{P + Q\cos\alpha}$ with the sense of P. The sense of the resultant is from A to C.

Cor. 1. If P and Q are known, then the value of R will be greatest when the value of $\cos\alpha$ is greatest.

Now, we know that the greatest value of $\cos\alpha$ is 1 when $\alpha = 0°$. Hence the greatest value of R is $\sqrt{P^2 + Q^2 + 2PQ.1}$ $= \sqrt{(P+Q)^2} = P + Q$ and then P and Q are inclined at an angle 0° *i.e.*, the forces act along the same line in the same sense.

Cor. 2. If the magnitudes P and Q are known, then the resultant R will be the least in magnitude when $\cos\alpha$ is least. We know that $\cos\alpha$ is least when α is 180° and the least value is -1.

Hence the least value of R is $\sqrt{P^2 + Q^2 + 2PQ(-1)}$ $= \sqrt{(P-Q)^2} = P - Q$ (taking $P > Q$) and that P and Q are inclined at an angle of 180° with each other, *i.e.*, the forces act along the same straight line in opposite senses. In this case, if $P = Q$ *i.e.*, if the forces be equal in magnitude, then the magnitude of the resultant is zero and the forces are in equilibrium. Hence if two equal forces acting at a point act along the same straight line in opposite senses, the forces balance each other.

Example. The greatest and least resultants of two concurrent forces of given magnitudes are 12 kg. and 2 kg. Find the magnitudes of the forces.

Let the forces be P and Q (P>Q).

Hence according to the question,

the greatest resultant $=P+Q=12$ Kg.$\cdots\cdots$(1)

and the least resultant $=P-Q=2$ Kg.$\cdots\cdots$(2)

Solving the equations (1) and (2) we obtain

$P=7$ kg. and $Q=5$ kg.

Cor. 3. If $\alpha=90^\circ$, *i.e.*, if the forces act at right angles, then $\cos\alpha=\cos 90^\circ=0$.

Hence $R^2=P^2+Q^2$, or, $R=\sqrt{P^2+Q^2}$.

Cor. 4. If the forces be equal in magnitude, *i.e.*, if $P=Q$, then $R^2=P^2+P^2+2P.P.\cos\alpha=2P^2+2P^2\cos\alpha$

$$=2P^2(1+\cos\alpha)=2P^2.2\cos^2\frac{\alpha}{2}. \quad \therefore \quad R=2P\cos\frac{\alpha}{2}.$$

Example. If the lines of action of two equal forces be inclined at an angle 2α, then their resultant is R. If the lines of action be inclined at an angle 2β, the resultant is 2R. Prove that $\cos\beta=2\cos\alpha$.

Let each of the two equal forces be of magnitude P.

So, when the forces are inclined at an angle 2α,

their resultant $R=2P\cos\dfrac{2\alpha}{2}=2P\cos\alpha\cdots\cdots$(1)

Again, when the forces are inclined at an angle 2β,

their resultant is $2R=2P\cos\dfrac{2\beta}{2}=2P\cos\beta\cdots\cdots$(2).

Dividing (2) by (1) we get,

$$2=\frac{\cos\beta}{\cos\alpha}, \quad \therefore \quad \cos\beta=2\cos\alpha.$$

Cor. 5. The resultant force is nearer to the greater force.

Let P>Q. Then in fig. 4, AB>BC.

$\therefore$ $m\angle CAB<m\angle ACB$. *i.e.*, $m\angle CAB<m\angle DAC$.

Hence the resultant is nearer to P, i.e., the greater force.

Cor. 6. As $R=\sqrt{P^2+Q^2+2PQ\cos\alpha}$ and the value of $\cos\alpha$ decreases as α increases, so the resultant of two forces of given magnitude decreases as the angle between the forces increases.

§ 2·3. (*a*) Vector Notation.

In fig. 4 as the two directed line segments $\overline{AD}$ and $\overline{BC}$ are equal and parallel having the same sense, so, they will represent the same force. Now from the parallelogram law of forces, we know that the resultant of the forces represented by $\overline{AB}$ and $\overline{AD}$ is represented by $\overline{AC}$. So, $\overline{AC}$ represents the resultant of the forces $\overline{AB}$ and $\overline{BC}$. Hence if two forces acting at a point are represented by the two sides $\overline{AB}$ and $\overline{BC}$ of a triangle taken in order, then their resultant will be represented by $\overline{AC}$ *i.e.*, by the third side taken in the reverse order. In vector notations, this is experessed as $\overline{AB}+\overline{BC}=\overline{AC}$.

Ex. 1. If the magnitude of the resultant of two equal forces of magnitude P acting at a point be also P. Find the angle between the equal forces. [P.U. 1930]

Let the required angle be α.

From the formula, $R^2=P^2+Q^2+2PQ\cos\alpha$,

we get $P^2=P^2+P^2+2P.P\cos\alpha$ [Here each of P, Q, R is P]

$$=2P^2(1+\cos\alpha)$$

or, $\dfrac{P^2}{2P^2}=1+\cos\alpha$, or, $1+\cos\alpha=\frac{1}{2}$

$\therefore\ \cos\alpha=-\frac{1}{2}=\cos 120°$, $\therefore\ \alpha=120°$.

Ex. 2. Prove that the line of action of the resultant of two equal forces acting at a point bisects the angle between the lines of action of the forces.

Let each of the equal forces be P and α be the angle between them. Now if θ be the angle which the line of action of the resultant makes with the line of action of one force P, then

$$\tan\theta=\frac{P\sin\alpha}{P+P\cos\alpha}=\frac{\sin\alpha}{1+\cos\alpha}=\frac{2\sin\frac{\alpha}{2}\cos\alpha}{2\cos^2\frac{\alpha}{2}}=\tan\frac{\alpha}{2}$$

$\therefore\ \theta=\frac{\alpha}{2}$. Hence the line of action of the resultant bisects the angle between the forces.

Alt. In fig. 4. if the equal forces be represented by the sides AB and AD of the parallelogram ABCD, then

$\because$ AB = AD = BC. So from $\triangle$ABC,

$\angle CAB = \angle ACB$. But $\angle ACB$ = alternate $\angle CAD$.

$\therefore$ $\angle CAB = \angle CAD$. So AC, the line of action of the resultant bisects the angle between the forces.

Ex 3. If the square of the resultant of two equal forces acting at a point be equal to twice the product of the forces, then find the angle between the forces.

Let each of the equal forces be P and R be their resultant. So, $R^2 = 2P^2$. If α be the angle between the forces, then

$$R = 2P\cos\frac{\alpha}{2} \quad \therefore \quad 2P^2 = R^2 = 4P^2\cos^2\frac{\alpha}{2}$$

$$\therefore \quad \cos^2\frac{\alpha}{2} = \frac{1}{2} \quad \therefore \quad \cos\frac{\alpha}{2} = \frac{1}{\sqrt{2}} = \cos 45^\circ$$

$$\therefore \quad \frac{\alpha}{2} = 45^\circ \quad \therefore \quad \alpha = 90^\circ.$$

Hence the angle between the forces is a right angle.

Ex. 4. The resultant of two forces P and Q acting at a point is $(2K+1)\sqrt{P^2+Q^2}$, when the forces are inclined at an angle α and the resultant is $(2K-1)\sqrt{P^2+Q^2}$ when the forces are inclined at an angle $90^\circ - \alpha$. Prove that $\tan\alpha = \dfrac{K-1}{K+1}$.

[B. H. U. 1946]

From the formula $R^2 = P^2 + Q^2 + 2PQ\cos\alpha$ we obtain, in the first case,

$$(2K+1)^2(P^2+Q^2) = P^2+Q^2+2PQ\cos\alpha$$

or, $(P^2+Q^2)\{(2K+1)^2-1\} = 2PQ\cos\alpha$

or, $(P^2+Q^2)\{4K^2+4K+1-1) = 2PQ\cos\alpha$

or, $4K(K+1)(P^2+Q^2) = 2PQ\cos\alpha \cdots\cdots(1)$

In the second case,

$$(2K-1)^2(P^2+Q^2) = P^2+Q^2+2PQ\cos(90^\circ-\alpha)$$

or, $(P^2+Q^2)\{(2K-1)^2-1\} = 2PQ\sin\alpha$

or, $(P^2+Q^2)(4K^2-4K+1-1) = 2PQ\sin\alpha$

or, $4K(K-1)(P^2+Q^2) = 2PQ\sin\alpha \cdots\cdots(2)$.

From $(2) \div (1)$, we get

$$\frac{K-1}{K+1} = \frac{\sin\alpha}{\cos\alpha}, \textit{ i.e, } \tan\alpha = \frac{K-1}{K+1}.$$

Ex. 5. Of two forces acting at a point one is double the other and the resultant is inclined at right angles to the smaller force. Find the angle between the forces.

Let the two forces be P and 2P and the required angle be α.

[Here the formula $\tan\theta = \dfrac{Q \sin\alpha}{P + Q\cos\alpha}$ is to be used.

Here $P = P$, $Q = 2P$, $\theta = 90°$, α is to be determined].

According to the problem, $\tan 90° = \dfrac{2P \sin\alpha}{P + 2P\cos\alpha}$;

Now, $\tan 90°$ is undefined. So, $P + 2P\cos\alpha = 0$.

or, $\cos\alpha = \dfrac{-P}{2P} = -\dfrac{1}{2} = \cos 120°$,

$\therefore$ $\alpha = 120°$ *i.e.*, the forces are inclined at an angle 120°.

Ex. 6. The resultant of two concurrent forces P and Q is $\sqrt{3}Q$ and is inclined at an angle 30° with the force P. Show that either $P = Q$ or $P = 2Q$.

From, Trigonometry, we know that

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}.$$

[See Fig. 4] So, in $\triangle ABC$,

$$\cos A = \frac{AC^2 + AB^2 - BC^2}{2AC.AB.}$$

Here $A = 30°$, $AC = \sqrt{3}Q$. $AB = P$ and $BC = Q$.

$\therefore$ $\cos 30° = \dfrac{3Q^2 + P^2 - Q^2}{2\sqrt{3}.Q.P}$, or, $\dfrac{\sqrt{3}}{2} = \dfrac{P^2 + 2Q^2}{2\sqrt{3}PQ}$,

or, $P^2 + 2Q^2 = 3PQ$; or, $P^2 - 3PQ + 2Q^2 = 0$.

or, $(P - Q)(P - 2Q) = 0$. $\therefore$ $P = Q$, or, $P = 2Q$.

Ex. 7. The least resultant of two forces of given magnitude is 31 kg. and the resultant is 41 kg. when the forces are at right angles. Find the manitudes of the forces.

Let the forces be P kg. and Q kg. and $P > Q$.

Hence their least resultant is $P - Q = 31 \cdots\cdots(1)$

Again, when the forces are at right angles, $\sqrt{P^2 + Q^2} = 41$,

or, $P^2 + Q^2 = 41^2 \cdots\cdots(2)$

Now from (1), $(P - Q)^2 = 31^2$, or, $P^2 + Q^2 - 2PQ = 31^2$

or, $41^2 - 2PQ = 31^2$ [From (2)]

or, $2PQ = 41^2 - 31^2 = (41+31)(41-31) = 720.$

$\therefore$ $(P+Q)^2 = P^2 + Q^2 + 2PQ = 41^2 + 720 = 1681 + 720 = 2401.$

$\therefore$ $P+Q = \sqrt{2401} = = 49 \cdots\cdots (3)$

[The magnitude of the resultant force cannot be negative ; so the negative sign is neglected]

Now solving the equations (1) and (3) we obtain $P=40$ and $Q=9$.

Hence the given forces are of magnitude 40 kg. and 9 kg.

Ex. 8. The resultant of two forces of magnitudes P_1 and Q_1 is inclined at right angles to the force P_1. Forces of magnitudes P_2 and Q_2 act respectively along the lines of action of the forces P_1 and Q_1 and their resultant is at right angles to the force Q_2. Show that $P_1P_2 = Q_1Q_2$.

Let the angle between the forces P_1, Q_1 or P_2, Q_2 be α.

Hence in the first case, $\tan 90° = \dfrac{Q_1 \sin \alpha}{P_1 + Q_1 \cos \alpha} \cdots (1)$

and in the second case, $\tan 90° = \dfrac{P_2 \sin \alpha}{Q_2 + P_2 \cos \alpha} \cdots (2)$

Now as tan 90 is undefined,

so from (1), $P_1 + Q_1 \cos \alpha = 0$, or, $P_1 = -Q_1 \cos \alpha \cdots (3)$;

and from (2), $Q_2 + P_2 \cos \alpha = 0$, or, $P_2 \cos \alpha - -Q_2 \cdots (4)$

From (3) $\times$ (4) we get, $P_1P_2 \cos \alpha = Q_1Q_2 \cos \alpha$

or, $P_1P_2 = Q_1Q_2$.

Ex. 9. Two forces acting at a point are inclined to each other at a right angle. The smaller force is 8 lbs. and the sum of their resultant and the larger force is 288 lbs. Find the resultant and the larger force. [*H. S.* '*67* (*Comp.*)]

Let the larger force be P and the resultant be R.

So, $P + R = 288 \cdots (1)$

Again R is the resultant of the forces P and 8 lbs. acting at a right angle.

$\therefore$ $R^2 = P^2 + 8^2$ or $(288 - P)^2 = P^2 + 8^2$ [from (1)]

or, $288^2 + P^2 - 2P.288 = P^2 + 8^2$

or, $288^2 - 8^2 = 2P.288$

or, $(288+8)(288-8) = 2P.288$

or, $P = \dfrac{296.280}{2.288} = 143\frac{8}{9}$ $\therefore$ $R = 288 - 143\frac{8}{9} = 144\frac{1}{9}$.

Again, if the resultant be inclined at an angle θ with the force P, then $\tan\theta = \dfrac{8}{143\frac{8}{9}} = \dfrac{72}{1295}$ $\therefore$ $\theta = \tan^{-1}\dfrac{72}{1295}$.

Ex. 10. Four equal forces each equal to P act along the sides $\overrightarrow{AB}$, $\overrightarrow{CB}$, $\overrightarrow{AD}$ and $\overrightarrow{DC}$ of the square ABCD. Find the resultant of the forces. [*U. P. 1954*]

The resultant of two equal forces P, P acting along $\overrightarrow{AB}$ and $\overrightarrow{AD}$ is a force $P\sqrt{2}$ acting along AC. Again, the resultant of the forces P, P acting along $\overrightarrow{DC}$ and $\overrightarrow{CB}$ is a force $P\sqrt{2}$ acting along $\overrightarrow{CF}$, the bisector of the angle between $\overrightarrow{DC}$ produced and $\overrightarrow{CB}$. Now the required resultant is the resultant of the two equal forces $P\sqrt{2}$, $P\sqrt{2}$ acting along $\overrightarrow{CE}$ and $\overrightarrow{CF}$. Now $\angle ECF = 90°$.

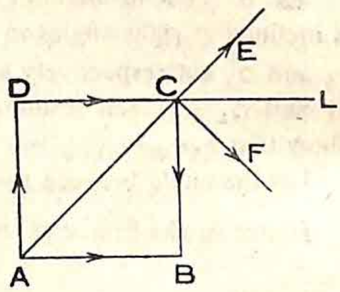


Fig. 5

Hence the required resultant is $2.P\sqrt{2}\cos 45° = 2P$.

The line of action of this resultant will bisect the angle ECF i.e., the resultant acts along $\overrightarrow{DC}$.

Ex. 11. A body is suspended by two cords of equal length attached to two points in the same horizontal line. Show that if the length of the cords be increased equally, then their tension will decrease.

Let T be the tension of each cord and the angle between the cords be α. Since the body is in equilibrium, its weight W = resultant of the two tensions $= 2T\cos\frac{\alpha}{2}$.

$$\therefore \quad T = \frac{W}{2}\sec\frac{\alpha}{2}.$$

Now, if the lengths of the cords be increased equally, then α and hence $\sec\frac{\alpha}{2}$ will both decrease. Hence the tension will also decrease.

Ex. 12. The resultant of two forces P and Q acting at a point and inclined at an angle 60° with each other is R. Prove that $2Q+P=\sqrt{4R^2-3P^2}$.

As R is the resultant of the two given forces P and Q, so

$$R^2=P^2+Q^2+2PQ\cos 60^\circ$$
$$=P^2+Q^2+PQ. \qquad [\because \cos 60^\circ=\tfrac{1}{2}]$$
$$\therefore \quad 4R^2=4P^2+4Q^2+4PQ$$
$$\therefore \quad 4R^2-3P^2=P^2+4Q^2+4PQ^2=(2Q+P)^2$$
$$\therefore \quad 2Q+P=\sqrt{4R^2-3P^2}.$$

Ex. 13. If R be the resultant of two forces P and Q acting at a point at an angle α, show that

$$\tan^2\frac{\alpha}{2}=\frac{(P+Q+R)(P+Q-R)}{(P-Q+R)(Q+R-P)}$$

As R is the resultant of the forces P and Q, so

$$R^2=P^2+Q^2+2PQ\cos\alpha$$

or, $$\cos\alpha=\frac{R^2-P^2-Q^2}{2PQ}$$

or, $$\frac{1-\cos\alpha}{1+\cos\alpha}=\frac{2PQ-R^2+P^2+Q^2}{2PQ+R^2-P^2-Q^2} \quad \text{[By Div. \& Comp.]}$$

or, $$\frac{2\sin^2\frac{\alpha}{2}}{2\cos^2\frac{\alpha}{2}}=\frac{(P+Q)^2-R^2}{R^2-(P-Q)^2}$$

or, $$\tan^2\frac{\alpha}{2}=\frac{(P+Q+R)(P+Q-R)}{(P-Q+R)(R+Q-P)}.$$

Ex. 14. The greatest and least resultant of two forces of constant magnitude acting at a point are F and G. Prove that if the forces be inclined at an angle 2α with each other, then their resultant is $\sqrt{(F^2\cos^2\alpha+G^2\sin^2\alpha)}$. [*G. U.* '67]

Let the constant magnitude of the forces be P and Q ($P>Q$).

Hence their greatest and least resultants are $P+Q$ and $P-Q$ respectively.

$$\therefore \quad F=P+Q \text{ and } G=P-Q.$$

Now, $$F^2\cos^2\alpha+G^2\sin^2\alpha=(P+Q)^2\cos^2\alpha+(P-Q)^2\sin^2\alpha$$
$$=(P^2+Q^2)(\cos^2\alpha+\sin^2\alpha)+2PQ(\cos^2\alpha-\sin^2\alpha)$$
$$=P^2+Q^2+2PQ\cos 2\alpha.$$

Again if R be the resultant of the forces when they are inclined at an angle 2α with each other, then

$$R^2=P^2+Q^2+2PQ\cos 2\alpha.$$

$$\therefore\ R^2=F^2\cos^2\alpha+G^2\sin^2\alpha.$$

i.e., $R=\sqrt{F^2\cos^2\alpha+G^2\sin^2\alpha}$.

Exercise 2A

1. The resultant of two forces of magnitudes P and Q inclined at an angle α is R. If,

(i) $P=6$ kg., $Q=10$ kg. and $\alpha=0°$, find R.

(ii) $P=22$ lbs., $Q=9$ lbs. and $\alpha=90°$, find R.

(iii) $P=10$ kg., $Q=6$ kg. and $R=14$ kg., then find α.

(iv) $P=28$ kg., $R=53$ kg. and $\alpha=90°$, find Q.

2. (a) Two equal forces act at a point. If the square of the resultant of the forces be three times the product of the forces, find the angle between the forces. [*P. U. 1930*]

(*b*) If the resultant of two forces acting on a particle be at right angles to one of them, and its magnitude be one-third of the other, show that the ratio of the larger force to the smaller is $3:2\sqrt{2}$. [*U. P. 1944*]

(*c*) The resultant of two forces P and Q acting at a point is R. If the sense of P is reversed, then the new resultant is perpendicular to R. Show that $P=Q$ (in magnitude) [*H. S. '64*]

(*d*) If one of two forces acting at a point be doubled, then the resultant of the forces is also doubled ; show that the angle between the forces is $\cos^{-1}\left(-\frac{3Q}{4P}\right)$.

(*e*) The resultant of two forces acting at a point is $\sqrt{10}$ lbs when they act at right angles. When they act at an angle 60°, then the resultant becomes $\sqrt{13}$ lbs. Find the magnitudes of the forces. [*U. P. B. 1950*]

3. The least resultant of two forces of given magnitude and acting at a point is 34 kg. and the resultant is 50 kg. when the forces are at right angles. Find the magnitudes of the forces.

4. The greatest and least resultants of two concurrent forces of given magnitudes are 100 kg. and 58 kg. respectively in magnitudes. Find the magnitudes of the forces.

5. (*a*) The magnitude of the greatest resultant of two concurrent forces of given magnitudes is 17 kg. If the forces be inclined at right angles, the magnitude of the resultant becomes 13 kg. Find the magnitudes of the forces.

(*b*) The least resultant of two forces acting at a point is 4 units and when the forces act at right angles to each other, their resultant becomes 20 units. Prove that the greatest resultant of the forces is 28 units. Also find the magnitude of the resultant when the forces are inclined at an angle 60° with each other.

(*c*) The resultant of two forces P and Q acting at a point is perpendicular to P; the resultant of two forces P and Q' acting at the same angle is perpendicular to Q'. Prove that $P^2 = QQ'$.

6. The resultant of two concurrent forces of magnitudes $3P$ and $2P$ is R in magnitude. If the magnitude of the first force is doubled, then that of the resultant is also doubled. Find the angle between the forces. [*C. U. 1932*]

7. The resultant of two forces P and Q in magnitude and inclined at a given angle is also P in magnitude. Prove that if two forces of magnitudes $2P$ and Q be inclined at the same angle, then their resultant is perpendicular to the line of action of Q.

8. The magnitude of the resultant of two forces of magnitudes $P+Q$ and $P-Q$ is $\sqrt{2(P^2+Q^2)}$. Find the angle between the forces.

9. P and R are the magnitudes of one of two forces and their resultant. If the forces be inclined at an angle 60°, show that the magnitude of the other force is,

$$\frac{\sqrt{4R^2-3P^2}-P}{2}.$$

10. The magnitudes of the resultants of two forces P and Q in magnitude are R and mR when the forces are inclined at angles of 120° and 60° respectively, Prove that,

$$P=\frac{R}{2\sqrt{2}}(\sqrt{3m^2-1}+\sqrt{3-m^2})$$

and $$Q=\frac{R}{2\sqrt{2}}(\sqrt{3m^2-1}-\sqrt{3-m^2}).$$

11. The resultant of two forces P and Q acting at a point is R; if the magnitude of the second force be doubled, the resultant is also doubled. Again, if the sense of the second force is reversed then also the resultant is doubled.

Show that $P:Q:R=\sqrt{2}:\sqrt{3}:\sqrt{2}$. [*Bombay, 1934*]

12. R is the resultant of two forces P and Q inclined at an angle α. If the magnitude of each of P and Q is increased by R, show that the tangent of the angle at which the line of action of the new resultant is inclined with the line of action of the old resultant R is $\dfrac{(P-Q)\sin\alpha}{P+Q+R+(P+Q)\cos\alpha}$.

[*P. U. 1943 ; B. H. U. 1943*]

13. If the resultant of the forces P and Q acting at a point be equal to that of $P+S$ and $Q-S$ acting at the same angle $(S\neq Q-P)$, find the magnitude of the resultant.

14. The resultant of two forces P, Q acting at a certain angle is X and that of P, R acting at the same angle is also X. The resultant of Q, R $(Q\neq R)$ acting at the same angle is Y. Show that $P=\sqrt{(X^2+QR)}=\dfrac{QR(Q+R)}{Q^2+R^2-Y^2}$.

Also if $P+Q+R=0$, then show that $X=Y$. [*P. U.*]

15. The resultant of two inter-secting forces P and Q is R, nR and $(n+2)R$ according as their angles of inclination are 90°, θ and $90°-\theta$ respectively. Prove that $(n-1)\tan\theta=n+3$.

16. If the greatest possible resultant of the forces P and Q acting at a point be n times the least, show that the angle α between them when their resultant is half their sum is given by $\cos\alpha=-\dfrac{n^2+2}{2(n^2-1)}$.

17. Of two forces P and Q acting at a point, P is given in magnitude and direction. If the magnitude only of Q is known, find the locus of the extremity of their resultant.

§ 2·4. Breaking up a given force into two Components.

A given force can be resolved into two components in two given directions.

Let R be a given force and $\overrightarrow{AB}$ and $\overrightarrow{AD}$ be two given directions. The force R is to be broken up into two components along $\overrightarrow{AB}$ and $\overrightarrow{AD}$. Let the directed line segment $\overline{AC}$ represent the force R and $m\angle BAD=\alpha$ and $m\angle BAC=\theta$ (See fig. 4). Since $\overrightarrow{AB}$ and $\overrightarrow{AD}$ as well as R are given, so α and θ are known. With $\overline{AC}$ as diagonal and adjacent sides along $\overrightarrow{AB}$ and $\overrightarrow{AD}$. complete the parallelogram ABCD. Now, by the parallelogram of forces R is the resultant of the forces represented by the directed line segments $\overline{AB}$ and $\overline{AD}$. In other words, the directed line segments $\overline{AB}$ and $\overline{AD}$ represent the components of R along $\overrightarrow{AB}$ and $\overrightarrow{AD}$. Let these components be P and Q.

Now in $\triangle ABC$, the lengths of the sides are proportional to the sines of the opposite angles.

$$\therefore \quad \frac{AB}{\sin BCA}=\frac{BC}{\sin BAC}=\frac{AC}{\sin ABC}.$$

$$\text{or,} \quad \frac{P}{\sin(\alpha-\theta)}=\frac{Q}{\sin\theta}=\frac{R}{\sin(\pi-\alpha)}.$$

[$\because$ $m\angle BAD=\alpha$ and $m\angle BAC=\theta$; $\therefore$ $m\angle CAD=\alpha-\theta$.
$\therefore$ $m\angle BCA=\alpha-\theta$.

Again, $\because$ $\overline{AD} \parallel \overline{BC}$, $\therefore$ $m\angle BAD+m\angle ABC=\pi$,

or, $m\angle ABC=\pi-m\angle BAD=\pi-\alpha$].

$$\therefore \quad P=\frac{R\sin(\alpha-\theta)}{\sin\alpha} \text{ and } Q=\frac{R\sin\theta}{\sin\alpha}.$$

Note. As with $\overline{AC}$ as diagonal an infinite number of parallelograms can be drawn, so a given force can be broken up into two components in an infinite number of ways.

Example. Resolve a force of magnitude 200 kg. into components making angles 45° and 60° with the line of action of the given force on opposite sides.

Let the components be P and Q.

Here $R=200$ kg. ; $\theta=45°$ and $\alpha=60°+45°=105°$.

$$\therefore \quad P = \frac{R \sin(\alpha - \theta)}{\sin \alpha} = \frac{200 \times \sin 60^\circ}{\sin 105^\circ}$$

$$= \frac{200 \times \cdot 8660}{\sin 105^\circ} = \frac{200 \times \cdot 8660}{\cos 15^\circ}$$

$$= \frac{200 \times \cdot 8660}{\cdot 9659} = 179 \cdot 32 \text{ kg.}$$

Also $Q = \dfrac{R \sin \theta}{\sin \alpha} = \dfrac{200 \times \sin 45^\circ}{\sin 105^\circ}$

$$= \frac{200 \times \cdot 7071}{\cos 15^\circ} = \frac{200 \times \cdot 7071}{\cdot 9659}$$

$$= 146 \cdot 42 \text{ Kg.}$$

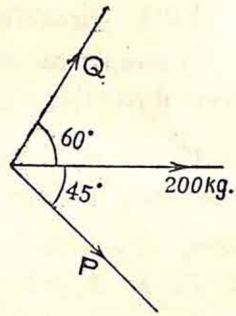


Fig. 6

§ 2·5. Resolved Part.

When the two components of a given force are inclined at a right-angle, then each component is said to be a resolved part of the given force in the direction of the component.

If the lines of action of the components are at right angles, then in the formula of § 2·4, $\alpha = 90^\circ$.

$$\therefore \quad P = \frac{R \sin(\alpha - \theta)}{\sin \alpha}$$

$$= \frac{R \sin(90^\circ - \theta)}{\sin 90^\circ} = R \cos \theta.$$

and $Q = \dfrac{R \sin \theta}{\sin \alpha} = \dfrac{R \sin \theta}{\sin 90^\circ}$

$$= R \sin \theta.$$

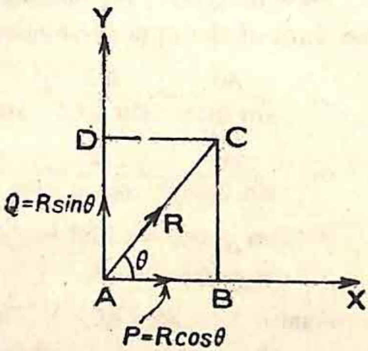


Fig. 7

Hence a given force R can be resolved along a given direction and its perpendicular direction into two resolved parts $R \cos \theta$ and $R \sin \theta$ respectively, clearly, θ is the inclination of the direction of R with the given direction.

Example. A force of magnitude 20 kg. is inclined at an angle of 60° with a given direction. Find the resolved parts of the force in the given direction and its perpendicular direction.

The resolved part of the force in the given direction is $20 \cos 60^\circ = 20 \times \frac{1}{2} = 10$ kg.

The resolved part in the perpendicular direction is

$$20 \sin 60^\circ = 20.\ \frac{\sqrt{3}}{2} = 10\sqrt{3} = 10 \times 1{\cdot}732 = 17{\cdot}32 \text{ Kg.}$$

§ 2·6. **Theorem. The algebraic sum of the resolved parts of any two forces acting at a point in a given direction is equal to the resolved part of their resultant in that direction.**

Let P and Q be two given forces acting at a point and $\overrightarrow{OX}$ a given direction. Represent P and Q by the two directed line segments $\overline{OA}$ and $\overline{OB}$ and complete the parallelogram OACB.

Hence the directed line segment $\overline{OC}$, will represent the resultant R of the forces P and Q.

Draw AL, BM, CN perpendiculars on $\overrightarrow{OX}$.

Hence the directed line segments $\overline{OL}$, $\overline{OM}$ and $\overline{ON}$ will respectively represent the resolved parts of P, Q and R in the direction of $\overrightarrow{OX}$. (As $OL = OA.\frac{OL}{OA} = P \cos \angle AOX$. etc.).

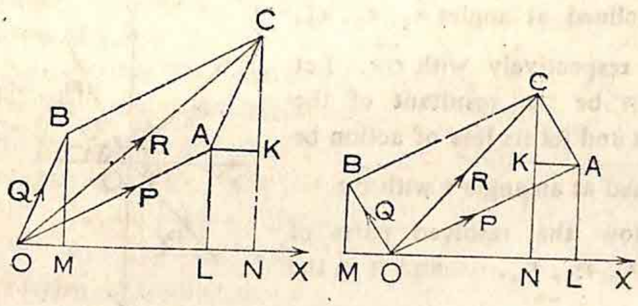


Fig. 7

Now in fig. (i) all the resolved parts are positive in the direction of $\overrightarrow{OX}$ (as the senses of the directed line segments $\overline{OL}$, $\overline{OM}$; $\overline{ON}$ are the same as that of $\overrightarrow{OX}$). In fig. (ii) the resolved part of Q in the direction of $\overrightarrow{OX}$ is negative (as $\overline{OM}$ and $\overrightarrow{OX}$ have opposite senses). In both figures draw $\overline{AK}$ perpendicular to $\overline{CN}$.

Now in both, figures the $\triangle BMO$ and $\triangle CKA$ are congruent.

$\therefore \quad OM = AK = LN.$

Now in fig. (i), $ON = OL + LN = OL + OM$

and in fig. (ii), $ON = OL - LN = OL - MO = OL - (-OM)$

$= OL + OM.$

Hence the algebraic sum of the resolved parts of P and Q in the direction of $\overrightarrow{OX}$ is equal to the resolved part of their resultant R in the same direction.

Corollary : On repeated application of the above theorem one can prove that the algebraic sum of the resolved parts of any finite number of forces acting at a point in any given direction is equal to the resolved part of their resultant in the same direction.

§ 2·7. Determination of the resultant of any finite number of co-planar forces acting at a point.

Let O be the point of application of the co-planar concurrent forces $P_1, P_2, P_3, P_4, \cdots\cdots$. We are to determine the resultant of these forces.

Through O draw two straight lines $\overrightarrow{XX'}$ and $\overrightarrow{YY'}$, at right angles to each other. Let the lines of action of $P_1, P_2, P_3, P_4, \cdots$ be inclined at angles $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \cdots$ respectively with $\overrightarrow{OX}$. Let also R be the resultant of the forces and let its line of action be inclined at an angle θ with $\overrightarrow{OX}$.

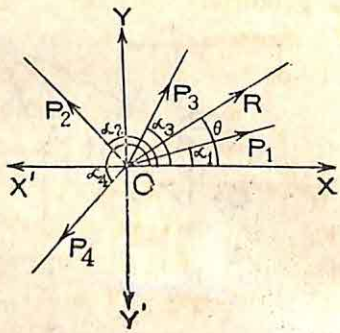


Fig. 8

Now the resolved parts of $P_1, P_2, P_3, P_4, \cdots\cdots$ and R in the direction $\overrightarrow{OX}$ are $P_1 \cos \alpha_1$, $P_2 \cos \alpha_2$, $P_3 \cos \alpha_3$, $P_4 \cos \alpha_4, \cdots$ and $R \cos \theta$ respectively. Again the resolved parts of $P_1, P_2, P_3, P_4, \cdots\cdots$ and R in the direction $\overrightarrow{OY}$ are $P_1 \sin \alpha_1$, $P_2 \sin \alpha_2$, $P_3 \sin \alpha_3$, $P_4 \sin \alpha_4, \cdots\cdots$ and $R \sin \theta$ respectively.

Hence, $R \cos \theta = P_1 \cos \alpha_1 + P_2 \cos \alpha_2 + P_3 \cos \alpha_3 + P_4 \cos \alpha_4 + \cdots = X \text{ (say)} \cdots\cdots (1)$

and $R \sin \theta = P_1 \sin \alpha_1 + P_2 \sin \alpha_2 + P_3 \sin \alpha_3 + P_4 \sin \alpha_4 + \cdots = Y \text{ (say)} \cdots\cdots (2).$

Now squaring equations (1) and (2) and then adding the results we get $R^2 = X^2 + Y^2 \cdots\cdots(3)$. $\therefore \quad R = \sqrt{X^2+Y^2}$

Again, dividing (2) by (1), we get, $\tan\theta = \frac{Y}{X}$, or, $\theta = \tan^{-1}\frac{Y}{X}$.

Corollary: If the forces be in equilibrium, then the magnitude of the resultant force will be zero *i.e.*, we shall get $R=0$.

Hence from (3), we obtain $X=0=Y$, *i.e.*, the algebraic sum of the resolved parts of the forces in the directions $\overrightarrow{OX}$ and $\overrightarrow{OY}$ will be separately zero.

Example. 1. P, Q, R are three coplanar concurrent forces and α, β, γ are the angles between (Q, R), (R, P) and (P, Q) respectively. Prove that the magnitude of the resultant of the forces is

$$\sqrt{P^2+Q^2+R^2+2QR\cos\alpha+2RP\cos\beta+2PQ\cos\gamma}.$$

Let the required resultant be F and its line of action be inclined at an angle θ with the direction of P. Resolving the forces in the direction of P and its perpendicular direction, we get by the theorem of § 2·6,

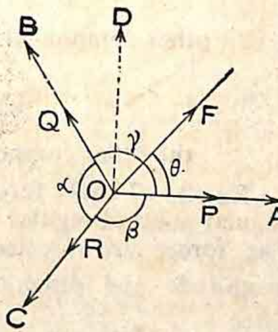


Fig. 9

$$\begin{aligned} F\cos\theta &= P+Q\cos\gamma+R\cos(\alpha+\gamma) \\ &= P+Q\cos\gamma+R\cos(2\pi-\beta) \\ &= P+Q\cos\gamma+R\cos\beta\,; \end{aligned}$$

$$\begin{aligned} \text{and } F\sin\theta &= Q\sin\gamma+R\sin(\alpha+\gamma) \\ &= Q\sin\gamma+R\sin(2\pi-\beta) \\ &= Q\sin\gamma-R\sin\beta. \end{aligned}$$

$$\begin{aligned} \therefore\quad & F^2\cos^2\theta+F^2\sin^2\theta \\ &= (P+Q\cos\gamma+R\cos\beta)^2 \\ &\quad +(Q\sin\gamma-R\sin\beta)^2 \end{aligned}$$

$$\begin{aligned} \text{or,}\quad F^2(\cos^2\theta+\sin^2\theta) &= P^2+Q^2\cos^2\gamma+R^2\cos^2\beta \\ &\quad +2QR\cos\beta\cos\gamma+2RP\cos\beta+2PQ\cos\gamma \\ &\quad +Q^2\sin^2\gamma+R^2\sin^2\beta-2QR\sin\beta\sin\gamma, \end{aligned}$$

$$\begin{aligned} \text{or,}\quad F^2 &= P^2+Q^2(\cos^2\gamma+\sin^2\gamma)+R^2(\cos^2\beta+\sin^2\beta) \\ &\quad +2QR(\cos\beta\cos\gamma-\sin\beta\sin\gamma) \\ &\quad +2RP\cos\beta+2PQ\cos\gamma \end{aligned}$$

$$=P^2+Q^2+R^2+2QR\cos(\beta+\gamma)+2RP\cos\beta+2PQ\cos\gamma$$
$$=P^2+Q^2+R^2+2QR\cos(2\pi-\alpha)+2PR\cos\beta+2PQ\cos\gamma$$
$$=P^2+Q^2+R^2+2QR\cos\alpha+2RP\cos\beta+2PQ\cos\gamma$$
$$\therefore\quad F=\sqrt{P^2+Q^2+R^2+2QR\cos\alpha+2RP\cos\beta+2PQ\cos\gamma}.$$

Ex. 2. A force 400 kg. acting vertically upwards is broken up into two components. One of the components, of magnitude 200 kg. acts in the horizontal direction. Find the magnitude and direction of the other component.

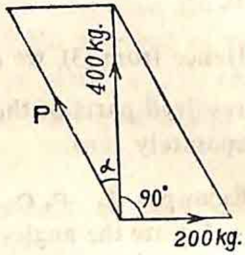


Fig. 10

Let the other component be P, inclined at an angle α with the vertical direction.

$$\therefore\quad 200=\frac{400\sin\alpha}{\sin(\alpha+90^\circ)}$$

(according to § 2·4).

$$=\frac{400\sin\alpha}{\cos\alpha}=400\tan\alpha.$$

$$\therefore\quad \tan\alpha=\frac{200}{400}=\frac{1}{2}.\quad\therefore\quad \cos\alpha=\frac{2}{\sqrt{5}}.$$

$$\therefore\ \text{other component } P=\frac{400\sin 90^\circ}{\sin(90^\circ+\alpha)}=\frac{400}{\cos\alpha}=\frac{400}{\frac{2}{\sqrt{5}}}$$
$$=200\sqrt{5}\ \text{kg}.$$

$\therefore$ the other component is $200\sqrt{5}$ kg.

Ex. 3. 7 equal forces each of magnitude 100 kg. act at an angular point of regular octagon, one side of which is horizontal. The forces are directed to the other angular points. Find the magnitude and direction of the resultant force.

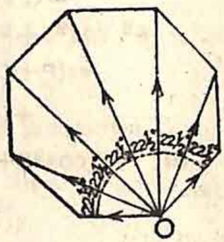


Fig. 12

Each internal angle of a regular octagon is of magnitude $\frac{(16-4)90^\circ}{8}=135^\circ$.

Hence each force is inclined to the next one at an angle of $\frac{135^\circ}{6}$ or, $22\frac{1}{2}^\circ$.

Let the resultant of the forces be F inclined at an angle θ with the horizontal side of the octagon. Hence resolving the forces along and perpendicular to the horizontal side we get

$$\text{F}\cos\theta=100\cos 0^\circ+100\cos 22\tfrac{1}{2}^\circ+100\cos 45^\circ +100\cos 67\tfrac{1}{2}^\circ+100\cos 90^\circ+100\cos 112\tfrac{1}{2}^\circ +100\cos 135^\circ$$

$$=100(1+\cdot 9239+\cdot 7071+\cdot 3827+0-\cdot 3827-\cdot 7071)$$
$$=100\times 1\cdot 9239=192\cdot 39 \text{ kg.}$$

and $$\text{F}\sin\theta=100\sin 0^\circ+100\sin 22\tfrac{1}{2}^\circ+100\sin 45^\circ +100\sin 67\tfrac{1}{2}^\circ+100\sin 90^\circ+100\sin 112\tfrac{1}{2}^\circ+100\sin 135^\circ$$
$$=100(0+\cdot 3827+\cdot 7071+\cdot 9239+1+\cdot 9239+\cdot 7071)$$
$$=100(4\cdot 6447)=464\cdot 47 \text{ kg. wts.}$$

$$\therefore\quad \text{F}=\sqrt{(100\times\cdot 19239)^2+(100\times 4\cdot 6447)^2} \text{ kg. wts.}$$
$$=100\times\sqrt{25\cdot 2648}=100\times 5\cdot 0266=502\cdot 66 \text{ kg. wts.}$$

and $\tan\theta=\dfrac{100\times 4\cdot 645}{100\times 1\cdot 9239}=2\cdot 41\quad\therefore\quad \theta=67\cdot 5^\circ$ (nearly).

Ex. 4. R is the resultant of two forces P and Q acting at a point. A transversal cuts the lines of action of P, Q and R at L, M and N respectively. Prove that

$$\frac{\text{P}}{\text{OL}}+\frac{\text{Q}}{\text{OM}}=\frac{\text{R}}{\text{ON}}. \qquad \text{[C. U.]}$$

From O, draw $\overline{\text{OK}}$, perpendicular to $\overleftrightarrow{\text{LM}}$.

Now the algebraic sum of the resolved parts of P and Q in the direction $\overrightarrow{\text{OK}}$ is equal to the resolved part of R in the same direction.

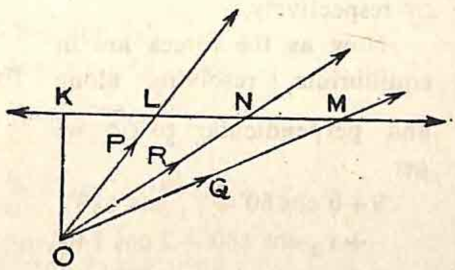


Fig. 12

$\therefore$ P cos KOL + Q cos KOM = R cos KON.

or, $\text{P}.\dfrac{\text{OK}}{\text{OL}}+\text{Q}.\dfrac{\text{OK}}{\text{OM}}=\text{R}.\dfrac{\text{OK}}{\text{ON}}$, or, $\dfrac{\text{P}}{\text{OL}}+\dfrac{\text{Q}}{\text{OM}}=\dfrac{\text{R}}{\text{ON}}$.

Ex. 5. Prove that a force acting in the plane of a triangle can be resolved into three componsnts acting along the sides, of the triangle.

[Construct the figure yourself]. Let F be a force acting in the plane of a triangle and its line of action intersect $\overline{\text{BC}}$ at D. Join AD. Now F can be resolved into two components P and F_1 along $\overrightarrow{\text{DC}}$ and $\overrightarrow{\text{DA}}$ respectively.

Again, F_1 can be resolved into components Q and R along $\overrightarrow{AC}$ and $\overrightarrow{AB}$ respectively.

Hence the given force F can be resolved into components P, $-Q$ and R along $\overrightarrow{BC}$, $\overrightarrow{CA}$ and $\overrightarrow{AB}$ respectively.

If the force does not intersect $\overleftrightarrow{BC}$, it will intersect $\overleftrightarrow{AC}$ or $\overleftrightarrow{AB}$ and we can proceed similarly.

Ex. 6. A ring placed at the centre of a regular hexagon has been kept in equilibrium by six cords tightly attached to the six angular points of the hexagon. The tensions in four consecntive cords are 2, 7, 9 and 6 lbs. wt. Find the tensions of the other two cords.

Let the hexagon be $ABCDEF$ and the ring is placed at its centre O. The tensions of the cords $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$ and $\overrightarrow{OD}$ are respectively 2, 7, 9 and 6 lbs wt.

Let T_1 and T_2 be the tensions of the cords OE and OF respectively.

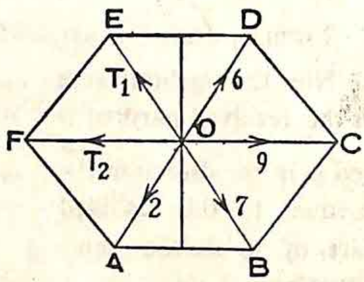


Fig. 13

Now as the forces are in equilibrium, resolving along and perpendicular to $\overrightarrow{OC}$ we get,

$$9+6\cos 60^\circ+T_1\cos 120^\circ+T_2\cos 180^\circ+2\cos 240^\circ+7\cos 300^\circ=0.$$

or, $9+6.\frac{1}{2}+T_1(-\frac{1}{2})+T_2.(-1)+2.(-\frac{1}{2})+7.\frac{1}{2}=0.$

or, $14\frac{1}{2}-\frac{T_1}{2}-T_2=0$ or, $T_1+2T_2=29\cdots(1).$

and $0+6\sin 60^\circ+T_1\sin 120^\circ+T_2\sin 180^\circ+2\sin 240^\circ+7\sin 300^\circ=0$

or, $6.\frac{\sqrt{3}}{2}+T_1.\frac{\sqrt{3}}{2}+T_2.0+2\left(-\frac{\sqrt{3}}{2}\right)+7.\left(-\frac{\sqrt{3}}{2}\right)=0$

or, $T_1\frac{\sqrt{3}}{2}=\frac{3}{2}\sqrt{3}$ or, $T_1=3$ lbs wt.

So, from (1) we get $2T_2=26$ or, $T_2=13$ lbs wt.

Hence the tensions of the other two cords are 3 lbs wt. and 13 lbs wt.

Ex. 7. If the line of action of the resultant of two forces P and Q acting at a point ($P>Q$) divides the angle between them in the ratio 1 : 2, show that the magnitude of their resultant is $\frac{P^2-Q^2}{Q}$ and the angle between them is $3\cos^{-1}\left(\frac{P}{2Q}\right)$.

[B. H. U. ,47]

Let the angle between the forces P and Q be 3α. Hence by question, the resultant R of the forces is inclined at angles α and 2α with P and Q respectively.

[Note as $P>Q$, so R is nearer to P].

$\therefore$ As in § 2·4

$$\frac{P}{\sin 2\alpha}=\frac{Q}{\sin\alpha}=\frac{R}{\sin 3\alpha}$$

$$\therefore \frac{P}{Q}=\frac{\sin 2\alpha}{\sin\alpha}=\frac{2\sin\alpha\cos\alpha}{\sin\alpha}=2\cos\alpha.$$

$$\therefore \cos\alpha=\frac{P}{2Q}\cdots(1)$$

Hence
$$\begin{aligned}R^2&=P^2+Q^2+2PQ\cos 3\alpha\\&=P^2+Q^2+2PQ(4\cos^3\alpha-3\cos\alpha)\\&=P^2+Q^2+2PQ\left(\frac{P^3}{2Q^3}-\frac{3P}{2Q}\right)\\&=P^2+Q^2+\frac{P^4}{Q^2}-3P^2=Q^2+\frac{P^4}{Q^2}-2P^2\\&=\frac{Q^4+P^4-2P^2Q^2}{Q^2}=\left(\frac{P^2-Q^2}{Q}\right)^2\end{aligned}$$

$$\therefore \quad R=\frac{P^2-Q^2}{Q}.$$

Again from (1) $\alpha=\cos^{-1}\frac{P}{2Q}$.

$\therefore$ The angle between the forces P and Q

$$=3\alpha=3\cos^{-1}\left(\frac{P}{2Q}\right).$$

Ex. 8. Two forces P and Q acting along two straight lines making an angle θ with each other, have a resultant R ; two other forces P' and Q' acting along the same lines have a

resultant R$'$. Prove that if ϕ be the angle between the lines of action of R and R$'$, then

$$\cos\phi = \frac{PP' + QQ' + (PQ' + P'Q)\cos\theta}{RR};$$

$$\sin\phi = \frac{(PQ' - P'Q)\sin\theta}{RR'}.$$

Let the lines of action of R and R$'$ be inclined at angles α and α' with the line of action of P or P. As R is the resultant of the two forces P and Q acting at a point, so resolving along and perpendicular to the line of action of P we get,

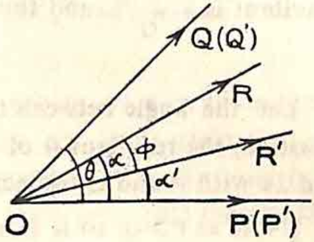


Fig. 14

$$R\cos\alpha = P + Q\cos\theta \cdots (1)$$

$$\text{and } R\sin\alpha = Q\sin\theta \cdots\cdots\cdots (2)$$

Again as R$'$ is the resultant of the forces P$'$ and Q$'$ acting at a point, so resolving the forces along and perpendicular to the line of action of P$'$ we get

$$R'\cos\alpha' = P' + Q'\cos\theta \cdots (3)$$

$$R'\sin\alpha' = Q'\sin\theta \cdots\cdots\cdots (4)$$

Now the angle ϕ between the lines of action of R and R$'$ is given by $\phi = \alpha - \alpha'$.

$$\therefore \quad \cos\phi = \cos(\alpha - \alpha') = \cos\alpha\cos\alpha' + \sin\alpha\sin\alpha'$$

$$= \frac{P + Q\cos\theta}{R} \cdot \frac{P' + Q'\cos\theta}{R'} + \frac{Q\sin\theta}{R} \cdot \frac{Q'\sin\theta}{R'}$$

$$= \frac{1}{RR'}\{PP' + (PQ' + P'Q)\cos\theta + QQ'\cos^2\theta + QQ'\sin^2\theta\}.$$

$$= \frac{PP' + QQ' + (PQ' + P'Q)\cos\theta}{RR'}$$

$$[\because \; QQ'\cos^2\theta + QQ'\sin^2\theta = QQ'(\cos^2\theta + \sin^2\theta) = QQ']$$

Again $\sin\phi = \sin(\alpha - \alpha') = \sin\alpha\cos\alpha' - \cos\alpha\sin\alpha'$

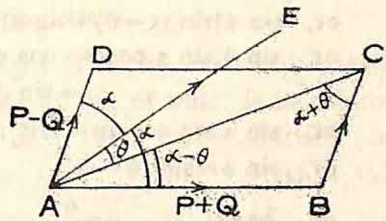
$$= \frac{Q\sin\theta}{R} \cdot \frac{P' + Q'\cos\theta}{R'} - \frac{Q'\sin\theta}{R'} \cdot \frac{P + Q\cos\theta}{R}$$

$$= \frac{(PQ' \quad P'Q)\sin\theta + QQ'(\cos\theta\sin\theta - \sin\theta\cos\theta)}{RR'}$$

$$= \frac{(PQ' - P'Q)\sin\theta}{RR'}.$$

Ex. 9. The line of action of the resultant of two forces $P+Q$ and $P-Q$ acting at a point is inclined at an angle θ with the bisector of the included angle of measure 2α between the forces. Prove that $P \tan \theta = Q \tan \alpha$. [P. U. '31]

Let the sides $\overline{AB}$ and $\overline{AD}$ of the parallelogram $ABCD$ represent respectively the forces $P+Q$ and $P-Q$. $\therefore$ the diagonal $\overline{AC}$ of the parallelogram will represent the resultant of the forces.



Let, $\overrightarrow{AE}$ be the bisector of $\angle BAD$.

$\therefore$ $m\angle BAE = \alpha$ and $m\angle CAE = \theta$.

Fig. 15

Since the resultant is nearer to the greater force, so $\overrightarrow{AE}$ is situated within $\angle CAD$.

$\therefore$ $m\angle BAC = \alpha - \theta$ and $m\angle CAD = \alpha + \theta$. So $m\angle ACB = \alpha + \theta$.

Now, according to the formula of § 2·4.

$$\frac{P+Q}{\sin(\alpha + \ )} = \frac{P-Q}{\sin(\alpha - \theta)}, \quad \text{or,} \quad \frac{P+Q}{P-Q} = \frac{\sin(\alpha+\theta)}{\sin(\alpha-\theta)}$$

$$\text{or,} \quad \frac{P+Q+P-Q}{P+Q-P+Q} = \frac{\sin(\alpha+\theta)+\sin(\alpha-\theta)}{\sin(\alpha+\theta)-\sin(\alpha-\theta)}$$

[By compo. & Divi.]

$$\text{or,} \quad \frac{P}{Q} = \frac{2\sin\alpha\cos\theta}{2\cos\alpha\sin\theta} = \frac{\tan\alpha}{\tan\theta}, \quad \therefore \quad P\tan\theta = Q\tan\alpha.$$

Ex. 10. If the resultant R of two forces P and Q, inclined at an angle of given measure, be inclined at an angle θ with the force P, then prove that the resultant of forces $P+R$ and Q inclined at the same angle will make an angle $\frac{1}{2}\theta$ with the direction of the force $P+R$. [B. U. '26, '29; B. U. '32]

Let α be the angle between the forces. Hence by § 2·4,

$$\frac{Q}{\sin\theta} = \frac{P}{\sin(\alpha-\theta)} = \frac{R}{\sin\alpha} = \frac{P+R}{\sin(\alpha-\theta)+\sin\alpha} \quad \ldots\ldots(1)$$

Let the resultant of the forces $P+R$ and Q be inclined with the force $P+R$ at an angle ϕ,

$$\text{Hence by § 2·4,} \quad \frac{Q}{\sin\phi} = \frac{P+R}{\sin(\alpha-\phi)} \quad \ldots\ldots(2)$$

Now, from (1) $\dfrac{P+R}{Q}=\dfrac{\sin(\alpha-\theta)+\sin\alpha}{\sin\theta}$.

and, from (2) $\dfrac{P+R}{Q}=\dfrac{\sin(\alpha-\phi)}{\sin\phi}$.

$\therefore \quad \dfrac{\sin(\alpha-\theta)+\sin\alpha}{\sin\theta}=\dfrac{\sin(\alpha-\phi)}{\sin\phi}$.

or, $\sin\phi\{\sin(\alpha-\theta)+\sin\alpha\}=\sin\theta\sin(\alpha-\phi)$

or, $\sin\phi\{\sin\alpha\cos\theta-\cos\alpha\sin\theta+\sin\alpha\}$
$$=\sin\theta\,(\sin\alpha\cos\phi-\cos\alpha\sin\phi)$$

or, $\sin\alpha\sin\phi=\sin\alpha\,(\sin\theta\cos\phi-\cos\theta\sin\phi)$

or, $\sin\phi=\sin(\theta-\phi) \quad \therefore \quad \phi=\theta-\phi$

or, $2\phi=\theta \quad \therefore \quad \phi=\dfrac{\theta}{2}$.

Hence the resultant of the forces $P+R$ and Q is inclined at an angle $\frac{1}{2}\theta$ with the force $P+R$.

Exercise 2B

1. Forces of magnitude 2 kg. wts., 4 kg. wts., 6 $\sqrt{3}$ kg. wts. and 8 kg. wts. act at a point. The angles included between the first and the second; the second and the third, the third and the fourth forces are 60°, 90° and 150° respectively. Find the magnitude and direction of their resultant.

2. Find the components of a force of magnitude 20 kg. in the directions making angles 30° and 45° with the direction of the force.

3. A force 300 kg. acting vertically upwards is resolved into two components. The magnitude of a component is 150 kg. and its line of action is in the horizontal direction. Find the magnitude and direction of the other component.

4. A truck-car is at rest on a rail line. A horizontal force of magnitude 100 lbs. is now pulling the car in a direction making angle 60° with the rail line. Find the force which will pull the car in the direction of the rail line.

5. The line of action of a force of magnitude 100 kg. bisects the angle between two given straight lines. If the angle between

the straight lines be 60°, (i) find the components and (ii) resolved parts of the force in the direction of those straight lines.

6. A force 50 kg. is acting towards north. The force is broken up into three components, one of magnitude $25\sqrt{2}$ kg. in the north-west dlrection, a second of magnitude 35 kg. towards west. Find the third component.

7. A force equal to the weight of 20 lbs. acting vertically upwards is resolved into two forces, one of which is horizontal and equal to the weight of 10 lbs. Find the magnitude and direction of the other component. [H. S. '66]

8. Three forces of magnitudes 1, 2 and $\sqrt{3}$ act at a point along the sides $\overrightarrow{AB}$, $\overrightarrow{AC}$ and the perpendicular $\overrightarrow{DA}$ on the side BC of an equilateral triangle ABC. Find the resultant of the forces.

9. Forces 2, $\sqrt{3}$, 5, $\sqrt{3}$ and 2 lbs. wt. respectively act at one of the angular points of a regular hexagon towards the five other angular points. Find the direction and magnitude of the resultant of the forces. [H. S. '73]

10. Three coplanar, concurrent forces of magnitudes P, 2P and 3P are inclined at angles of 120° with one another. Find the magnitude and direction of the resultant of the forces. [C. U.]

11. Forces $R-S$, R, $R+S$ act at a point in directions parallel to the sides of an equilateral triangle taken in order. Find their resultant.

12. A force F is resolved into two components ; if one of them be equal to F and act in a direction perpendicular to F, find the magnitude and direction of the other component.

12. (*a*). The resolved part of the resultant R of two forces P and Q in the direction of P is of magnitude Q. Show that the angle between the forces is $2\sin^{-1}\sqrt{\frac{P}{2Q}}$.

13. One of two forces acting on a particle is double of the other. The resultant of the forces make an angle θ with the greater force. Show that $\theta \ngtr \frac{\pi}{6}$.

14. A transversal cuts the lines of action OA_1, OA_2, OA_3, $\cdots$, OA_n of forces $P_1, P_2, P_3, \cdots, P_n$ at the points $A_1, A_2, A_3 \cdots, A_n$ respectively. If the forces are in equilibrium, show that $\frac{P_1}{OA_1}+\frac{P_2}{OA_2}+\frac{P_3}{OA_3}+\cdots+\frac{P_n}{OA_n}=0.$

15. Of two forces acting on a particle, one is doubled and the magnitude of the other is increased by R. Prove that if the direction of the resultant force remains unaltered, then the magnitude of the second force is R.

16. (*a*). Three forces, each of magnitude P, act on a particle in the direction and sense of the sides of a triangle taken in order. Prove that the magnitude of the resultant of the forces is

$$P\sqrt{3-2\cos A-2\cos B-\cos C}.$$

(*b*) Forces act through the angular points of a triangle perpendicular to the opposite sides and are proportional to the cosines of the corresponding angles, show that the resultant is proportional to

$$\sqrt{1-8\cos A\cos B\cos C}.$$

(*c*) Forces P, Q, R act at a point along the sides of an equilateral triangle taken in order ; show that the resultant of the forces is $\sqrt{(P^2+Q^2+R^2-2QR-2RP-2PQ)}$.

17. The length of each side of a regular hexagon is *a*. Five forces acting at the point A can be represented by $\overline{AB}$, $\overline{AC}$, $2\overline{AD}$, $5\overline{AE}$ and $4\overline{AF}$. Find their resultant.

18. A, B, C are three points on the circumference of a circle. Two forces inversely proportional to AB and BC act along $\overrightarrow{AB}$ and $\overrightarrow{BC}$. Prove that the line of action of the resultant force is tangent to the circle at the point B. [U. P. 1941]

19. The resultant F of two forces P and Q inclined at an angle $\alpha(\neq\pi)$ is inclined at an angle θ with the line of action of P. The resultant F′ of two forces P and Q′, acting respectively along the same lines as P and Q, is inclined at an angle ϕ with the line of action of P. Prove that $F'\sin(\alpha-\phi)=F\sin(\alpha-\theta)$.

20. F is the resultant of two forces R and S acting along two given straight lines inclined at an angle θ. F′ is the resultant of two forces R′ and S′ respectively acting along the same two straight lines. If ϕ be the angle between the lines of action of F and F′, then show that $(1-\cos\phi)(1+\cos\phi)$

$$=\frac{(R^2S'^2-2RR'SS'+R'^2S^2)(1-\cos\theta)(1+\cos\theta)}{F^2F'^2}.$$

[C. U. 1946]

21. Two forces P and Q are inclined at an angle θ. If the lines of action of P and Q are interchanged, show that the line of action of the resultant is rotated through an angle ϕ such that

$$\tan\frac{\phi}{2}=\frac{P-Q}{P+Q}\tan\frac{\theta}{2}.$$

§ 2·8. **The lines of action of two forces acting at a point O are along $\overrightarrow{OA}$ and $\overrightarrow{OB}$ and their magnitudes are $m.OA$ and $n.OB$ respectively. The segment $\overline{AB}$ is divided at C in the ratio $n : m$. Join OC. Prove that the line of action and magnitude of the resultant of P and Q are $\overrightarrow{OC}$ and $(m+n).$ OC respectively.**

[A force of magnitude, *m* times the magnitude of the force represented by $\overline{AB}$ and acting along $\overrightarrow{AB}$ is expressed as $m\,\overline{AB}$.]

Through O draw a straight line XY parallel to $\overrightarrow{AB}$. Draw $\overline{AX}$ and $\overline{BY}$ parallel to CO intersecting $\overleftrightarrow{XY}$ at X and Y respectively.

Hence XOCA and YOCB are two parallelograms.

Now, according to the parallelogram law of forces, the force represented by the directed line segment $\overline{OA}$ can be resolved into components represented by $\overline{OC}$ and $\overline{OX}$. Hence the force $m.\overline{OA}$ can be resolved into components $m.\overline{OC}$ and $m.\overline{OX}$.

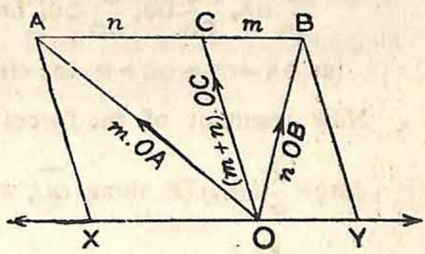


Fig. 16

Similarly, the force $n.\overline{CB}$ can be resolved into conponents $n.\overline{OC}$ and $n.\overline{OY}$. Hence the esultant of the forces P and Q is the same as the resultant of

the forces $m.\overline{OC}$, $m.\overline{CX}$, $n.\overline{OC}$ and $n.\overline{OY}$ along $\overrightarrow{OC}$, $\overrightarrow{OX}$, $\overrightarrow{OC}$, $\overrightarrow{OY}$ respectively.

Now, as $\frac{AC}{BC}=\frac{n}{m}$, $\therefore$ $m.AC=n.BC$, or $m.OX=n.OY$.

Hence the forces $m.\overline{OX}$ and $n.\overline{OY}$ balance each other (for, they have the same line of action but opposite senses). Hence the resultant of the forces P and Q is the resultant of the forces $m.\overline{OC}$ and $n.\overline{OC}$ *i.e.*, $(m+n).\overline{OC}$.

Cor. In the above theorem putting $m=n=1$ we get, if C be the middle point of $\overline{AB}$, then the resultant of forces represented by $\overline{OA}$ and $\overline{CB}$ along $\overrightarrow{OA}$ and $\overrightarrow{OB}$ is a force along $\overrightarrow{OC}$ represented by $2.\overline{OC}$.

Ex. 1. If G be the centroid of $\triangle ABC$, the resultant of forces acting at the point O and represented by $\overline{OA}$, $\overline{OB}$, $\overline{OC}$ is a force acting at O and represented by $3.\overline{OG}$.

The resultant of forces represented by $\overline{OB}$ and $\overline{OC}$ is a force represented by $2.\overline{OD}$, where $\frac{BD}{CD}=\frac{1}{1}$, or, $BD=CD$ *i.e.*, D is the middle point of $\overline{BC}$. Again, the resultant of forces represented by $\overline{OA}$ and $2.\overline{OD}$ is a force represented by $(1+2).\overline{OG}=3.\overline{CG}$, where G is a point on AD such that $\frac{AG}{DG}=\frac{2}{1}$ *i.e.*, the point G is the centroid of the triangle ABC.

Ex. 2. O is the circum-centre of the triangle ABC. Prove that the resultant of forces along $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$ and respectively proportional to the lengths BC, CA, AB passes throught the in-centre of the triangle.

Let $BC=a$, $CA=b$, $AB=c$ and the forces be $K.a$, $K.b$ and $K.c$ in magnitude.

Now these magnitudes can be written as

$$\frac{Ka}{OA}.OA,\ \frac{K.b}{OB}.OB,\ \frac{Kc}{OC}.OC,\ i.e.,\ \frac{Ka}{R}.OA,\ \frac{Kb}{R}.OB \text{ and } \frac{Kc}{R}.OC$$

[as $OA=OB=OC=R$, the circum-radius of the triangle.]

Now resultant of the forces $\frac{K.b}{R}.\overline{OB}$ and $\frac{Kc}{R}.\overline{OC}$ acting at O is a force $\frac{K}{R}(b+c).\overline{OD}$ along $\overrightarrow{OD}$, where D is a point on BC such that $\frac{BD}{CD}=\frac{\frac{Kc}{R}}{\frac{Kb}{R}}=\frac{c}{b}=\frac{AB}{AC}$.

Hence if AD is joined, then $\overrightarrow{AD}$ is the bisector of $\angle BAC$.

Again, the resultant of forces $\frac{K}{R}(b+c).\overline{OD}$ and $\frac{K}{R}.a.\overline{OA}$ acting along $\overline{OD}$ and OA respectively is a force $\frac{K}{R}(a+b+c).\,\overline{OI}$ acting at O along $\overrightarrow{OI}$, where I on $\overline{AD}$ is such that $\frac{AI}{DI}=\frac{\frac{K(b+c)}{R}}{\frac{Ka}{R}}=\frac{b+c}{a}$.

Now, $\frac{AB}{BD}=\frac{AC}{CD}=\frac{AB+AC}{BD+CD}=\frac{c+b}{BC}=\frac{c+b}{a}$.

$\therefore \frac{AI}{DI}=\frac{AB}{BD}$. $\therefore$ $\overrightarrow{AI}$ is the bisector of $\angle ABC$.

So, I is the in-centre of $\triangle ABC$ and the resultant passes through the in-centre of the triangle.

Ex. 3. I is the in-centre of a triangle ABC and O is a point in the plane of the triangle. Prove that the resultant of forces $\overline{OA}.\sin A$; $\overline{OB}.\sin B$ and $\overline{OC}.\sin C$ acting along $\overrightarrow{OA}$, $\overrightarrow{OB}$, and $\overrightarrow{OC}$ respectively is a force $4\overline{OI}.\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$ acting along $\overrightarrow{OI}$.

The resultant of forces $\overline{OB}.\sin B$ and $\overline{OC}.\sin C$ acting along $\overrightarrow{OB}$ and $\overrightarrow{OC}$ respectively is a force $(\sin B+\sin C).\overline{OD}$ acting along $\overrightarrow{OD}$ where D is a point in $\overline{BC}$ such that $\frac{BD}{CD}=\frac{\sin C}{\sin B}=\frac{c}{b}=\frac{AB}{AC}$.

$\therefore$ $\overrightarrow{AD}$ bisects $\angle BAC$.

Again, the resultant of forces $\sin A.\overline{OA}$ and $(\sin B+\sin C).\overline{OD}$ acting along $\overrightarrow{OA}$ and $\overrightarrow{OD}$ is a force $(\sin A+\sin B+\sin C).OI'$ acting along $\overrightarrow{OI'}$ where I′ is a point on $\overline{AD}$ such that

$$\frac{AI'}{DI'}=\frac{\sin B+\sin C}{\sin A}=\frac{b+c}{a}.$$

Again, $\because$ $\frac{BD}{CD}=\frac{AB}{AC}$, so $\frac{AI'}{DI'}=\frac{AB}{BD}$ (as in Ex. 2)

$\therefore$ $\overrightarrow{AI'}$ is the bisector of $\angle ABC$.

$\therefore$ I′ is the in-centre of $\triangle ABC$. $\therefore$ $I\equiv I'$.

Also from Trigonometry,

$\sin A+\sin B+\sin C=4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$ (This is an identity)

Hence the resultant of the forces is a force

$4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}\,\overline{OI}$ acting along $\overrightarrow{OI}$.

Ex. 4. Find the position of the point within a quadrilateral ABCD, so that forces represented by $\overline{PA}$, $\overline{PB}$, $\overline{PC}$ and $\overline{PD}$ will keep a particle placed at the point in equilibrum.

Let M and N be the middle points of AC and BD. Now $\overline{PA}+\overline{PC}=2\overline{PM}$ and $\overline{PB}+\overline{PD}=2\overline{PN}$.

So if the forces $2\overline{PM}$ and $2\overline{PN}$ be equal and opposite, then the particle placed at P will be in equilibrium. Now $2\overline{PM}$ and $2\overline{PN}$ will be equal and opposite if P be the middlepoint of MN.

Ex. 5. Three forces P, Q, R are represented by $\overline{AB}$, $\overline{AC}$ and $\overline{AD}$ respectively where D is the middle point of BC. Show that $R^2=\frac{1}{4}(P^2+Q^2+2PQ\cos A)$.

As the forces P and Q are represented by $\overline{AB}$ and $\overline{AC}$ respectively, so their resultant is represented by $(1+1)\,\overline{AD}=2\overline{AD}$, where D is the middle point of BC.

Hence the resultant of the two forces P and Q is 2R.

$\therefore\ (2R)^2=P^2+Q^2+2PQ\cos A.$

or $\ 4R^2=P^2+Q^2+2PQ\cos A$

or $\ R^2=\frac{1}{4}(P^2+Q^2+2PQ\cos A)$.

Exercise 2C.

1. Prove that the resultant of forces $\dfrac{k}{\cos A}$ and $\dfrac{k}{\cos B}$ acting along $\overrightarrow{CA}$ and $\overrightarrow{CB}$ is a forces $k(\tan A+\tan B)$ acting along $\overrightarrow{CF}$ where the points A, B, C are the vertices of the triangle ABC and F is the foot of the perpendicular from C on AB.

2. Prove with the help of the theorem of § 2·8 that if a transversal intersects the lines of action of two forces P and Q acting at a point O, and their resultant R at the points L, M and N respectively, then $\dfrac{P}{OL}+\dfrac{Q}{OM}=\dfrac{R}{ON}$.

3. P is a point in the plane of a hexagon ABCDEF. Prove that if O be the circum-centre of the hexagon, the resultant of the forces represented by $\overline{PA}$, $\overline{PB}$, $\overline{PC}$, $\overline{PD}$ $\overline{PE}$ and $\overline{PF}$ will be represented by 6 $\overline{PO}$.

4. $\overline{AB}$ and $\overline{CD}$ are two equal and parallel chords of a circle. P is a point on the circumference equidistant from A and B. Prove that the resultant of the forces acting at the point P and represented by $\overline{PA}$, $\overline{PB}$, $\overline{PC}$ and $\overline{PD}$ is of constant magnitnde. [C. U. 1943]

5. PQRS is a quadrilateral. Prove that the resultant of the forces completely represented by the directed line segments $\overline{PQ}$, $\overline{QR}$, $\overline{PS}$, $\overline{SR}$ is represented in magnitude, direction and sense by $2\overline{PR}$ and that its line of action bisects QS. [C. U. 1941]

6. ABC is a right angled triangle and B is the right angle. BE is perpendicular an CA. Prove that resultant of the forces $\frac{K}{BA}$ and $\frac{K}{BC}$ acting along $\overrightarrow{BA}$ and $\overrightarrow{BC}$ is a force $\frac{K}{BE}$ along $\overrightarrow{BE}$.

7. Show that the resultant of forces represented by $\overline{PA}$, $\overline{PB}$, $\overline{PC}$, $\overline{AQ}$, $\overline{BQ}$, $\overline{CQ}$ is represented by $3\overline{PQ}$ and it passes through the centroid of the triangle ABC.

8. Forces of magnitudes proportional to cos A, cos B act along the sides CA, CB of a triangle ABC. Prove that their resultant is proportional to sin C and its line of action divides the angle C into two portions $\frac{1}{2}(C+B-A)$ and $\frac{1}{2}(C+A-B)$.

9. The points O and H are respectively the circumcentre and orthocentre of the triangle ABC. Prove that the resultant of the forces represented by (i) $\overline{OA}$, $\overline{OB}$, $\overline{OC}$ is represented by $\overline{CH}$ (ii) $\overline{HA}$, $\overline{HB}$, $\overline{HC}$ is represented by $2\overline{HO}$ and (iii) $\overline{AH}$, $\overline{HB}$, $\overline{HC}$ is represented in magnitude and direction by the diameter through A of the cricumcircle of the triangle.

10. P is a moving point in the plane of three fixed points A, B, C. The resultant of the forces represented by $\overline{PA}$ and $\overline{PB}$ always passes through the point C. Prove that the locus of the point C is the straight line $\overleftrightarrow{CD}$ where D is the middle point of AB.

11. P is a moving point in the plane of the quadrilateral ABCD such that the resultant of the forces represented by $\overline{PA}$, $\overline{PB}$, $\overline{PC}$ and $\overline{PD}$ is of constant magnitude. Prove that the locus of the point P is a circle.

12. If the forces represented by $(m-n)$ $\overline{OP}$, $(n-l)$ $\overline{OQ}$ and $(n-l)$ $\overline{CR}$ be in equilibrium, prove that the points P, Q, R are collinear.

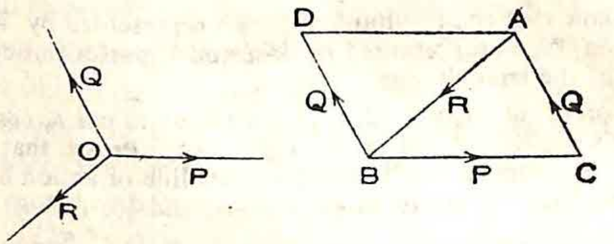
CHAPTER THREE

CONDITIONS OF EQUILIBRIUM OF CONCURRENT FORCES

§ 3·1. Theorem of Triangle of Forces. If three forces acting at point can be represented in magnitude, direction and sense by the three sides of a triangle taken in order, then the forces are in equilibrium.

Three forces P, Q, R acting at a point O are represented in magnitude, direction and sense by the three sides $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ of a triangle ABC taken in order. To prove that the forces are in equilibrium.

Proof. Complete the parallelogram BCAD. Now as $\overline{BD}$ and $\overline{CA}$ are equal and parallel and $\overline{CA}$ represents the force Q so $\overline{BD}$ will also represent Q.



(i) Fig. 73 (ii)

Now, the two adjacent sides $\overline{BC}$ and $\overline{BD}$ of the parallelogram BCAD represent the two forces P and Q in magnitude, direction and sense. So by parallelogram of forcs, the diagonal $\overline{BA}$ drawn from B will represent the resultant of the forces P and Q. Again the force R acting at the same point is represented by $\overline{AB}$. Hence the resultant of the forces P and Q is equal and opposite to the force R acting at the same point and so they balance each other. Hence the forces P, Q, R are in equilibirum.

§ 3·2. Converse of the theorem of triangle of forces : If three forces acting at a point be in equilibrium, then the three forces can be represented in magnitude, direction and sense by the three sides of a triangle taken in other.

See the figure of § 3·1. The three forces P, Q, R acting at a point O are in equilibrium. To prove that the three forces can be represented in magnitude, direction and sense by the three sides of a triangle taken in order.

Let according to a certain scale the directed line segments $\overline{BC}$ and $\overline{CA}$ represent the forces P and Q. Join AB and complete the parallelogram BCAD.

As $\overline{BD}$ and $\overline{CA}$ are equal and parallel, so the directed line segment $\overline{BD}$ also represents the force Q in the same scale. Now the two adjacent sides $\overline{BC}$ and $\overline{BD}$ of the parallelogram BCAD represent the forces P and Q acting at a point. Hence their resultant is represented by the diagonal $\overline{BA}$ of the parallelogram. Now since the forces P, Q and R are in equilibrium, the force R must be equal and opposite to the resultant of P and Q. So, as the resultant of P and Q is represented by $\overline{BA}$, R can be represented by $\overline{AB}$. Hence the forces P, Q, R can be represented in mangnitude, direction and sense by the three sides $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ of the triangle ABC taken in order. Hence the theorem is proved completely.

Note. The sides of any other triangle parallel to the sides of the traingle can also represent the forces according to a suitable scale. For, the triangles are similar and the lengths of corresponding sides will be proportional.

Example 1. The magnitudes of three forces are in the ratio 4 : 5 : 8. If the three forces act at a point, can they keep the point in equilibrium under any circumstances ? In this question if the magnitudes of the forces be (i) 4 : 5 : 9 or, (ii) 4 : 5 : 10, then discuss the equilibrium of the forces.

Since the sum of any two of 4, 5 and 8 is greater than the third, so the three forces can be represented by the three sides of a triangle taken in order. Hence in that case the forces will be in equilibrium.

(i) As $4+5=9$, so the three forces cannot be represented in magnitude, direction and sense by the three sides of a triangle taken in order. But if the forces of magnitudes proportional to 4 and 5 act along the same straight line in the

same sense and the other force, act along the same line in the opposite sense, then the resultant of the first two forces will balance the third force. Hence the three forces will be in this case in equilibrium.

(ii) As $4+5<10$, so the three forces cannot be represented by the sides of a triangle nor any force can be balanced as in (i) above by the resultant of the other two. Hence in this case the forces cannot remain in equilibrium under any circumstances.

Ex. 2. The magnitudes of three forces are in the ratio $2 : \sqrt{2} : \sqrt{3}+1$. If the forces act at a point and be in equilibrium, find the angle between the forces whose magnitudes are proportional to 2 and $\sqrt{3}+1$.

Since the three force acting at a point are in equilibrium, so the forces can be represented in magnitude, direction and sense by the three sides of a triangle ABC taken in order. Let the forces proportional to 2 and $(\sqrt{3}+1)$ are represented by $\overline{BC}$ and $\overline{CA}$ respectively. Hence if θ be the angle between these forces, then $m\angle ACB(=C)=\pi-\theta$.

Now, $\cos C=\dfrac{a^2+b^2-c^2}{2ab}$ [Here a, b, c are the lengths of the sides of the triangle, so let $a=2k$; $b=(\sqrt{3}+1)k$ and $c=\sqrt{2}k$]

$$=\frac{k^2\{(2)^2+(\sqrt{3}+1)^2-(\sqrt{2})^2\}}{k^2.2.2.(\sqrt{3}+1)}=\frac{4+3+1+2\sqrt{3}-2}{4(\sqrt{3}+1)}$$

$$=\frac{6+2\sqrt{3}}{4(\sqrt{3}+1)}=\frac{2\sqrt{3}(\sqrt{3}+1)}{4(\sqrt{3}+1)}=\frac{\sqrt{3}}{2}=\cos 30^\circ$$

$\therefore$ $C=30^\circ$. $\therefore$ θ i.e., the required measure of the angle is $180^\circ-30^\circ=150^\circ$.

Ex. 3. Two weightless cords of lengths 2·5 meters and 3 meters are respectively attached to two points A and B of a horizontal rod and are knotted with each other at the point C. A weight of 10 kg. is suspended from the point C. The distance AB is 4 meters and the weight is in equilibrium. Find the tensions of the string.

Let the tensions of the strings be T_1 and T_2. Now the three forces T_1, T_2 and the weight 10 kg. acting at the point C

are in equilibrium. So these forces can be represented by the three sides of a triangle taken in order. Let on a certain scale XYZ be the triangle of forces and $\overline{ZY}$, $\overline{YX}$ and $\overline{XZ}$ respectively represent the forces T_1, T_2 and 10 kg. in magnitude, direction and sense $\therefore$ $\overline{YX}$, $\overline{ZY}$ and $\overline{XZ}$ are respectively parallel to $\overline{CB}$, $\overline{CA}$ and $\overline{CE}$.

$\therefore$ $m\angle YZX = m\angle ACD = \theta$ (say) and $m\angle YXZ = \angle BCD = \phi$ (say).

Now in $\triangle ABC$, $AB = 4m$, $BC = 2{\cdot}5m$ and $CA = 3m$.

$$\therefore \cos A = \frac{4^2+3^2-(2{\cdot}5)^2}{2.4.3} = \frac{25}{32} = {\cdot}7815 = \cos 38°27$$

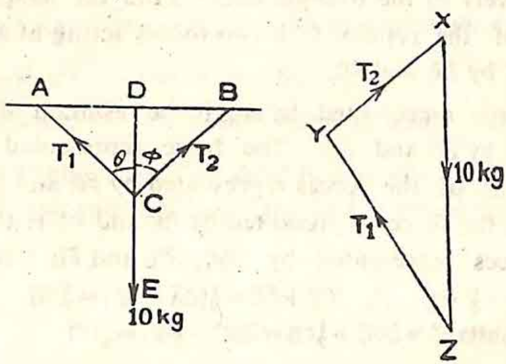


Fig. 17

$\therefore$ $A = 38°27'$; $\therefore$ $\theta = 90° - 38°27' = 51°33'$.

Similarly $B = 48°30'$ and $\phi = 90° - 48°30' = 41°30'$

$\therefore$ $m\angle XYZ = 180° - (\theta + \phi) = 86°57'$.

Now, from $\triangle XYZ$, $\dfrac{YX}{XZ} = \dfrac{\sin 51°33'}{\sin 86°57'}$

But $XZ = 10$ ($\because$ it represents the force 10 kg. weight)
$\therefore$ $YX = 7{\cdot}8$ (nearly).

Similarly from $\dfrac{YZ}{10} = \dfrac{\sin 41°30'}{\sin 87°7'}$,

$\therefore$ $YZ = 6{\cdot}65$ (nearly).

Hence the required tensions are 7·8 kg. wts. and 6·65 kg. wts,

Ex. 4. If three forces acting at a point be proportional to the three sides of a triangle taken in order and their

directions be perpendicular to the sides of the triangle (all inwards or all outwards), then prove that the three forces are in equilibrium.

Rotate the triangle through a right angle in such a way that the sides of the triangle become respectively parallel to the forces. Since the magnitudes of the forces are proportional to the sides of the triangle, so in the new position, the three sides of the triangle taken in order, will represent the forces in magnitude, direction and sense. Hence by the theorem of triangle of forces the three forces will be in equilibrium.

Ex. 5. F and E are the middle points of the sides AB and AC respectively of the triangle ABC. Find the magnitude and direction of the resultant of two forces acting at a point and represented by $\overline{BE}$ and $\overline{FC}$.

The force represented by $\overline{BE}$ is the resultant of the forces represented by $\overline{BC}$ and $\overline{CE}$. The force represented by $\overline{FC}$ is the resultant of the forces represented by $\overline{FB}$ and $\overline{BC}$. So the resultant of the forces represented by $\overline{BE}$ and $\overline{FC}$ is the resultant of the forces represented by $2\overline{BC}$, $\overline{CE}$ and $\overline{FB}$. Now, $CE=\frac{1}{2}$ CA and $FB=\frac{1}{2}$ AB. $\therefore$ $\overline{CE}+\overline{FB}=\frac{1}{2}(\overline{CA}+\overline{AB})=\frac{1}{2}\overline{CB}$. Hence the required resultant $=2\overline{BC}+\frac{1}{2}\overline{CB}=2\overline{BC}-\frac{1}{2}\overline{BC}=\frac{3}{2}\overline{BC}$

Ex. 6. E is the middle point of the side DC of the parallelogram ABCD. If three forces acting at the point A be represented by $\overline{AB}$, $\overline{AC}$ and $2\overline{AE}$, show that their resultaut is represented by $3\overline{AC}$.

From the theorem of triangle of forces we get

$$\overline{AE}+\overline{ED}=\overline{AD} \text{ and } \overline{AE}+\overline{EC}=\overline{AC}$$

$$\therefore\ 2\overline{AE}+\overline{ED}+\overline{EC}=\overline{AD}+\overline{AC} \quad \cdots \quad \cdots \quad (1)$$

Now as E is the middle point of CD, so $\overline{ED}+\overline{EC}=0$.

So from (1) we get $2\overline{AE}=\overline{AD}+\overline{AC}$.

Again, as $\overline{AD}$ and $\overline{BC}$ are equal, parallel and of the same sense, so $\overline{AD}=\overline{BC}$.

$$\therefore\ 2\overline{AE}=\overline{BC}+\overline{AC}.$$

$$\therefore\ 2\overline{AE}+\overline{AB}+\overline{AC}=\overline{BC}+\overline{AC}+\overline{AB}+\overline{AC}=\overline{AB}+\overline{BC}+\overline{BC}+\overline{AC}=\overline{AC}+\overline{AC}+\overline{AC}=3\overline{AC}.$$

Ex. 7. R and S are respectively the least and greatest resultants of two forces acting at a point and of given magnitudes P and Q. If the forces P, Q and $\sqrt{RS}$ acting at a point be in equilibrium, then prove that two of these later three forces are perpendicular to each other. [*Nagpur 1940*]

As R and S are the least and greatest resultants of the forces P and Q,

So $R=P-Q$ and $S=P+Q$ (talking $P>Q$).

$\therefore \quad RS=P^2-Q^2$ or, $\sqrt{RS}=\sqrt{P^2-Q^2}$.

Now the three forces P, Q and $\sqrt{RS}$ acting at a point are in equilibrium, so they can be represented in magnitude direction and sense be the three sides of a triangle ABC.

Let $BC=P$, $CA=Q$ and $AB=\sqrt{RS}$.

$\therefore \quad AB^2=RS=P^2-Q^2=BC^2-CA^2$ or $AB^2+AC^2=BC^2$.

So, the triangle ABC is right angled and $\angle BAC$ is a right angled. So AB and CA i.e., $\sqrt{RS}$ and Q are perpendicular to each other.

Exercise 3A

1. The magnitudes of three forces acting at a point are proportional to (*i*) 5, 6, 8 ; (*ii*) 5, 6, 11 and (*iii*) 5, 6, 12.

In which of the above cases the forces can remain in equilibrium ?

2. The magnitudes of three forces acting at a point are proportional to 3 : 2 : 1. State whether equilibrium of the forces is possible under any circumstances. [*C.U. 1943*]

3. Three forces of magntiudes a, a, $\sqrt{2}a$ act on a particle. If the three forces remain in equilibrium, then prove that the equal forces act at right angles.

4. One end of each of two thin, light wires is attached to the points A and B of a horizontal rod AB, and the other two ends are knotted at the point C. A weight of mass 80 kg. is

suspended at the point C. If $AB = 50$ cms. and the lenghts of the wires be 40 cms. and 30 cms. determine the tensions of the wires when the system is in equilibrium.

5. The length of the jib of a crane is 6 meters ; the chain or tie of the crane is 5 meters long and is attached to a point of the post 4 meters above the bottom of the post. If the jib cannot bear pressure more than 200 kg. then find the greatest weight that the crane can bear.

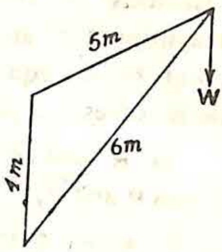


Fig. 18

6. A weight 10 kg. has been kept in equilibrium on an inclined plane of inclination 30° with the horizontal by a horizontal string. Construct triangle of forces to determine the tension of the string and the reaction of the inclined plane.

7. Three forces P, Q, R acting at a point are in equilibrium. The angle between P and Q is 105° and that between P and R is 120°. If the magnitude of the force P be 10 kg. wts., determine Q and R.

8. ABCD is a parallelogram and P is a point inside it. Prove that for any position of P inside the quadrilateral, forces represented by $\overline{PA}$, $\overline{PC}$, $\overline{BP}$ and $\overline{DP}$ will be in equilibrium.

9. ABCD is a parallelogram. Prove that the resultant of the two forces represented by the diagonals $\overline{AC}$ and $\overline{BD}$ acting at a point is represented by $2\overline{BC}$.

10. ABCD is a parallelogram. Prove that four forces acting at a point and represented by $\overline{AB}$, $\overline{CD}$, $\overline{AC}$ and $\overline{DB}$ is represented by $2\overline{AB}$.

11. D, E, F are the middle points of the sides BC, CA, AB respectively of the triangle ABC. Prove that the three concurrent forces represented by $\overline{AD}$, $\overline{BE}$ and $\overline{CF}$ are in equilibrium.

§ 3·3. Polygon of forces.

If a finite number (here more than three) of forces acting at a point can be represented in magnitude, direction and sense by the sides of a closed polygon taken in order, then the forces are in equilibrium.

Let the forces P, Q, R, S, T acting at a point O be represented in magnitude, direction and sense by the sides $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DE}$ and $\overline{EA}$ of the closed polygon ABCDE taken in order. Join AC and AD.

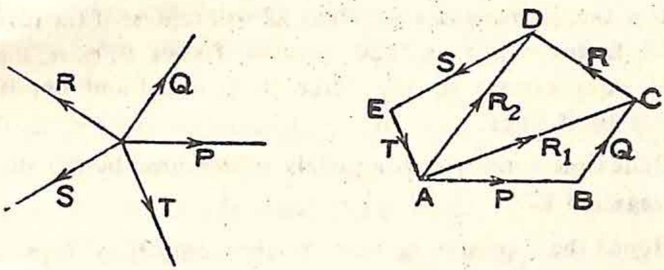


Fig. 19

Since the forces P and Q are represented in magnitude, direction and sense by $\overline{AB}$ and $\overline{BC}$, so $\overline{AC}$ will represent the resultant R_1 of P and Q in magnitude, direction and sense.

Similarly the resultant R_2 of the forces R_1 and R will be represented by $\overline{AD}$, and the resultant of R_2 and S will be represented by $\overline{AE}$. Hence the resultant of the forces P, Q, R and S are represented by $\overline{AE}$.

Again, the remaining force T is represented by $\overline{EA}$. Now the forces represented by $\overline{AE}$ and $\overline{EA}$ having the same magnitude and direction but opposite sense are in equilibrium.

Hence the forces P, Q, R, S, T are in equilibrium.

Note : If magnitude, direction and sense of several forces acting at a point are known and the forces are represented in magnitude, direction and sense by the directed line segments $\overline{AB}$, $\overline{BC}$, $\overline{CD}$ etc, then the directed line segment $\overline{AF}$ joining the initial point of the first line segment and the terminal point of the last one will represent in the same scale the magnitude, direction and sense of the resultant of the several forces.

§ 3·4. Converse of Polygon of forces :

If a number of forces (more than three) acting at a point be in equilibrium, then the forces can be represented in magnitude, direction and sense by the sides of a closed polygon taken in order.

Let the number of forces be five and the forces be P, Q, R, S and T. Represent on a certain scale the forces P, Q, R, S by the directed line segments $\overline{AB}$, $\overline{BC}$, $\overline{CD}$ and $\overline{DA}$ taken in order in magnitude, direction and sense. Join $\overline{EA}$.

Now the directed line segment $\overline{AE}$ will represent the resultant of the forces P, Q, R, S, and since the forces P, Q, R, S and T are in equilibrium, so the force T is equal and opposite to this resultant force.

Hence the force T is completely represented by the directed line segment $\overline{EA}$.

Hence the forces P, Q, R, S, T are completely represented by the sides of the closed polygon ABCDE, taken in order.

Similarly the proposition can be proved for any finite number of forces.

Note : Since if the corresponding sides of two polygons be parallel, then the sides of the two polygons are not necessarily proportional, so any polygon with sides parallel to the given forces cannot represent the forces.

§ 3·5. Lami's Theorem :

If three forces acting at a point be in equilibrium, then each force is proportional to the sine of the angle between the other two.

Let three forces P, Q, R acting at a point O be in equilibrium and $\overrightarrow{OX}$, $\overrightarrow{OY}$, $\overrightarrow{OZ}$ be the respective lines of action of the forces. To prove that

$$\frac{P}{\sin YOZ} = \frac{Q}{\sin ZOX} = \frac{R}{\sin XOY}$$

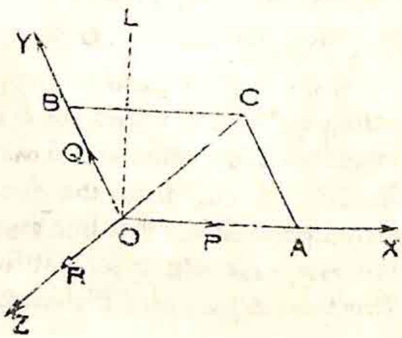


Fig. 20

Let the line segments $\overline{OA}$ and $\overline{OB}$ of $\overrightarrow{OX}$ and $\overrightarrow{OY}$ respectively represent the forces P and Q on a certain preassigned scale. Complete the parallelogram OACB. Join OC.

Hence by the parallelogram law of forces, the directed line segment $\overline{OC}$ will represent the resultant of the forces P and Q in magnitude, direction and sense.

Now since the forces P, Q, R are in equilibrium, so this resultant and R will be in equilibrium and hence they must have the same magnitude, direction and line of action but opposite sense.

Hence $\overline{CO}$ will represent R and the three points C, O, Z must lie in the same straight line.

Again, $\overline{AC}$ and $\overline{OB}$ are equal and parallel.

Hence $\overline{AC}$ will also represent the magnitude, direction and sense of the force Q.

Now, from $\triangle OAC$,

$$\frac{OA}{\sin OCA}=\frac{AC}{\sin COA}=\frac{CO}{\sin OAC}$$

Now, OA, AC and CO represent the magnitudes of the forces P, Q, R respectively and

$$\sin OCA=\sin COB=\sin(180^\circ-YOZ)=\sin YOZ$$

$$\sin COA=\sin(180^\circ-ZOX)=\sin ZOX$$

$$\sin OAC=\sin(180^\circ-XOY)=\sin XOY.$$

$$\therefore \quad \frac{P}{\sin YOZ}=\frac{Q}{\sin ZOX}=\frac{R}{\sin XOY}.$$

Alternative method : Let $\overrightarrow{OL}$ be perpendicular on $\overrightarrow{OX}$.

Now, since the forces are in equilibrium, so the algebraic sum of the resolved parts of the forces along $\overrightarrow{OL}$ is zero.

So, $Q \sin XOY - R \sin XOZ = 0$

or, $\dfrac{Q}{\sin XOZ}=\dfrac{R}{\sin XOY}$.

Similarly resolving the forces along a direction perpendicular to $\overrightarrow{OY}$, we obtain $\dfrac{P}{\sin YOZ}=\dfrac{R}{\sin XOY}$

$$\therefore \quad \frac{P}{\sin YOZ}=\frac{Q}{\sin ZOX}=\frac{R}{\sin XOY}.$$

§ 3·6. Converse of Lami's Theorem :

If the sense of three forces acting at a point be such that each of the forces lies within the angle opposite to that in which the resultant of the other two forces lies and each of the forces is proportional to the sine of the angle between the other two, then, the forces are in equilibrium.

Let, $\overrightarrow{OX}$, $\overrightarrow{OY}$, $\overrightarrow{OZ}$ be the lines of action of three forces acting at a point O and $\dfrac{P}{\sin YOZ}=\dfrac{Q}{\sin ZOX}=\dfrac{R}{\sin XOY}$ ... (1)

[See the figure of § 3·5]

To prove that the forces are in equilibrium.

Produce $\overrightarrow{ZO}$ and from produced $\overrightarrow{ZO}$ cut off the line segment $\overline{OC}$ so that according to any chosen scale, the length $\overline{CO}$ represents the force R.

Now complete the parallelogram OACB with $\overline{OC}$ as diagonal and adjacent sides $\overline{OA}$ and $\overline{OB}$ along $\overrightarrow{OX}$ and $\overrightarrow{OY}$.

Now, from $\triangle OAC$,

$$\frac{OA}{\sin OCA}=\frac{AC}{\sin AOC}=\frac{OC}{\sin OAC}=\frac{R}{\sin OAC}.$$

Again, $\sin OCA=\sin BOC=\sin(180°-YOZ)=\sin YOZ$

$\sin AOC=\sin(180°-ZOX)=\sin ZOX$

and $\sin OAC=\sin(180°-XOY)=\sin XOY$

$$\therefore \frac{OA}{\sin YOZ}=\frac{AC}{\sin ZOX}=\frac{R}{\sin XOY} \quad \text{......(2)}$$

From (1) and (2) we get $P=OA$, $Q=OB$.

Hence the directed line segments $\overline{OA}$ and $\overline{OB}$ represent the forces P and Q respectively in magnitude, direction and sense.

Hence by parallelogram of forces $\overline{OC}$ will represent the resultant of the forces P and Q.

Again, $\overline{CO}$ represents the force R acting at the point O.

Hence the force R and the resultant of the forces P and Q are equal and opposite *i.e.*, the resultant of P and Q balances the given force R.

Hence the three forces P, Q, R are in equilibrium.

Ex. 1. The point I is the in-centre of $\triangle ABC$ and three forces P, Q, R acting along $\overrightarrow{IA}$, $\overrightarrow{IB}$, $\overrightarrow{IC}$ are in equilibrium. Prove that

$$\frac{P}{\cos\frac{A}{2}}=\frac{Q}{\cos\frac{B}{2}}=\frac{R}{\cos\frac{C}{2}}.$$

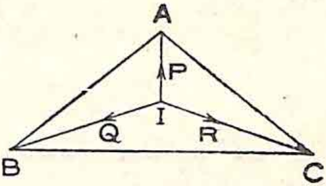


Fig. 21

As the three forces P, Q, R acting at the point I are in equilibrium, hence by Lami's Theorem, $\frac{P}{\sin BIC}=\frac{Q}{\sin CIA}=\frac{R}{\sin AIB}$

or, $$\frac{P}{\sin\left(90^\circ+\frac{A}{2}\right)}=\frac{Q}{\sin\left(90^\circ+\frac{B}{2}\right)}=\frac{R}{\sin\left(90^\circ+\frac{C}{2}\right)}$$

[$\because$ I is the in-centre of the triangle]

or, $$\frac{P}{\cos\frac{A}{2}}=\frac{Q}{\cos\frac{B}{2}}=\frac{R}{\cos\frac{C}{2}}.$$

Ex. 2. The point O is the circum-centre of the $\triangle ABC$ and three forces P, Q, R acting at the point O along $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$ respectively are in equilibrium.

Prove that, $$\frac{P}{a^2(b^2+c^2-a^2)}=\frac{Q}{b^2(c^2+a^2-b^2)}=\frac{R}{c^2(a^2+b^2-c^2)}$$

Since the forces P, Q, R acting at a point O along $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\overrightarrow{OC}$ are in equilibrium, hence by Lami's Theorem,

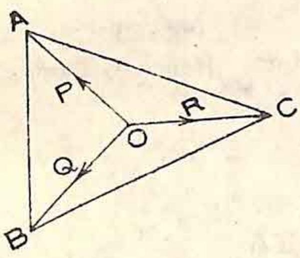


Fig. 22

$$\frac{P}{\sin BOC}=\frac{Q}{\sin COA}=\frac{R}{\sin AOB}$$

or, $$\frac{P}{\sin 2A}=\frac{Q}{\sin 2B}=\frac{R}{\sin 2C}$$

[$\because$ O is the circum-centre of the triangle and on the same arc, the angle at the centre is twice the angle at the circumference.]

or, $$\frac{P}{2\sin A\cos A}=\frac{Q}{2\sin B\cos B}=\frac{R}{2\sin C\cos C}.$$

or, $$\frac{P}{\frac{a}{2R'}\cdot\frac{b^2+c^2-a^2}{2bc}}=\frac{Q}{\frac{b}{2R'}\cdot\frac{c^2+a^2-b^2}{2ca}}=\frac{R}{\frac{c}{2R'}\cdot\frac{a^2+b^2-c^2}{2ab}}$$

[R' is the circum-radius of $\triangle$ABC]

or, $$\frac{P}{\frac{a(b^2+c^2-a^2)}{bc}}=\frac{Q}{\frac{b(c^2+a^2-b^2)}{ca}}=\frac{R}{\frac{c(a^2+b^2-c^2)}{ab}}$$

or, $$\frac{P}{\frac{a^2(b^2+c^2-a^2)}{abc}}=\frac{Q}{\frac{b^2(c^2+a^2-b^2)}{abc}}=\frac{R}{\frac{c^2(a^2+b^2-c^2)}{abc}}$$

or, $$\frac{P}{a^2(b^2+c^2-a^2)}=\frac{Q}{b^2(c^2+a^2-b^2)}=\frac{R}{c^2(a^2+b^2-c^2)}.$$

Ex. 3. A weight W is suspended by two cords from the point of their intersection. One of the cords is inclined at an angle 30° with the vertical; what should be the inclination of the other cord with the vertical so that its tension may be the least? Determine the tension of both the cords in this case.

Let $\overline{AB}$ and $\overline{AC}$ be the two cords and the line $\overleftrightarrow{EF}$ through A represent the vertical direction and $m\angle BAF=30°$.

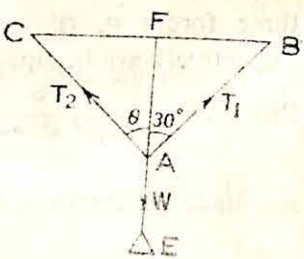


Fig. 23

Let T_1 and T_2 be the tensions of the two cords and $m\angle CAF=\theta$.

$\therefore\ m\angle BAE=180°-30°=150°$;

$m\angle CAE=180°-\theta$, and $m\angle BAC=30°+\theta$.

Now the three forces T_1, T_2 and W acting at the point A are in equilibrium. Hence by Lami's Theorem,

$$\frac{T_1}{\sin CAE}=\frac{T_2}{\sin BAE}=\frac{W}{\sin BAC}$$

or, $$\frac{T_1}{\sin(180-\theta)}=\frac{T_2}{\sin 150°}=\frac{W}{\sin(30°+\theta)}$$

$$\therefore\ T_1=\frac{\sin(180°-\theta)W}{\sin(30°+\theta)}=\frac{\sin\theta}{\sin(30°+\theta)}W$$

and $$T_2=\frac{\sin 150°}{\sin(30°+\theta)}\cdot W=\frac{W}{2\sin(30°+\theta)}$$

Now, T_2 will be least when $\sin(30°+\theta)$ will be greatest i.e., 1 i.e., $30°+\theta=90°$. $\therefore\ \theta=60°$.

Hence in this case,

$$T_1=\frac{\sin 60°}{\sin 90°}W=\frac{\sqrt{3}W}{2} \text{ and } T_2=\frac{W}{2.\sin 90°}=\frac{W}{2}.$$

Ex. 4. A body of mass 10 lbs. is suspended by two cords 7 and 24 inches long, their other ends being fastend to the extremities of a rod of length 25 inches. If the rod be so held that the body hangs immediately below its middle point, find the tensions of the cord. [U. P. 1943]

Let the rod be $\overline{AB}$ and the cords be $\overline{CA}$ and $\overline{CB}$ so that the weight is suspended from C. Let also D be the middle point of $\overline{AB}$; so $\overleftarrow{CD}$ is a vertical line.

Now $AB=25''$, $BC=24''$ and $AC=7''$.

Again $25^2=24^2+7^2$.

$\therefore\ AB^2=BC^2+AC^2$

and so $\angle ACB$ is a right angle.

$\therefore\ CD=\frac{1}{2}AB=BD=AD$

$\therefore\ \angle DBC\cong\angle DCB$

and $\angle DAC\cong\angle DCA$.

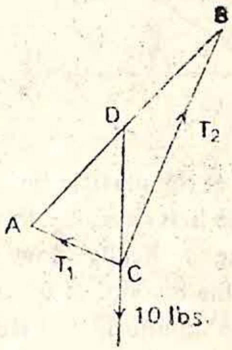


Fig. 24

Now the weight W acting vertically downwards and the tensions T_1 and T_2 of the cords along $\overrightarrow{CA}$ and $\overrightarrow{CB}$ all acting at the point C are in equilibrium. Hence by Lami's Theorem,

$$\frac{T_1}{\sin(180°-DCB)}=\frac{T_2}{\sin(180°-DCA)}=\frac{10 \text{ lbs. wt.}}{\sin ACB}$$

or, $$\frac{T_1}{\sin DCB}=\frac{T_2}{\sin DCA}=\frac{10 \text{ lbs wt.}}{\sin 90°}$$

or, $$\frac{T_1}{\sin DBC}=\frac{T_2}{\sin DAC}=10 \text{ lbs wt.}$$

or, $$\frac{T_1}{\frac{7}{25}}=\frac{T_2}{\frac{24}{25}}=10 \text{ lbs wt.}$$

$$\therefore\ T_1=\frac{7}{25}\times 10 \text{ lbs wt.}=\frac{14}{5} \text{ lbs wt. and}$$

$$T_2=\frac{24}{25}\times 10 \text{ lbs wt.}=\frac{48}{5} \text{ lbs wt.}$$

Ex. 5. A particle whose weight is W may be supported on a smooth inclined plane by a force P acting horizontally or by a force Q acting along the plane. If R and S be the pressures on the plane respectively in the two cases, show that

$$RS = W^2 \text{ and } \frac{1}{Q^2} = \frac{1}{P^2} + \frac{1}{W^2}.$$

Let the inclined plane be inclined with the horizontal at an angle α.

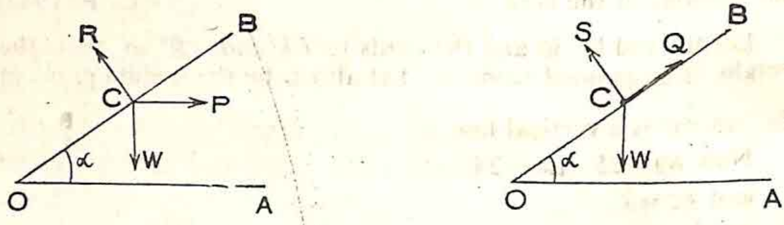


Fig. 25

Let the particle be at the point C on the inclined plane. Now in the first case, the three forces viz, the weight of the particle acting vertically downwards, the force P acting horizontally and the reaction R of the inclined plane all acting at the point C are in equilibrium. Hence by Lami's Theorem,

$$\frac{P}{\sin(180^\circ - \alpha)} = \frac{W}{\sin(90^\circ + \alpha)} = \frac{R}{\sin 90^\circ}$$

or, $\dfrac{P}{\sin\alpha} = \dfrac{W}{\cos\alpha} = R.$ $\therefore$ $P = W\tan\alpha$ and $R = \sec\alpha$.

In the second case, W acting vertically dowhwards, the force Q acting along the plane and the pressure S of the inclined plane, all acting at the point C are in equilibrium. Hence by Lami's Theorem,

$$\frac{Q}{\sin(180^\circ - \alpha)} = \frac{W}{\sin 90^\circ} = \frac{S}{\sin(90^\circ + \alpha)}$$

$\therefore$ $\dfrac{Q}{\sin\alpha} = \dfrac{W}{1} = \dfrac{S}{\cos\alpha}.$

$\therefore$ $Q = W\sin\alpha$ and $S = W\cos\alpha$.

$\therefore$ $RS = W\sec\alpha . W\cos\alpha = W^2$.

and $\frac{1}{Q^2}-\frac{1}{P^2}=\frac{1}{W^2}\operatorname{cosec}^2\alpha-\frac{1}{W^2}\cot^2\alpha.$

$$=\frac{1}{W^2}(\operatorname{cosec}^2\alpha-\cot^2\alpha)=\frac{1}{W^2}.1=\frac{1}{W^2}\quad\therefore\quad\frac{1}{Q^2}=\frac{1}{P^2}+\frac{1}{W^2}.$$

Ex. 6. A uniform lamina in the form of a rhombus is kept in equilibrium by two forces P and Q ($P>Q$) acting at the centre of the rhombus along its diagonals. If one angle of the rhombus is 120° and one of its sides is horizontal then prove that $P^2=3Q^2$.

Let ABCD be the uniform lamina in the form of a rhombus and the side AB be horizontal and also $m\angle BAD=120$. Let the diagonals of the rhombus intersect at O and W be the weight of the rhombus. Then W acts at the point O vertically downwards. If the line of action of W intersects AB at E, then $m\angle AEO=90°$. Now, as $m\angle BAD=120°$, so $\overline{BD}$ is the longer diagonal and P and Q act along $\overline{AC}$ and $\overline{BD}$. Now the three forces, P, Q and W acting at the point O are in equilibrium. Hence by Lami's Theorem,

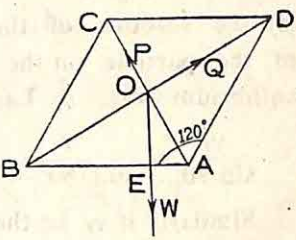


Fig. 26

$$\frac{P}{\sin EOD}=\frac{Q}{\sin COE}=\frac{W}{\sin COD}\cdots\cdots(1)$$

Now as $m\angle BAD=120°$, $\therefore$ $m\angle EAO=60°$

$\therefore$ $m\angle AOE=30°$; again $m\angle AOD=90°$.

$\therefore$ $m\angle EOD=120°$ and $m\angle COE=150°$;

for, $m\angle COE=m\angle COB+m\angle BOE=90°+60°=150°$

$\therefore$ From (1) we obtain $\frac{P}{\sin 120°}=\frac{Q}{\sin 150°}$ or, $\frac{P}{\frac{\sqrt{3}}{2}}=\frac{Q}{\frac{1}{2}}$

or, $\frac{P}{\sqrt{3}}=Q$ or, $P=\sqrt{3}Q$, $\therefore$ $P^2=3Q^2$.

Ex. 7. Two particles are placed on two smooth inclined planes having a common height. The particles are kept in equilibrium by a string attached to them and passing over a smooth pulley lying on the line of intersection of the planes.

Prove that weights of the particles are in the ratio of the lengths of the inclined planes.

Let the smooth inclined planes be AB and AC and their inclinations to the horizontal be α and β respectively. As the pulleyes are smooth, so the tensions of the string on the two sides of the string are equal.

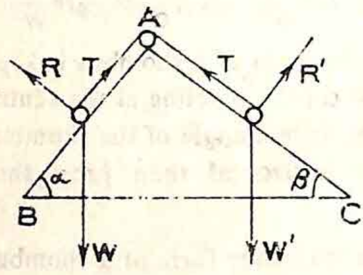


Fig. 27

On each particle act three forces viz (a) its weight (b) the tension T of the string and (c) the reaction of the inclined plane. So, if the weight of the particle on the plane AB be W, then considering its eqilibrium we get by Lami's Theorem,

$$\frac{W}{\sin 90^\circ}=\frac{T}{\sin(180^\circ-\alpha)} \quad \therefore \quad W=\frac{T}{\sin\alpha} \quad \dots \quad \dots \quad (1)$$

Similarly if W' be the weight of the second particle, then

$$\frac{W'}{\sin 90^\circ}=\frac{T}{\sin(180^\circ-\beta)} \quad \text{or,} \quad W'=\frac{T}{\sin\beta} \quad \dots \quad \dots \quad (2)$$

From (1) and (2) we get

$$\frac{W}{W'}=\frac{\dfrac{T}{\sin\alpha}}{\dfrac{T}{\sin\beta}}=\frac{\sin\beta}{\sin\alpha}=\frac{AB}{AC} \quad [\text{ From } \triangle ABC]$$

Hence the ratio of the weights is equal to the ratio of the lengths of the inclined planes.

Ex. 8. A light string is fastened to two points A and D in the same horizontal line, the length of the string being greater than the distance AD. Particles of weights 2 lbs. and 1 lbs. are fastened to it at two points B and C respectively. If AB, BC, CD make angles α, β, γ respectively with the horizontal,

show that $\tan\alpha=2\tan\gamma\pm3\tan\beta$.

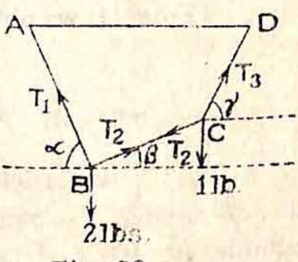


Fig. 28

First let the point B is below the point C and the tensions of the portions AB, BC, CA of the string be T_1, T_2, T_3 respectively.

Now, the particle at B is in equilibrium under the action of the forces (i) the tension T_1 along $\overrightarrow{BA}$, (ii) the tension T_2 along $\overrightarrow{BC}$ and (iii) the weight 2 lbs. of the particle acting vertically downwards. Hence by Lami's Theorem we get

$$\frac{T_2}{\sin(90^\circ+\alpha)}=\frac{2}{\sin(180^\circ-\alpha-\beta)} \quad \text{or,} \quad \frac{T_2}{\cos\alpha}=\frac{2}{\sin(\alpha+\beta)}$$

$$\therefore \quad T_2=\frac{2\cos\alpha}{\sin(\alpha+\beta)} \quad \ldots \quad \ldots \quad (1)$$

Similarly considering the equilibrium of the particle at the point C we get by Lami's Theorem,

$$\frac{T_2}{\sin(90^\circ+\gamma)}=\frac{1}{\sin(180^\circ-\gamma+\beta)} \quad \text{or,} \quad \frac{T_2}{\cos\gamma}=\frac{1}{\sin(\gamma-\beta)}$$

$$\therefore \quad T_2=\frac{\cos\gamma}{\sin(\gamma-\beta)} \quad \ldots \quad \ldots \quad (2)$$

So, from (1) and (2) we get,

$$\frac{2\cos\alpha}{\sin(\alpha+\beta)}=\frac{\cos\gamma}{\sin(\gamma-\beta)} \quad \text{or,} \quad \frac{\sin(\alpha+\beta)}{\cos\alpha}=\frac{2\sin(\gamma-\beta)}{\cos\gamma}$$

or $\tan\alpha\cos\beta+\sin\beta=2\tan\gamma\cos\beta-2\sin\beta$

or, $\tan\alpha+\tan\beta=2\tan\gamma-2\tan\beta$

or, $\tan\alpha=2\tan\gamma-3\tan\beta$.

Again, if the point C be below the point B, then similarly it can be proved that $T_2=\dfrac{2\cos\alpha}{\sin(\alpha-\beta)}=\dfrac{\cos\gamma}{\sin(\beta+\gamma)}$

or $\tan\alpha=2\tan\gamma+3\tan\beta$.

So, $\tan\alpha=2\tan\gamma\pm3\tan\beta$.

Ex. 9. Forces acting at a point are represented in magnitude and direction by $2\overline{AB}$, $3\overline{BC}$, $2\overline{CD}$, $\overline{DA}$, $\overline{CA}$ and $\overline{DB}$. Show that the forces are in equilibrium. [H. S. '66]

Forces $2\overline{AB}$, $3\overline{BC}$, $2\overline{D}$, $\overline{DA}$, $\overline{CA}$ and $\overline{DB}$ act along the sides AB, BC, CD, DA and the diagonals CA and DB respectively of the quadrilateral ABCD.

Now, $2\overline{AB}+3\overline{BC}+2\overline{CD}+\overline{DA}+\overline{CA}+\overline{DB}$

$=(\overline{AB}+\overline{BC}+\overline{CD}+\overline{DA})+(\overline{AB}+\overline{BC}+\overline{CA})$

$+(\overline{BC}+\overline{CD}+\overline{DB})$ ... ... (1)

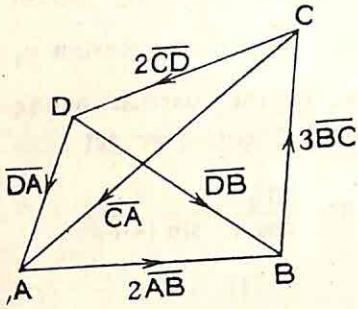


Fig. 29

Now according to the polygon of forces $\overline{AB}+\overline{BC}+\overline{CD}+\overline{DA}=0$ and also by triangle of forces $\overline{AB}+\overline{BC}+\overline{CA}=0$

and $\overline{BC}+\overline{CD}+\overline{DB}=0$.

Hence from (1) $2\overline{AB}+3\overline{BC}+2\overline{CD}+\overline{DA}+\overline{CA}+\overline{DB}=0$

i.e., the forces are in equilibrium.

Ex. 10. ABCDE is a regular pentagon and forces acting at a point are represented in magnitude and direction by $\overline{AB}$, $\overline{AC}$, $\overline{AD}$, $\overline{AE}$, $\overline{BC}$, $\overline{BD}$, $\overline{BE}$, $\overline{CD}$, $\overline{CE}$ and $\overline{DE}$. Prove that their resultant and is represented by $4\overline{AE}+2\overline{BD}$. [C. U.]

By the polygon of forces $\overline{AB}+\overline{BC}+\overline{CD}+\overline{DE}=\overline{AE}$. By the triangle of forces, $\overline{AC}+\overline{CE}=\overline{AE}$, $\overline{AD}+\overline{DE}=\overline{AE}$ and $\overline{BE}+\overline{ED}=\overline{BD}$.

$\therefore$ $\overline{AB}+\overline{BC}+\overline{CD}+\overline{DE}+\overline{AC}+\overline{CE}+\overline{AD}+\overline{DE}+\overline{BE}+\overline{ED}=3\overline{AE}+\overline{BD}$.

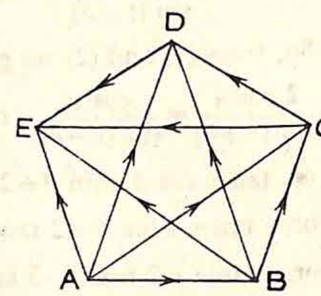


Fig. 30

Now the forces $\overline{DE}$ and $\overline{ED}$ cancel each other :

$\therefore$ $\overline{AB}+\overline{BC}+\overline{CD}+\overline{DE}+\overline{AC}+\overline{CE}+\overline{AD}+\overline{BE}=3\overline{AE}+\overline{BD}$

or, $\overline{AB}+\overline{BC}+\overline{CD}+\overline{DE}+\overline{AC}+\overline{CE}+\overline{AD}+\overline{BE}+\overline{AE}+\overline{BD}$

$=3\overline{AE}+\overline{BD}+\overline{AE}+\overline{BD}$ (Introducing the forces $\overline{AE}$ and $\overline{BD}$ on both sides).

$=4\overline{AE}+2\overline{BD}$.

Ex. 11. A body of weight 30 lbs. has been kept in equilibrium on a smooth inclined plane of inclination 15° by a string on the plane. The string cannot bear a load more

than 15 lbs. The inclination of the plane with the horizontal is gradually increased. Determine when the string will break.

Let the body is in equilibrium when the inclination of the plane to the horizontal is 15°. The forces acting on the body at this time are (i) the tension T of the string along the plane (ii) the weight 30 lbs. of the body acting vertically down-wards and (iii) the reaction of the smooth plane acting along the normal to the plane. As the body is in equilibrium under the action of these forces, so we get by Lami's Theorem

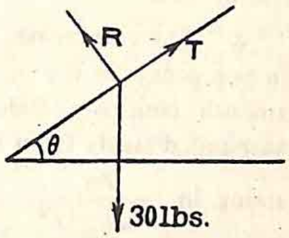


Fig. 31

$$\frac{T}{\sin(180°-\theta)}=\frac{30 \text{ lbs.}}{\sin 90°} \quad \text{or} \quad \frac{T}{\sin\theta}=30 \text{ lbs.}$$

$\therefore\ \sin\theta=\frac{T}{30° \text{ lbs.}}$; Here the maximum value of $T=15$ lbs.

$\therefore$ The maximum value of $\sin\theta$ before the string breaks is $\frac{1}{2}=\sin 30°\quad\therefore\ \theta=30°$.

As $15°<30°$, so at the begining the body was in equilibrium and will remain in equilibrium till the inclination of the plane is 30°. If the inclination of the plane then increases slightly, the string will break.

Exercise 3(B)

1. ABCD is a cyclic quadrilateral ; forces X, Y, Z acting along $\overrightarrow{AB}$, $\overrightarrow{AD}$ and $\overrightarrow{CA}$ are in equilibrium. Prove that $X:Y:Z=CD:CB:BD$.

2. O is the circum-centre of the $\triangle ABC$ and D, E, F are the feet of the perpendiculars from A, B, C respectively on the opposite sides. Three forces P, Q, R acting along $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\overrightarrow{OC}$ are in equilibrium. Prove that,

$$\frac{P}{EF}=\frac{Q}{FD}=\frac{R}{DE}.$$

3. Prove that three forces acting at a point and represented by the directed line segments joining the vertices of the triangle with the middle points of the opposite sides remain in equilibrium.

4. The extremities of a fine string of length l are attached to two points in the same horizontal line at a distance c. A smooth ring can slide along the string and a weight W is suspended freely from the ring. Show that the tension of the string in $\dfrac{lW}{2\sqrt{l^2-c^2}}$.

5. Three weightless cords are knotted with one another to form an equilateral triangle. From the point A is suspended a weight W. Both the triangle and the weight are kept in equilibrium by two other strings knotted at B and C and inclined at angles of 135° with $\overline{BC}$. If $\overline{BC}$ remains horizontal, show that the tension of BC is $\dfrac{W}{6}(3-\sqrt{3})$.

6. Three forces acting at a point are in equilibrium. If the forces are equally inclined with one another, show that the forces are equal in magnitude.

7. A weight is supported on a smooth plane of inclination α to the horizon by a string inclined to the vertical at an angle γ. If the slope of the plane be increased to β and the slope of the string is unaltered, the tension of the string is doubled to support the weight. Prove that $\cot\alpha - \cot\gamma = 2\cot\beta$.

[C. U. 1945]

8. Three forces acting at a point within a triangle along straight lines perpendicular to the sides of the triangle are in equilibrium. Prove that the forces are proportional to the corresponding sides of the triangle.

9. The three equal coplanar line segments $\overline{OA}$, $\overline{OB}$ and $\overline{OC}$ are not all on the same side of any straight line passing through O. Forces P, Q, R act respectively along $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\overrightarrow{OC}$. If

$$\frac{P}{\text{area OBC}} = \frac{Q}{\text{area COA}} = \frac{R}{\text{area AOB}}$$

prove that the forces are in equilibrium.

10. Three forces acting at the point O are in equilibrium. A circle with centre O intersects the lines of action of the forces at A, B and C. Prove that the forces are proportional to the sides of the triangle ABC.

11. Two extremities of two strings are knotted with each other at the point O and from O is suspended a weight 65 kg. The length of the strings are 50 cms. and 120 cms. The other two extremities of the strings are at a distance of 130 cms. and are in the same horizontal line. Find the tensions of the two strings if the system remains in equilibrinm.

12. The extremities A and B of a string are tied to two fixed points in the same horizontal line ; from a given point C of the string is suspended a weight W. Prove that if the system remains in equilibrium, then the tension in the portion CA of the string is $\frac{Wb}{4c\triangle}(c^2+a^2-b^2)$, where a, b, c are the lengths of the sides and $\triangle$ the area of the triangle ABC.

13. The ends of a string of length l are attached to two points A and B on a fixed horizontal beam at a distance a $(a<l)$ apart. A smooth ring of weight W slides on the string and is in equilibrium under horizontal force P when W is vertically below B. Prove that $P=\frac{aW}{l}$. [H. S. Comp. '67]

14. Three forces P, Q, R acting at a point O are in equilibrium. A transversal cuts the lines of action of the forces at the points A, B, C respectively. Show that with a convention regarding sign $\frac{P}{OA.BC}=\frac{Q}{OB.CA}=\frac{R}{OC.AB}$.

15. A body is supported on a smooth plane inclined at an angle α to the horizontal by a force P_1 acting along the plane together with a horizontal force P_2. The body is found to be in equilibrium when both the forces P_1 and P_2 as also the inclination α are all halved. Prove that $P_1 : P_2 = 2\cos^2\frac{\alpha}{4} : 1$.

16. A series of equal weights are knotted at different points of a string, the two extremities of which are tied to two fixed points. Prove that if the system remains in equilibrium, the tangents of the inclination to the horizontal of the successive portions of the string will form an arithmetical progression.

17. A body of mass 80 lbs. is suspended by strings whose lengths are 6 and 8 ft, respectively, from two points in a horizontal line whose distance apart is 10 ft. ; find the tensions of the strings. [H. S. '69]

CHAPTER FOUR

PARALLEL FORCES

§ 4·1. **Parallel forces**: If the lines of action of two or more forces be parallel, then the forces are said to be parallel forces. If two parallel forces have the same sense, then the forces are said to be *like parallel forces* and if they have opposite senses, then the forces are said to *be unlike parallel forces.* In the previous chapter we have discussed concurrent forces which act on a particle or at a single point of body. Two or more parallel forces cannot act on a particle or at a point of a body, because then they will be intersecting.

In the next two articles we shall determine the resultant of two like parallel forces or two unequal and unlike parallel forces. Two equal and unlike parallel forces constitute a couple. We shall discuss couples in Chapter six.

§ 4·2. To find the resultant of two like parallel forces:

Two like parallel forces P and Q respectively act at the point A and B of a rigid body. To find the resultant of the forces.

Join AB. Introduce two equal and opposite forces F and F at the points A and B along $\overrightarrow{AB}$ and $\overrightarrow{BA}$ respectively. Since the equal forces act along the same line in opposite directions, they balance each other and will have no effect on the body. [See the axiom of § 1·4].

Let according to any predetermined scale the directed line segments $\overline{AL}$ and $\overline{AD}$ respectively represent the forces P and F acting at A and $\overline{BM}$ and $\overline{BE}$ represent respectively the forces Q and F acting at B.

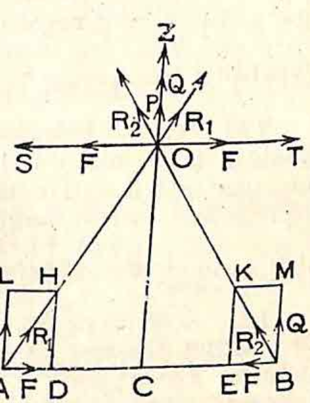


Fig. 32

Complete the parallelograms ADHL and BEKM. Hence by the parallelogram law of forces the directed line segments $\overline{AH}$

and $\overline{BK}$ respectively represent the resultants of the forces P and F acting at A and of the forces Q and F acting at B.

Let these resultants be R_1 and R_2.

Let the line segments $\overline{AH}$ and $\overline{BK}$ when produced intersect at O. Through O draw $\overrightarrow{CO}$ parallel to the lines of action of P and Q and let $\overrightarrow{CO}$ intersect $\overline{AB}$ at C. Now shift the point of application of the forces R_1 and R_2 to the point O according to the principle of transmissibility of forces and draw $\overrightarrow{ST}$ through O parallel to $\overline{AB}$.

Resolve the force R_1 acting at O into two components P along $\overrightarrow{CO}$ and F along $\overrightarrow{OT}$ parallel to $\overline{AL}$ and $\overline{AB}$. Also, resolve the force R_2 acting at O into two components Q along $\overrightarrow{CO}$ and F along $\overrightarrow{OS}$ parallel to $\overline{BM}$ and $\overline{BA}$ respectively.

Now the two equal and opposite forces F and F acting at O balance each other and we are left with two forces P and Q acting at O both along $\overrightarrow{CO}$. Since these forces P and Q have the same line of action, direction, and sense, so their resultant is a force $R=P+Q$ along $\overrightarrow{CO}$. Now, shift the point of application of this resultant force $R=P+Q$ to the point C on its line of action.

Hence the resultant of the given like parallel forces P and Q acting at the points A and B respectively is a like parallel force $P+Q$ acting at the point C on $\overline{AB}$ along $\overrightarrow{CO}$.

Let us now determine the position of C. The triangles ADH and ACO are similar.

$$\therefore \frac{AD}{DH}=\frac{AC}{CO}, \quad \therefore \frac{F}{P}=\frac{AC}{CO}, \quad \text{or,} \quad P.AC=F.CO \quad \cdots\cdots(1)$$

Again the triangles BEK and BCO are similar.

$$\text{So, } \frac{BE}{EK}=\frac{BC}{CO}, \quad \therefore \frac{F}{Q}=\frac{BC}{CO}, \quad \text{or,} \quad Q.BC=F.CO \quad \cdots\cdots(2)$$

From (1) and (2) we obtain, $P.AC = Q.BC$.

or, $\frac{AC}{BC} = \frac{Q}{P}$; *i.e.* the line segment $\overline{AB}$ is divided at C in the inverse ratio of the forces.

Note. 1. As $\frac{AC}{BC} = \frac{Q}{P}$, so the resultant is nearer to the greater force.

2. If $P = Q$, then C is the middle point of $\overline{AB}$.

3. As, $\frac{AC}{BC} = \frac{Q}{P}$, $\therefore$ $\frac{P}{BC} = \frac{Q}{AC} = \frac{P+Q}{BC+AC} = \frac{R}{AB}$ and this is the working formula.

One can remember this formula as follows :

$$\frac{P}{\text{distance between the other two forces}} = \frac{Q}{\text{distance between the other two forces}} = \frac{R}{\text{distance between the other two forces}}$$

4. The magnitude and point of application of the resultant force depends on the magnitudes and points of application of the two given forces and not on their direction and sense (provided they are like).

5. In this article, the principle of transmisibility of forces has been used assuming that the points of application concerned are rigidly connected.

§ 4·3. **To find the resultant of two unequal and unlike Parallel Forces.**

P and Q $(P > Q)$ are two unequal and unlike parallel forces acting at points A and B of a rigid body. To find the resultant of the forces. Join AB. Introduce two equal and opposite forces F and F along $\overrightarrow{AB}$ and $\overrightarrow{BA}$ acting at the points A and B respectively. As these equal forces are acting on a rigid body along the same line in opposite senses, so by the axiom stated in § 1·4, the forces will balance each other and their introduction will not affect the state of the body.

Let according to any pre-determined scale the directed line segments $\overline{AL}$ and $\overline{AD}$ respectively represent the forces P and F acting at A and the directed line segments $\overline{BM}$ and $\overline{BE}$ respectively represent the forces Q and F acting at B. Complete the parallelograms ADHL and BEKM. Hence according to the parallelogram law of forces the directed line segments $\overline{AH}$ and $\overline{BK}$ will respectively represent the resultants R_1 and R_2 of the forces P and F acting at A and the forces Q and F acting at B. Hence the resultant of the forces P and Q acting on the rigid body is the same as the resultant of the forces R_1 and R_2 acting on the body.

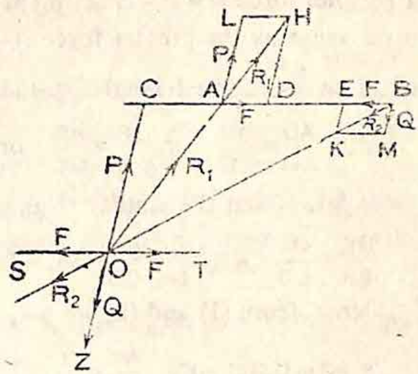


Fig. 33

Produce $\overrightarrow{HA}$ and $\overrightarrow{BK}$ to intersect at O. Through O draw $\overrightarrow{OZ}$ parallel to the lines of action of P and Q to intersect $\overleftrightarrow{AB}$ at the point C on $\overline{BA}$ produced. Now shift the points of applications of R_1 and R_2 to the point O on their lines of action (by the principle of transmissibility of forces). Draw $\overleftarrow{ST}$ through O parallel to $\overline{AB}$.

Now resolve the force R_1 acting at O into components P and F parallel to $\overline{AL}$ and $\overline{AB}$ along $\overrightarrow{OC}$ and $\overrightarrow{OT}$ respectively. Also, resolve the force R_2 acting at O into components Q and F parallel to $\overline{BM}$ and $\overline{BA}$ and along $\overrightarrow{CO}$ and $\overrightarrow{OS}$ respectively.

Now, the forces F acting at O along $\overrightarrow{ST}$ and $\overrightarrow{TS}$ are equal and opposite and so they balance each other and we are left with two forces P and Q acting at O along $\overrightarrow{OC}$ and $\overrightarrow{CO}$ respectively. The resultant of the two forces is $P-Q$ along $\overrightarrow{OC}$, since they act along the same line in opposite senses and $P > Q$. Now transfer the point of application of this resultant force to the point C on its line of action by the principle of transmissibility of forces.

Hence the resultant of the two given unequal and unlike parallel forces acting at the points A and B of the rigid body is a parallel force $R=P-Q$ acting at a point C on $\overleftrightarrow{AB}$ having the same sense as the greater force P. Let us now find the position of C on $\overleftrightarrow{AB}$. The triangle ADH and ACO are similar.

$$\therefore \quad \frac{AD}{DH}=\frac{AC}{CO} \quad \text{or,} \quad \frac{F}{P}=\frac{AC}{OC} \quad \text{or,} \quad F.OC=P.AC \quad \cdots\cdots(1)$$

Again, from the similar triangles BEK and BCO we obtain,

$$\frac{BE}{EK}=\frac{BC}{CO} \quad \text{or,} \quad \frac{F}{Q}=\frac{BC}{OC} \quad \text{or,} \quad F.OC=Q.BC \quad \cdots\cdots(2)$$

Now, from (1) and (2) we get,

$$P.AC=Q.BC \quad \text{or,} \quad \frac{AC}{BC}=\frac{Q}{P}.$$

Hence C divides $\overline{AB}$ externally in the inverse ratio of P and Q.

Note. (1) It is evident that the resultant is nearer to the greater force.

(2) As, $\frac{AC}{BC}=\frac{Q}{P}$, $\therefore \quad \frac{P}{BC}=\frac{Q}{AC}=\frac{P-Q}{BC-AC}=\frac{R}{AB}.$

and this is the working formula. This formula also can be remembered as,

$$\frac{P}{\text{distance between the other two forces}}$$
$$=\frac{Q}{\text{distance between the other two forces}}$$
$$=\frac{R}{\text{distance between the other two forces}}$$

3. In this article also whereever we have used the principle of transmissibility of forces, we have assumed that the points of application under consideration are rigidly connected,

4. If $P=Q$, then $\overline{AH}$ and $\overline{BK}$ will not intersect.

Here $P \neq Q$; If $P=Q$, then the forces constitute a couple.

§ 4·4. Resultant of more than two parallel forces.

Suppose we are to determine the resultant of forces P, Q; R; S etc.

Now there are two possibilities. (i) All the forces are like and (ii) some of the forces are like to one another and others are unlike to this group but like to one another.

Now (i) if the forces are all like, first determine the resultant $P+Q$ of the forces P and Q. Next determine the resultant $P+Q+R$ of the forces $P+Q$ and R, then the resultant $P+Q+R+S$ of the forces $P+Q+R$ and S and so on.

Continue this process till all the forces are exhausted and finally the resultant of all the forces $F=P+Q+R+S+\cdots\cdots$ will be obtained.

(ii) Determine the resultants F_1 and F_2 of the two groups in the process shown in (i) above.

Evidently F_1 and F_2 are two unlike parallel forces.

Now, if $F_1=F_2$ and if they act along the same straight line, then they will cancel each other and the given forces will be in equilibrium.

If $F_1=F_2$ and if their lines of action be different, then the forces will constitute a couple.

If $F_1=F_2$ and (*a*) $F_1>F_2$, then the resultant force will be F_1-F_2, a force, like parallel to the forces in the first group.

If (*b*) $F_1<F_2$, then the resultant force will be F_2-F_1, a force like parallel to the forces of the second group.

Note. If the system of parallel forces possesses a resultant, then the magnitude and point of application of the resultant will depend on the magnitude and points of application of the forces of the system and not on the direction of the forces.

Example 1. Two parallel forces 14 kg. and 10 kg. act at two points at a distance of 36 cms. Find the magnitude and point of application of the forces if (i) the forces be like and (ii) the forces be unlike.

(i) Since the forces are like parallel, their resultant is a like parallel force $14+10=24$ kg. The point of application will divide the line joining the points of application of the given forces in the ratio $\frac{10}{14}=\frac{5}{7}$. Hence the point of application of the resultant force is at a distance $\dfrac{5}{5+7}\times 36$ cms. $=15$ cms. from the point of application of the force of magnitude 14 kg.

(ii) Let the points of application of the forces 14 kg. and 10 kg. be A and B respectively. Since the forces are unlike parallel, so the resultant is a force $14-10=4$ kg. and it is like parallel of the bigger force 14 kg. The point of application C of the resultant will divide $\overline{AB}$ externally in the ratio :

$$10 : 14 \text{ or, } 5 : 7 \quad i.e. \ \frac{AC}{BC}=\frac{5}{7},$$

$$\text{or, } \frac{AC}{AB+AC}=\frac{5}{7}, \quad \text{or, } 7AC=5AB+5AC$$

$$\text{or, } 2AC=5AB, \quad \text{or, } AC=\frac{5\times 36 \text{ cms.}}{2}=90 \text{ cms.}$$

Hence the point of application of the resultant force is at distances 90 cms. from A and $90+36=126$ cms. from B.

Ex. 2. Two like parallel forces act at the extremities of a rod of length 100 cms. The resultant of these parallel forces is 75 kg. and its point of application divides the rod in the ratio 2 : 3. Find the magnitudes of the forces.

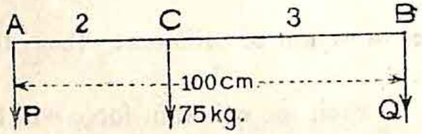


Fig. 34

Let P and Q be the like parallel forecs acting at the extremities A and B respectively of the rod AB. The resultant of the forces is $P+Q=75$ kg......(1).

Also, $\frac{AC}{BC}=\frac{Q}{P}$; so by question,

$$\frac{Q}{P}=\frac{2}{3} \quad \therefore \quad 3Q=2P\cdots\cdots(2).$$

Solving equations (1) and (2) we obtain

$$P=45 \text{ kg. and } Q=30 \text{ kg.}$$

Hence the required forces are 45 kg. and 30 kg.

Ex. 3. The magnitudes P and Q ($P>Q$) of two unlike parallel forces are each increased by F. Show that the magnitude of their resultant will remain unaltered but its point of application will move farther from the force P.

As each of the forces is increased by F, so the increased magnitudes are $P+F$ and $Q+F$. Hence after increase the

resultant is $(P+F)-(Q+F)=P-Q.=$ resultant of the original forces P and Q.

Hence the magnitude of the resultant is not changed.

Now let the resultant of the forces P and Q act at C before they are increased in magnitude and D be the point of application after each in increased by F.

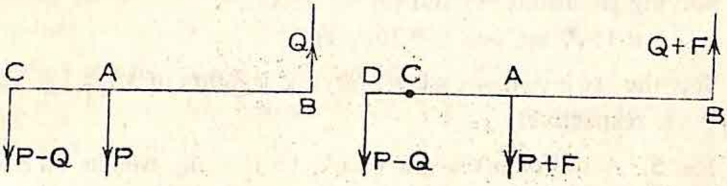


Fig. 35

$\therefore \quad \frac{AC}{BC}=\frac{Q}{P} \quad \cdots \quad \cdots(1)$ and $\frac{AD}{BD}=\frac{Q+F}{P+F} \quad \cdots \quad \cdots(2)$

From (1) we get, $P.AC=Q.BC=Q(BA+AC)$

$\therefore \quad (P-Q).AC=Q.BA \quad \cdots \quad \cdots(3)$

From (2) we get,

$(P+F).AD=(Q+F).BD=(Q+F)(BA+AD)$

or, $(P-Q).AD=(Q+F).BA \quad \cdots \quad \cdots(4)$

So, dividing equation-(4) by equation-(3)

we obtain, $\frac{AD}{AC}=\frac{Q+F}{Q}>1 \quad \therefore \quad AD>AC.$

Hence the point of application of the resultant is moved farther from A.

Ex. 4. Two persons carry between them a uniform rod of length 10 metres and weight 42 kg. They were carrying the rod at distances $1\frac{1}{2}$ metre and 3 metres from the two extremities respectively. Determine, how much weight each person was carrying.

Let the rod be AB.

Since the rod is uniform, its weight 42 kg. act at its middle point O.

Let $AC=1\frac{1}{2}$ m. and $BD=3$ m. and one person was carrying a weight P at C and the other was carrying a weight Q at D.

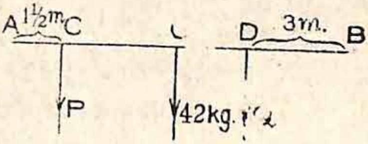


Fig. 36

$\therefore$ $P+Q=42$ kg. ... ...(1)

and $P.CO=Q.DO$ ... ...(2).

Now $AO=5$ m. $\therefore$ $CO=5$ m.$-1\frac{1}{2}$ m.$=3\frac{1}{2}$ m.

and $DO=5$ m.-3 m.$=2$ m.

$P.3\frac{1}{2}=Q.2$ or, $\frac{P}{Q}=\frac{4}{7}$ ... ...(3).

Solving equations (1) and (3) we obtain,

$P=15\frac{3}{11}$ kg. and $Q=26\frac{8}{11}$ kg.

So, the two persons were carrying weights of $15\frac{3}{11}$ kg. and $26\frac{8}{11}$ kg. respectively.

Ex. 5. A uniform see-saw plank, 16 ft. long, weighs 16 cwt. Find the position of the support when two children weighing 44 lbs. and 68 lbs. respectively sit at the two ends. [P.U. 1945]

Let the plank be AB and its length and weight are 16 ft. and 1 cwt.$=112$ lbs.

Let C be the middle point of the rod. So the weight 112 lbs. of the rod acts at C vertically downwards. Let the two children of weights 68 lbs. and 44 lbs. sit respectively at the extremities A and B. Hence these like parallel forces 68 lbs. at A, 44 lbs. at B and 112 lbs. at C are acting on the plank. The support is to be placed at the point on the plank in which the resultant of these forces act.

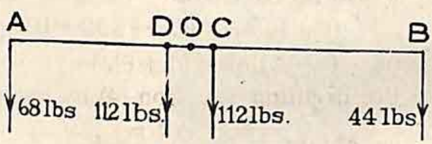


Fig. 37

Now, let the resultant $68+44=112$ lbs. of the weights of the two children act at the point D on AB.

$\therefore$ $\frac{AD}{BD}=\frac{44}{68}=\frac{11}{17}$, or, $\frac{BD}{AD}=\frac{17}{11}$,

or, $\frac{BD}{AD}+1=\frac{17}{11}+1$ or, $\frac{AB}{AD}=\frac{28}{11}$

$\therefore$ $AD=\frac{11}{28}\times AB=\frac{11}{28}\times 16$ ft.$=6\frac{2}{7}$ ft.

$\therefore$ $CD=8-6\frac{2}{7}=\frac{12}{7}$ ft.

Now, the resultant $112+112=224$ lbs. of the like parallel forces 112 lbs. at D and 112 lbs. at C act at the middle point O of CD. Hence the support is to be placed at O.

Now, $OD = \frac{1}{2}CD = \frac{12}{7} \div 2 = \frac{6}{7}$ ft.

$AO = AD + DO = 6\frac{2}{7} + \frac{6}{7} = 7\frac{1}{7}$ ft.

Hence the see-saw plank is to be supported at a point O which is at a distance $7\frac{1}{7}$ ft. from the extremity in which the heavier child weighing 68 lbs. sit.

Ex. 6. A man was carrying a weight of 24 kg. at the extremity of a stick, 4 meters long which is placed on his shoulder ; the other extremity was within his hand.

Find the pressure on his shoulder when (i) the weight is 1 metre behind his shoulder and (ii) the weight is $1\frac{1}{2}$ metre behind his shoulder.

(i) Let the man exert a downward pressure P by his hand and the pressure on his shoulder be Q.

$\therefore \quad P + 24 = Q \quad \cdots \quad \cdots (1)$

and $P \times 4 = Q \times 1$; or, $Q = 4P \quad \cdots \quad \cdots (2)$.

$\therefore$ From (1) $P + 24 = 4P$; or, $3P = 24$

or, $P = 8$ kg, $\therefore \quad Q = 32$ kg.

(ii) In the second case, let the pressure of the hand and the pressure on the shoulder be P_1 and Q_1 respectively.

$\therefore \quad P_1 + 24 = Q_1 \quad \cdots \quad \cdots (1)$

and $P_1 . 4 = Q_1 \times 1{\cdot}5$; or, $\dfrac{P_1}{Q_1} = \dfrac{3}{8}$.

$\therefore \quad P_1 + 24 = \frac{8}{3}P_1 \quad \therefore \quad \frac{5}{3}P_1 = 24$ or, $P_1 = \frac{72}{5}$ kg.

$\therefore \quad Q_1 = \frac{8}{3}P_1 = \frac{72}{5} \times \frac{8}{3} = \frac{192}{5} = 38\frac{2}{5}$ kg.

Ex. 7. The extremities of a weightless straight bamboo pole 8 ft. long rests on two smooth pegs P and Q. A heavy load hangs from a point R of the pole. If $PR = 3RQ$ and the pressure at Q be 325 lbs. more than that at P, find the weight of the load.

[C. U. 1941]

Let the weight of the load be W and the pressures at P and Q be W_1 and $W_1 + 325$ lbs. respectively.

$\therefore \quad W = W_1 + W_1 + 325$ lbs. $= 2W_1 + 325$ lbs.

Also $W_1 . PR = (W_1 + 325 \text{ lbs.}) QR$

or, $W_1 . 3QR = (W_1 + 325 \text{ lbs.}) QR$.

or, $3W_1 = W_1 + 325$ lbs., *i.e.* $2W_1 = 325$ lbs.

Hence $W = 2W_1 + 325$ lbs. $= 325$ lbs. $+ 325$ lbs. $= 650$ lbs.

Ex. 8. Three like parallel forces P, Q, R respectively act at the angular points A, B, C of a triangle ABC. If the resultant of the forces passes through the circumcentre of the triangle for all directions of the forces show that,

$$\frac{P}{\sin 2A}=\frac{Q}{\sin 2B}=\frac{R}{\sin 2C}.$$

Let O be the circum centre of the triangle ABC and join AO.

Let $\overrightarrow{AO}$ produced intersect $\overline{BC}$ at D.

Now the resultant $Q+R$ of the parallel forces Q and R acting at B and C respectively acts at a point E of BC such that $Q.BE=R.CE$ ...(i)

Again, the resultant of the like parallel forces P at A and $Q+R$ at E is a like parallel forces $P+Q+R$ and acts at a point O′ of $\overline{AE}$, such that $P.AO'=(Q+R)EO'$ ...(ii)

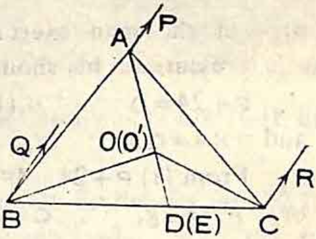


Fig. 38

Now, as the resultant passes through O, so the line of action of this resultant is $\overleftrightarrow{OO'}$. Hence $\overleftrightarrow{OO'}$ is parallel to the direction of the parallel forces. So whatever be the direction of the forces P, Q, R their resultant will always act along OO′. This is possible if and only if O and O′ are the same point (for, the point O′ is independent of the direction of P, Q, R).

$\therefore$ O and O′ will be coincident and so D and E will also coincide.

$\therefore$ From (i) we get, $\dfrac{Q}{R}=\dfrac{DC}{BD}=\dfrac{\frac{DC}{OD}}{\frac{BD}{OD}}$

$$=\frac{\frac{\sin COD}{\sin OCD}}{\frac{\sin BOD}{\sin OBD}} \quad [\text{From } \triangle COD \text{ and } \triangle BOD]$$

Now as O is the circumcentre of the triangle,

So, $m\angle \text{OBD} = m\angle \text{OCD}$. $\therefore$ $\sin \text{OCD} = \sin \text{OBD}$.

$$\therefore \quad \frac{Q}{R} = \frac{\sin \text{COD}}{\sin \text{BOD}} = \frac{\sin(180° - \angle \text{AOC})}{\sin(180 - \angle \text{AOB})} = \frac{\sin \angle \text{AOC}}{\sin \angle \text{AOB}}.$$

Again, $\angle \text{AOC} = 2\angle \text{B}$ and $\angle \text{AOB} = 2\angle \text{C}$.

[As angle at the centre is double of the angle at the circumference].

$$\therefore \quad \frac{Q}{R} = \frac{\sin 2B}{\sin 2C} \quad \text{or,} \quad \frac{Q}{\sin 2B} = \frac{R}{\sin 2C}$$

Similarly it can be proved that

$$\frac{P}{\sin 2A} = \frac{Q}{\sin 2B} \quad \therefore \quad \frac{P}{\sin 2A} = \frac{Q}{\sin 2R} = \frac{R}{\sin 2C}.$$

Ex. 9. If the points of application A and B of two like parallel forces P and Q ($P > Q$) acting on a rigid body be interchanged in position; show that the point of application of the resultant will be displaced along $\overline{AB}$ through a distance d, where $d = \frac{P-Q}{P+Q}.AB$ [C. U. 1950]

Let at first the point of application of the resultant be C.

$$\therefore \quad \frac{P}{BC} = \frac{Q}{AC}, \text{ or, } \frac{Q}{P} = \frac{AC}{BC}; \text{ or, } \frac{Q}{P} + 1 = \frac{AC}{BC} + 1$$

$$\text{or, } \frac{P+Q}{P} = \frac{AB}{BC}. \quad \text{or, } BC = \frac{P}{P+Q}.AB.$$

Let now, the points of applications of P and Q be changed to B and A respectively and the resultant act at the point D on $\overline{AB}$.

$$\therefore \quad \frac{P}{AD} = \frac{Q}{BD}; \quad \text{or, } \frac{P}{Q} = \frac{AD}{BD}; \quad \text{or, } \frac{P}{Q} + 1 = \frac{AD}{BD} + 1$$

$$\text{or, } \frac{P+Q}{Q} = \frac{AB}{BD}; \quad \therefore \quad BD = \frac{Q}{P+Q}.AB.$$

Now, as $P > Q$ $\therefore$ $BC > BD$.

Hence the displacement of the resultant force

$$= d = BC - BD = \frac{P}{P+Q}AB - \frac{Q}{P+Q}.AB = \frac{P-Q}{P+Q}.AB$$

Ex. 10. Three like parallel forces P, Q, R act at the angular points of a triangle. If the resultant of the parallel

forces always pass through the in centre of the triangle, prove that $\frac{P}{\sin A}=\frac{Q}{\sin B}=\frac{R}{\sin C}$ or, $\frac{P}{a}=\frac{Q}{b}=\frac{R}{c}$.

If the resultant of the like parallel forces Q and R acting at B and C respectively act at the point D on BC; then

$$\frac{BD}{CD}=\frac{R}{Q}\cdots\cdots(1).$$

Now, the resultant of the like parallel forces P at A and Q+R at D will pass through point I′ on AD such that

$$\frac{AI'}{DI'}=\frac{Q+R}{P}\cdots\cdots(2).$$

Now, as the resultant of the forces P, Q, R pass through the incentre I of the triangle, the resultant which is parallel to P, Q, R will act along II′ what ever be the direction of P, Q, R. But this is possible if and only if I and I′ coincide. So I and I′ are the same point. ∴ AD is the bisector of $\angle BAC$.

$$\therefore \frac{BD}{DC}=\frac{AB}{AC}=\frac{c}{b}=\frac{\sin C}{\sin B}.$$

So, from (1), $\frac{R}{Q}=\frac{\sin C}{\sin B}$ or, $\frac{Q}{\sin B}=\frac{R}{\sin C}$.

Similarly it can be proved that $\frac{P}{\sin A}=\frac{Q}{\sin B}$.

$$\therefore \frac{P}{\sin A}=\frac{Q}{\sin B}=\frac{R}{\sin C}$$

or, $\frac{P}{\frac{a}{2R'}}=\frac{Q}{\frac{b}{2R'}}=\frac{R}{\frac{c}{2R'}}$ [Where R′ is the circum—radius of the triangle.]

or, $\frac{P}{a}=\frac{Q}{b}=\frac{R}{c}$.

Ex. 11. Three like parallel forces P, Q, R act at the angular points A, B, C of a triangle ABC. If the resultant of these forces always pass through the ortho centre of the triangle, then show that $\frac{P}{\tan A}=\frac{Q}{\tan B}=\frac{R}{\tan C}$

or, $P(b^2+c^2-a^2)=Q(c^2+a^2-b^2)=R(a^2+b^2-c^2)$.

Let O be the orthocentre of the $\triangle ABC$. Let AO produced intersect BC at D. Then AD is perpendicular on BC.

Now the resultant of the like parallel forces Q and R acting at B and C respectively is a like parallel force Q+R acting at a point E on BC such that $\frac{Q}{R}=\frac{CE}{BE}$......(1).

Again the resultant of the like parallel forces P and Q+R acting at A and E respectively is a like parallel force P+Q+R acting at a point O′ on AE such that $\frac{AO'}{EO'}=\frac{Q+R}{P}$......(2).

As the resultant of the forces P, Q, R passes through O, so OO′ is the line of action of this resultant which is parallel to P, Q, R.

So, whatever be the direction of the forces, the resultant will act along OO′ and this is possible if and only if O and O′ coincide. So O and O′ are the same point and so D and E coincide.

$\therefore$ From (1) $\frac{R}{Q}=\frac{BD}{DC}=\frac{AD\cot B}{AD\cot C}=\frac{\tan C}{\tan B}$.

$\therefore \quad \frac{Q}{\tan B}=\frac{R}{\tan C}$.

Similarly, it can be proved that $\frac{P}{\tan A}=\frac{Q}{\tan B}$.

$\therefore \quad \frac{P}{\tan A}=\frac{Q}{\tan B}=\frac{R}{\tan C}$.

or, $\dfrac{P}{\frac{\sin A}{\cos A}}=\dfrac{Q}{\frac{\sin B}{\cos B}}=\dfrac{R}{\frac{\sin C}{\cos C}}$

or, $\dfrac{P}{\frac{a}{2R'}\cdot\frac{1}{\frac{b^2+c^2-a^2}{2bc}}}=\dfrac{Q}{\frac{b}{2R'}\cdot\frac{1}{\frac{c^2+a^2-b^2}{2ca}}}=\dfrac{R}{\frac{c}{2R'}\cdot\frac{1}{\frac{a^2+b^2-c^2}{2ab}}}$

[Where R′ is the circum-radius of the triangle.]

or, $\frac{P(b^2+c^2-a^2)}{abc}=\frac{Q(c^2+a^2-b^2)}{abc}=\frac{R(a^2+b^2-c^2)}{abc}$

or, $P(b^2+c^2-a^2)=Q(c^2+a^2-b^2)=R(a^2+b^2-c^2)$.

Exercise 4

1. In each of the examples find the magnitude and point of application of the resultant of the given like parallel forces.

(i) Magnitude : 4 kg ; 6 kg. distance : 30 cms.

(ii) Magnitude : 600 gms ; 200 gms ; distance : 80 cms.

(iii) Magnitude : 3 lbs. ; 11 lbs. ; distance : 56 inches.

2. In each of the examples find the magnitude and point of application of the unlike parallel forces whose magnitude and distance of the lines of action are given.

(i) $7\frac{1}{2}$ kg. and $1\frac{1}{2}$ kg. ; 96 cms.

(ii) 4 kg. and 16 kg. ; 90 cms.

(iii) 1000 lbs. and 800 lbs. ; 450 ft.

3. (*a*) Two like parallel forces 8 kg. and 16 kg. act at an extremity and the middle point of a rod. Find the magnitude and point of application of the force that will balance the rod.

(*b*) Find the magnitude and point of application of the force in the above example, if the forces be unlike.

4. Find the line of action of the resultant of two unlike parallel forces, the distance between whose lines of action is 18 cms. and whose magnitudes are in the ratio 4 : 5.

5. The magnitudes of the smaller of two unlike parallel forces and their resultant are 12 and 8 dynes respectively. If the distance between the lines of action of the smaller force and the resultant be 18 cms, find the magnitude and line of action of the other force.

6. A man was carrying two weights by suspending them from the extrimities of a weightless rod placed on his shoulder. If the length of the rod be 39 cms. and one weight be half of the other find in which point of the rod it should be supported on the shoulder.

7. A man carries a bundle at the end of a stick which is placed horizontally over his shoulder ; if the distance between his hand and his shoulder be changed, how does the pressure on his shoulder change ?

8. A 2·5 metre long uniform rod weighing 24 kg. is placed on a table so that 0·25 metre of the rod is outside the table. Find what maximum weight can be suspended from the extremity, that is outside the table, so that the rod will remain in equilibrium.

9. The distance between the points of application of two unlike parallel forces P, C($P>Q$) is 3 metres and their resultant is 5 kg. If the point of application of the resultant force be at a distancc of 2 metres from that of the larger force, find the magnitudes of P and Q.

10. A heavy uniform rod is placed on two smooth pegs whose distance is 2 metres. If the pressures on the pegs be in the ratio 1 : 2, find the distance of the pegs from the middle point of the rod.

11. Two persons have to carry a stone of weight 300 lbs. by suspending it from a 6 ft. long weightless rod. The weaker of the two persons cannot carry more than 100 pounds. From which point of the rod, the stone should be suspended so that the weaker person can carry his maximum share.

12. A mass 50 kg. is placed on a weightless horizontal plank, 10 metres long at a point 1 metre from one end. The plank is then supported at its two ends. If the weight is now placed at the middle point of the plank, determine the change of pressure at each end.

13. Two like parallel forces P and Q act at given points of a body ; if Q be changed to $\frac{P^2}{Q}$ or the forces are simply inter-changed, show that the line of action of the resultant is the same in both the cases.

14. If the line of action of the resultant of two like parallel forces P and Q is unaltered, when the points of application of P and Q are interchanged, prove that $P=C$.

15. A heavy uniform rod, 4 metres long is placed horizontally on two smooth pegs at a distance of 1 metre. The rod is on the point of overturn if weights 10 kg. and 4 kg. are placed at the two ends in succession. Determine the weight of the rod and the distance of the pegs from the centre of the rod.

16. Prove that the algebraic sum of the resolved parts of two parallel forces (which do not constitute a couple) along any line in their plane is equal to the resolved part of their resultant along the same line.

17. Three equal like parallel forces act (i) at the angular points of a triangle or (ii) at the middle points of the sides of a triangle. Show that in both cases the resultant passes through the centroid of the triangle.

18. If the resultant of three like parallel forces acting at the vertices of a triangle, passes in all cases whatever be their common direction through the centroid of the triangle, show that the forces are equal in magnitude.

19. R is the resultant of two like parallel forces P and Q. If P is moved parallel to itself through a distance d, show that R is displaced through a distance $\frac{Pd}{P+Q}$.

20. If the resultant of every pair of like parallel forces acting at A and B (i) P and Q, (ii) P+R and Q+S and (iii) Q and R passes through a fixed point of $\overline{AB}$, then $\frac{(Q-R)^2}{P-Q}=R-S$.

21. The resultant of two parallel forces P, Q acting at A, B acts at C when the forces are like and acts at D when the forces are unlike. If parallel forces whose magnitudes are equal to these resultant forces, act simultaneusly at C and D, then prove that the point of application of the resultant force is A or B according as the forces are like or unlike.

22. Three like parallel forces of magnitude, K.BC, K.CA and K.AB are applied at the vertices A, B, C respectively of a triangle ABC. Prove that the resultant of the forces passes through the in-centre of the triangle.

23. O is a point within the triangle ABC. Like parallel forces of magnitudes K.$m\triangle$BOC, K.$m\triangle$COA and K.$m\triangle$AOB act at the vertices of the triangle. Show that the resultant of the forces passes through the point O.

24. A uniform heavy rod, 8 metres long is placed on two smooth pegs in the horizontal position. If the distance of the

pegs be 6 metres and one end of the rod is placed on one of the pegs, show that the pressure on one peg is twice that on the other.

25. O is a point between A and B on the line segment $\overline{AB}$. Two like parallel forces P and Q act at the mid points of $\overline{AO}$ and $\overline{BO}$ and their resultant passes through the point O. Prove that if the points of application of P and Q are mutually interchanged, then the resultant force will pass through the middle point of $\overline{AB}$.

26. P and Q ($P > Q$) are two like parallel forces and their resultant is R. Find how much the magnitude of P must be decreased so that the distance of the line of action of the resultant from that of P will be equal to the distance between the lines of action of the forces R and Q.

CHAPTER FIVE

MOMENT

§ 5·1. The moment of a force about a point is the product of the magnitude of the force and the perpendicular distance of the point from the line of action of the force.

ON is perpendicular from the point O on the line of action of the force P. If $ON = p$, then the moment of the force P about the point $O = P.ON = P.p$.

Since moment is the product of two factors, so the moment vanishes if any one of the factors is zero. Hence if P or the perpendicular distance of the point from the line of action of the force is zero, i.e., if the point lies on the line of action of the force, then the moment is zero. Hence if the moment of a force about any point vanishes, then the line of action of the force passes through the point and conversely the moment of a force about any point on its line of action vanishes.

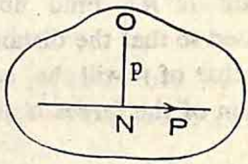


Fig. 39

Since the moment is the product of a force and the perpendicular distance of the point from the line of action of the force, so the unit of moment = unit of force × unit of length.

In the C.G.S. system the unit of moment is gramme-weight—Centimetre where as in the F.P.S. system it is pound-weight—foot. In the M.K.S. system the unit of moment is kg.-weight-metre.

§ 5·2. **Physical Concept of moment.**

Suppose a plane lamina is fixed at the point O on a table with the help of a pin. Now apply a force P at any point A of the lamina. You will find that the lamina undergoes a rotation about the point O. It will also be found that if the line of action of the force P passes through the point O, then the lamina cannot undergo any rotation. In Fig. 40 the line of action of P and the position of O

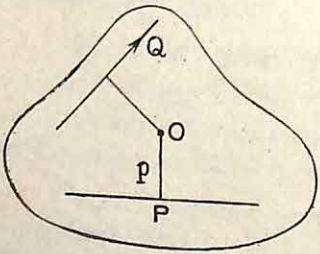


Fig. 40

have been shown. In this position the lamina undergoes a rotation in the counter-clock wise direction. But instead of the force P if the force Q along the line shown in the figure is applied on the lamina, then it undergoes a rotation in the clock-wise direction.

Let us now consider the case when both the forces P and Q are applied on the lamina simultaneously. The forces P and Q will have tendencies to rotate the lamina respectively in the counter clock-wise and clock-wise directions. Let p and q be the perpendicular distances of the lines of action of the forces P and Q respectively from the point O. It will be found that if $P.p > Q.q$, then the lamina will rotate in the counter clock-wise direction. If $P.p = Q.q$, then the lamina will not undergo any rotation. Let now $P = Q$. It will be found what the lamina undergoes rotation in the counter-clock-wise or clock-wise directions according as $p > q$ or $p < q$. Similarly if $p = q$, then the lamina undergoes rotation in the counter clock-wise or clock-wise directions according as $P > Q$ or $P < Q$. Thus, we find that if a body is fixed at a point, then the body can be rotated about the point by the application of force. This rotation may be in the counter clock-wise or clock-wise directions. The amount of rotation depends upon the product of the force and the perpendicular distance of the point from the line of action of the force. This product is the moment of the force about the point. If any of the factors of the moment increases or decreases, then the rotation will also respectively increase or decrease. Hence if a point of a body is fixed, then the body undergoes rotation about the point and the amount of rotation depends upon the moment of the force about the point.

§ 5·3. Sign of moments.

In the previous section we have seen that if a point of a body is fixed and if a force is applied on the body, then the body undergoes rotation and the rotation may be in the counter clock-wise or in the clockwise directions. Conventionally, when the rotation of the body is in the counter clock-wise sense, then the moment of the force about the point is taken to be positive

and the moment is negative when the rotation is in the clockwise direction;

§ 5·4. Geometrical representation of moments.

In the figure, the directed line segment $\overline{AB}$, represents the force P in magnitude, direction and sense. The perpendicular distance of the point O from AB is ON$=p$. Join OA and OB.

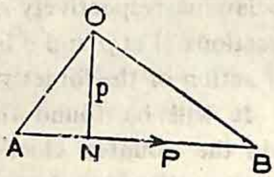


Fig. 41

Now, the moment of the force P about the point O is P.ON$=$P.p.

Again, P.$p=$ AB.ON$=2.\frac{1}{2}$.AB.ON

$=2m\triangle$AOB.

Hence the magnitude of the moment of a force about any point is twice the area of the triangle constructed with the point as vertex and the line-segment representing the force as base.

§ 5·5. **Varignon's Theorem.** *The algebraic sum of the moments of any two forces (which do not form a couple), about any point in their plane is equal to the moment of their resultant about the given point.*

P and Q are two given forces and O is a given point in their plane. To prove that the algebraic sum of the moments of the forces about the point O is equal to the moment of their resultant about the point. Since the forces do not form a couple, so the forces are (*i*) concurrent or (*ii*) like parallel or unequal and unlike parallel forces.

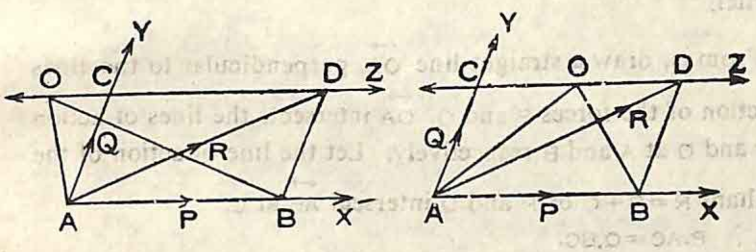
(*i*) First let the forces be concurrent.

Here there are two possibilities.

The point O may be (*a*) on the same side [Fig. (*i*)] or (*b*) on opposite sides of the forces P and Q [Fig. (ii)].

Let $\overrightarrow{AX}$ and $\overrightarrow{AY}$ be the lines of action of P and Q respectively. Through O, draw the straight line $\overleftrightarrow{OZ}$ parallel to $\overrightarrow{AX}$. Let $\overleftrightarrow{OZ}$ intersect $\overrightarrow{AY}$ at C. Now, choose a scale $\left(\text{unit force}=\frac{AC}{Q}\right)$, so

that the directed line segment $\overline{AC}$ represents the force Q. Let the directed line segment $\overline{AB}$ represents the force P according to the same scale.



(i) Fig. 42 (ii)

Complete the parallelogram ABDC. Hence by the parallelogram of forces the directed line segment $\overline{AD}$ will represent the resultant R of the forces P and Q.

Now, $2m\triangle OAB$, $2m\triangle OAC$ and $2m\triangle OAD$ are respectively the moments of P, Q, R about the point O.

Now, in Fig. (*i*) since the point O is on the same side of all the forces P, Q, R, so the moments of the forces about O are all of the same sign and are positive according to Fig. (i).

Hence the algebraic sum of the moments of P and Q about O
$=2m\triangle OAB+2m\triangle OAC.$
$=2m\triangle ABD+2m\triangle OAC$ [$\because$ the triangles ABD and OAB lie on the same base AB and between the same parallel straight lines AB and CD, so $m\triangle OAB=m\triangle ABD$].
$=2m\triangle ACD+2m\triangle OAC$ [as the diagonals of a parallelogram bisect the parallelogram, so $m\triangle ABD=m\triangle ACD$].
$=2m\triangle OAD=$ moment of the force R about O.

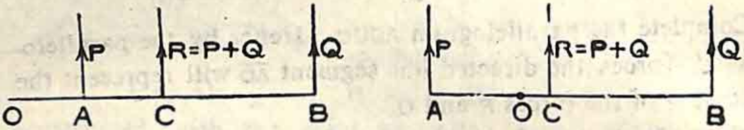
Again, in Fig. (*ii*), the forces P and R and the force Q are on opposite sides of the point O.

According to this figure, the moments of the forces P and R about O are positive and that of the force Q about O is negative. Hence the algebraic sum of the moments of P and Q about the point O $=2m\triangle OAB-2m\triangle OAC$
$=2m\triangle ABD-2m\triangle OAC$
$=2m\triangle ACD-2m\triangle OAC$
$=2m\triangle AOD=$ moment of the force R about the point O.

(*ii*) Now, let the forces P and Q be parallel. Here we prove the theorem taking P and Q as like parallel. The theorem can similarly be proved when the forces are unequal and unlike parallel.

From O, draw a straight line $\overleftrightarrow{OA}$, perpendicular to the lines of action of the forces P and Q. $\overleftrightarrow{OA}$ intersects the lines of action of P and Q at A and B respectively. Let the line of action of the resultant $R=P+Q$ of P and Q intersect $\overleftrightarrow{AB}$ at C.

$\therefore \quad P.AC=Q.BC.$



(*i*) Fig. 43 (*ii*)

First, let the point O be on the same side of the forces P and Q. So, O is on the same side of all the forces P, Q and R and the moments of the forces about the point O are of the same sign and are positive according to fig. (*i*). Now, the algebraic sum of the moments of the forces P and Q about $O=P.OA+Q.OB$

$=P.(OC-AC)+Q(OC+CB)=P.OC-P.AC+Q.OC+Q.CB$

$=P.OC-P.AC+Q.OC+P.AC \quad [\because \quad P.AC=C.CB]$

$=P.OC+Q.OC=(P+Q).OC=R.OC.$

$=$ moment of the force R about the point O.

Next let the point O be situated on the opposite sides of the forces P and Q. In the figure (Fig. *ii*) the moment of the force P about O is negative and those of the force Q and R are positive. Now, the algebraic sum of the moments of the forces P and Q about the point $O=-P.OA+Q.OB$

$$=-P(AC-OC)+Q.(BC+OC)$$

$$=-P.AC+P.OC+Q.OC+Q.BC$$

$$=P.OC+Q.OC \quad [\because \quad P.AC=Q.BC]$$

$$=(P+Q).OC=R.OC$$

$=$ moment of R about O

Cor. 1. If the point O lies on the line of action of the resultant force, then the moment of the resultant about the point O will be zero, so the algebraic sum of the moments of P and Q about O will be zero *i.e.*, the moments of P and Q about O will be of the same magnitude but of opposite signs. Hence one can say, the moments of any two forces (not constituting a couple) about any point on the line of action of their resultant are of the same magnitude but of opposite signs *i.e.*, the algebraic sum of the moments is zero.

Cor. 2. If the algebraic sum of the moments of any two forces about every point in their plane be zero, *i.e.*, if the moments be of the same magnitude but of opposite signs, then the point is situated on the line of action of the resultant of the forces or else, the forces are in equilibrium (for, if the magnitude of the resultant is zero, its moment about any point is zero.)

§ 5·6. Generalisation of Varignon's Theorem.

If a finite set of coplanar forces have a resultant, then the algebraic sum of the moments of the forces about any point in their plane is equal to the moment of their resultant about the point.

Let P_1, P_2, P_3,······be a finite set of co-planar forces and O be any point in their plane.

Now, the algebraic sum of the moments of P_1 and P_2 about the point O = moment of R_1, the resultant of P_1 and P_2 about O. Again, the algebraic sum of the moments of R_1 and P_3 about O = moment of the resultant R_2 of P_1, P_2, P_3 about O. Proceeding similarly until all the forces are exhausted, the theorem can be proved.

Cor. 1. If a finite set of co-planar forces have a resultant, then the algebraic sum of the moments of the set of forces about any point on the line of action of their resultant is zero.

Cor. 2. If the algebraic sum of the moments of a finite set of forces about any point in their plane be zero, then the point will lie on the line of action of their resultant or else the forces will be in equilibrium.

§ 5·7. Determination of the moment of a force acting at a point (α, β) about the point (h, k).

Let the co-ordinates of the point of application A of the force P be (α, β) and one has to determine the moment of P about the point O′ (h, k).

Let X, Y be respectively the resolved parts of P along the directions parallel to the x and y-axes.

Hence the force P is the resultant of the forces X and Y and the algebraic sum of the moments of X and Y about O′ = moment of P about O′.

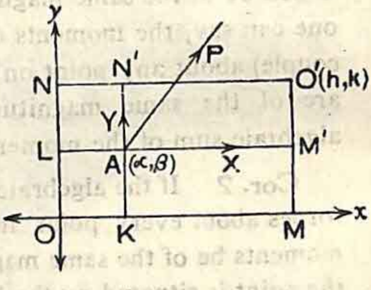


Fig. 44

Now, draw perpendiculars O′M and O′N respectively on the axes of x and y from O′. Let O′M and O′N intersect the lines of action of X and Y at M′ and N′ respectively.

From A draw AK and AL perpendiculars on the axes of x and y respectively.

Now, $O'M' = O'M - M'M = O'M - AK = k - \beta$

$O'N' = O'N - N'N = O'N - AL = h - \alpha.$

Now the moment of the force P about point O′ = Algebraic sum of the moments of X and Y about O′

$= X.O'M' - Y.O'N' = X(k-\beta) - Y(h-\alpha).$

Example 1. In each of the following cases the magnitude of a force and the perpendicular distance of the line of action of the force from a point O is given. Determine the magnitudes of the moments of the forces about the point.

(*i*) 4 kg. ; 5 meters. (*ii*) 100 gms. ; 1000 cms,

(*i*) Required moment = magnitude of the force × perpendicular distance kg. wt. metre.

$= 4 \times 5$ kg. wt. metre = 20 kg. wt. metre.

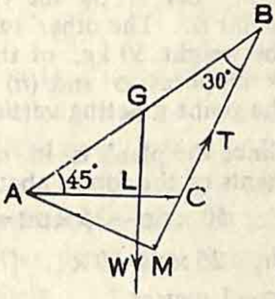
(*ii*) Required moment = 100×1000 gm. wt. cm.

= 100000 gm. wt. cm.

Ex. 2. The weight of a uniform rod is W and it is hinged at the end A. The rod is kept in a position inclined at an angle 45° with the horizontal by a string attached to the end B. If the string be inclined at an angle 30° with the rod, find the tension of the string.

Let AB be the rod, the string be BC and its tension be T. Also let G be the mid-point of the rod.

Now, the forces acting on the rod are (*i*) the weight W of the rod acting at its centre of gravity G, vertically downwards, (*ii*) the tension T of the string acting at the end B and (*iii*) the reaction R of the rod at the point A (not shown in the figure).



Fin. 45

Since the forces are in equilibrium, there is no rotation of the rod about the point A; also as the reaction R acts at the point A, so the moment of R about A is zero. Hence the moments of T and W about the point A are equal in magnitude but are of opposite signs. Draw perpendicular AL and AM respectively on the lines of action of W and T.

$\therefore \quad W.AL = T.AM \quad \cdots \quad \cdots(1)$

Now, $\dfrac{AL}{AG} = \cos 45°$, $\therefore \quad AL = AG \cos 45° = AG.\dfrac{1}{\sqrt{2}}$.

Again, $\dfrac{AM}{AB} = \sin 30°$, or, $AM = AB \sin 30° = 2\,AG.\frac{1}{2} = AG$.

Hence from (1), we get $W.AG.\dfrac{1}{\sqrt{2}} = T.AG$

$\therefore \quad T = \dfrac{W}{\sqrt{2}} = \dfrac{W\sqrt{2}}{2}$.

Ex. 3. A uniform plank AB, 10 metres long, is of weight 50 kg. The plank has been kept in the horizontal position on two supports C and D. The supports C and D are respectively at distances 2 metres and 4 metres from the ends A and B. A boy started walking along the plank from the support D towards the end B. If the weight of the boy be 25 kg., how far can the boy walk safely ?

Suppose the boy can walk safely upto the point E between D and B and let DE $=x$. When the boy will come to this point, the plank will be about to over-turn and the plank will be about to lose contact with the support C ; so the reaction at the point C will be zero. Let R be the reaction (not shown in the figure) at the point D. The other two forces acting on the plank are (*i*) the weight 50 kg. of the plank acting vertically downwards at its midpoint G and (*ii*) and the weight 25 kg. of the boy at the point E acting vertically downwards.

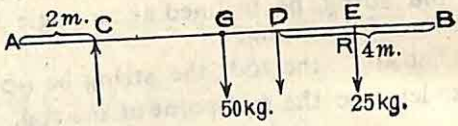


Fig. 46

Since the plank is in equilibrium, the algebraic sum of the moments of the forces about the point D is zero.

$\therefore\quad 50\times GD-25\times DE=0.$

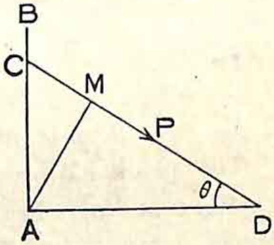
or, $25\times x=50\times 1$. [$\because$ BG $=5$ metres and BD $=4$ metres so GD $=1$ metres].

$\therefore\quad x=2$ meters.

Hence the boy can walk safely upto the point E, at a distance 2 metres from the support D.

Ex. 4. A man, standing on the ground tries to uproot a pillar with the help of a rope of length 20 meters, by fastening one extremity of the rope at some point of the pillar and pulling at the other end with a force of given magnitude. Find at what height from the foot of the pillar the rope must be fastend so that the man may uproot the pillar most conveniently.

Let the pillar be AB and the rope CD be fastened at the point C. Also let $m\angle CDA=\theta$ and AM be perpendicular on the rope CD.



Fin. 47

$\therefore\quad AM=AD\sin\theta$

$=CD\cos\theta\sin\theta$

$=\frac{1}{2}CD\sin 2\theta$

$=\frac{1}{2}\times 20\sin 2\theta=10\sin 2\theta.$

Now, the magnitude of the moment of the force of given magnitude P (say) along $\overrightarrow{CD}$ about $A=P.AM=P.10\sin 2\theta=10P\sin 2\theta.$

Hence it will be most convenient for the man to exert force when 10 P sin 2θ will be greatest in value. Now this value is greatest when sin 2θ is maximum *i.e.* $\theta = 45°$.

$\therefore \quad m\angle ACD = 45°. \quad \therefore \quad AC = AD = CD \cos 45°$

$$= 20 \times \frac{1}{\sqrt{2}} = 10 \sqrt{2} \text{ meteres.}$$

Hence the rope is to be fastend at a height $10\sqrt{2}$ metres from the ground.

Ex. 5. A uniform beam, 4 metres long is simply supported at the ends. It carries a concentrated load of 1·5 ton at a distance of 1 meter from the left end and another load of 2 tons at a point 2 meters from the right end. The weight of the beam is 0·8 ton and there is a unifromly distributed load of 1 ton per metre on a length of 2 metres from the right hand end. Find the reactions at the supports.

[State Council, W. Bengal 1975]

The point C is at a distance of 1 metre from the left end A and the point E is at a distance 1 metre from the right end B of the uniform beam AB. The point D is the mid-point of the rod. Now the forces acting on the rod are,

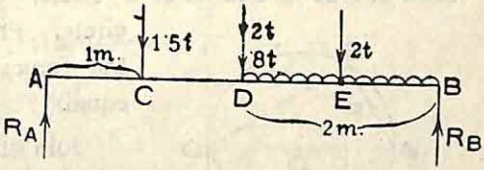


Fig. 48

(1) The reaction R_A acting at A vertically upwards.

(2) the concentrated load $1 \cdot 5t$ acting at the point C vertically downwords. [t for ton]

(3) the weight $0 \cdot 8t$ of the uniform rod acting vertically downwards at the point D.

(4) the weight $2t$ acting at the point D vertically downwards.

(5) the resultant force $2t$ of the uniformly distributed load 1 ton per metre on a length of 2 metres from the right hand end acting at the point E vertically downwards.

(6) the upward reaction R_B at B.

Now since the forces are in equilibrium, so

the resultant of the downward forces = the resultant of the upward forces.

$$\therefore \quad (1{\cdot}5+2+0{\cdot}8+2)t = R_A + R_B$$

$$\text{or,} \quad R_A + R_B = 6{\cdot}3t \qquad \cdots \qquad \cdots(1)$$

Again, since the forces are in equilibrium, so the algebraic sum of the moments of the forces about every point of their plane is zero.

Hence taking moments of the forces about the point A, we get $R_B \times 4 - 1{\cdot}5 \times 1 - (2+{\cdot}8) \times 2 - 2 \times 3 = 0$

$$\therefore \quad 4R_B = 1{\cdot}5 + 5{\cdot}6 + 6. \quad \therefore \quad R_B = \frac{13{\cdot}1}{4}t = 3{\cdot}28t \text{ (nearly)}$$

$\therefore$ From (1) we get $R_A = (6{\cdot}3 - 3{\cdot}28)t = 3{\cdot}02$ tons.

Ex. 6. The forces P and Q are represented by the perpendicular chords $\overline{AB}$ and $\overline{AC}$ of a circle. $\overline{AD}$ is a diameter of the circle. Prove that the moments of the forces about the point D are equal.

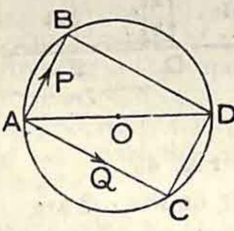


Fig. 49

Join BD and CD.

Now, ABCD is a rectangle

$\therefore \quad m\triangle ABD = m\triangle ACD,$

or, $\quad 2m\triangle ABD = 2m\triangle ACD.$

Now, $2m\triangle ABD$ and $2\,m\triangle ACD$ represent respectively the moments of P and Q about the point D.

Hence the two moments are equal.

Ex. 7. A uniform plank of length a and weight W is placed in a horizontal position on two pegs C and D. The greatest weights that can be placed at the extremities successively, without upsetting the plank are P and Q respectively.

If CD $= l$, prove that $\dfrac{P}{W+P} + \dfrac{Q}{W+Q} = \dfrac{2l}{a}$.

[Draw the figure yourself].

Let the plank be AB and G its mid-point.

First, let the weight P be placed at the extremity A. Since P is the greatest weight that can be placed at A, so all the pressures will act at C and the reaction at D will be zero due to loss of contact. Hence taking moment about C we get,

$$P.AC-W.CG=0, \quad \text{or,} \quad P.AC=W.CG \quad \cdots \quad \cdots(1)$$

Similarly when the weight Q is placed at the extremity B, then taking moments about D we obtain $Q.BD=W.DG \quad \cdots \quad \cdots(2)$

From (1) we get $\frac{CG}{AC}=\frac{P}{W}$, or, $\frac{CG}{AC}+1=\frac{P}{W}+1$

$$\text{or,} \quad \frac{CG+AC}{AC}=\frac{P+W}{W}, \quad \text{or,} \quad \frac{AG}{AC}=\frac{P+W}{W}$$

$$\text{or,} \quad \frac{\frac{a}{2}}{AC}=\frac{P+W}{W} \quad \therefore \quad AC=\frac{W}{P+W}.\frac{a}{2}.$$

Similarly from (2) we obtain, $BD=\frac{W}{W+Q}.\frac{a}{2}$

Now, from (1) and (2)

$$P.AC+Q.BD=W(CG+DG)=W.CD=W.l$$

$$\text{or,} \quad \left(\frac{PW}{P+W}+\frac{QW}{Q+W}\right)\frac{a}{2}=W.l$$

$$\text{or,} \quad \frac{Wa}{2}\left(\frac{P}{P+W}+\frac{Q}{Q+W}\right)=W.l, \quad \text{or,} \quad \frac{P}{P+W}+\frac{Q}{Q+W}=\frac{2l}{a}.$$

Ex. 7(*a*). **A unifrom rod 5 metres long and weighing 4 kg. can turn freely about a point on it at a distance of 1 metre from one end from which a mass of weight 10 kg. is suspended. Find the weight that should be suspended from the other end to keep the rod horizontal.**

Let the required weight be W and P be the point about which the rod can turn freely. If the rod remains horizontal, then the algebraic sum of the moments of the forces acting on the rod, viz. W, the weight 4 kg. of the rod and the weight 10 kg. about the point P must be zero.

$$\therefore \quad 4\times 1\tfrac{1}{2}+W\times 4=10\times 1 \quad \text{or,} \quad 4W=10-6=4.$$

$$\therefore \quad W=1 \text{ kg. wt.}$$

Ex. 7(*b*). An weightless rod of length 20 metres rest on two smooth pegs at a distance of 10 metres from each other. Two weights of mass 8 kg. and 12 kg. are suspended from the two ends. If the pressures on the two pegs be equal find the distance of the pegs from the two ends.

Let x-metres be the distance of the peg A nearer to the end from which the weight 8 kg. has been suspended. Now the four forces, viz., the two reactions of the two pegs A and B and the two suspended weights are in equilibrium. Let the two equal reactions be R.

$\therefore\ R+R=8+12$ or, $2R=20$ $\therefore\ R=10$ kg.

Now taking moments about A we get, for the equilibrium of the forces $8x+0+10\times10-12.(20-x)=0$

or, $20x+100-240=0$; or, $20x=140$. $\therefore\ x=7$.

So, the peg nearer to the end from which the weight 8 kg. has been suspended is at a distance 7 metres from this end. Similarly it can be shown that the distance of the other peg from its nearer end is 3 metres.

Ex. 7(*c*) A uniform rod weighing 120 lbs. is 14 ft. long. The rod is supported on two smooth props at distances 5 ft. and 3 ft. from the two ends. If no prop can support weights more than 200 lbs., then what maximum and equal weights can be placed at the two ends ? [C. U. 1938]

Let P and Q be the two props and equal weights W and W be placed at the two ends A and B. The resultant 2W of these weights act at the middle point C of the rod. Hence the resultant force acting at C is (120+2W) vertically downwards. This is equivalent to the resultant of the two pressures T_1 and T_2 acting at P and Q respectively vertically down wards.

$\therefore\ T_1+T_2=120+W$.

Now $PC=2$ ft. and $QC=4$ ft. Now as the rod is in equilibrium taking moments about C, we get $T_1.PC=T_2.QC$

or, $T_1.2=T_2.4$ or, $T_1=2T_2$.

As no prop can support a weight more than 200 lbs, so the maximum values of T_1 and T_2 will be 200 lbs. and 100 lbs. respectively.

Now $T_1+T_2=120+2W$ or, $3T_2=120+2W$.
$[\because T_1=2T_2]$

Now as maximum value of $T_2=100$ lbs, so $300=120+2W$
or, $W=90$ lbs.

Hence the maximum weight that can be placed at each end is 90 lbs.

Ex. 8. A carriage wheel of weight W and radius r is to be dragged over an obstacle of height h, by a horizontal force applied at the centre of the wheel. Show that F must be slightly greater than $W.\dfrac{\sqrt{2rh-h^2}}{r-h}$.

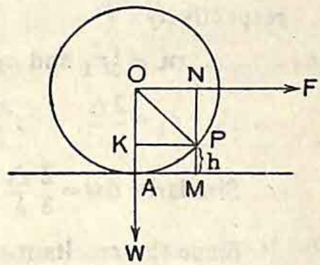


Fig. 50

Let O be the centre of the wheel and PM be the obstacle. PM being perpendicular to the horizontal ground. Let ON be perpendicular from O on PM and PK be perpendicular from P on the vertical radius OA. $\therefore$ PK = ON.

So, $PM=h$, $MN=r$. $\therefore$ $PN=MN-PM=r-h$.

Again, $ON=\sqrt{OP^2-PN^2}=\sqrt{r^2-(r-h)^2}=\sqrt{2rh-h^2}$.

Now if the obstacle can be overcome by the application of a horizontal force F, then the moment of the force F about P must be slightly greater than the moment of W about the point P and their signs should be opposite.

$\therefore$ $F.PN > W.PK$, or, $F.PN > W.ON$

or, $F(r-h) > W\sqrt{2rh-h^2}$.

or, $F > \dfrac{W\sqrt{2rh-h^2}}{r-h}$.

Ex. 9. The forces P, Q, R act along the sides BC, CA, AB of the triangle ABC taken in order. Prove that, if their resultant,

(i) passes through the centroid of the triangle ABC, then
P cosec A + Q cosec B + R cosec C = 0 ;

(ii) passes through the orthocentre of the triangle ABC, then, P sec A + Q sec B + R sec C = 0 ;

(iii) passes through both the centroid and the orthocentre;

then $\dfrac{P}{\sin 2A \sin(B-C)}=\dfrac{Q}{\sin 2B \sin(C-A)}=\dfrac{R}{\sin 2C \sin(A-B)}$.

The forces P, Q, R act along the sides $\overrightarrow{BC}$, $\overrightarrow{CA}$, $\overrightarrow{AB}$ of the triangle ABC taken in order. Let the resultant of the three forces be F.

(i) Let G be the centroid of the traingle and draw GL, GM, GN perpendiculars respectively on BC, CA, AB. Let the lengths of the perpendiculars from A, B, C on $\overleftrightarrow{BC}$, $\overleftrightarrow{CA}$, $\overleftrightarrow{AB}$ be p_1, p_2, p_3 respectively.

$\therefore$ $GL=\frac{1}{3}p_1$ and $ap_1=2m\triangle ABC=2\triangle$ (say)

$\therefore$ $p_1=\dfrac{2\triangle}{a}$, $\quad\therefore\quad GL=\dfrac{2}{3}.\dfrac{\triangle}{a}$.

Similarly $GM=\dfrac{2}{3}.\dfrac{\triangle}{b}$ and $GN=\dfrac{2}{3}.\dfrac{\triangle}{c}$.

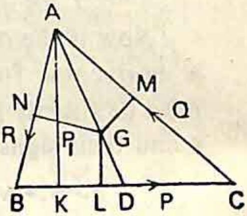
Since the resultant of the forces passes through the point G, so, the algebraic sum of the moments of the forces about the point G is zero.

$\therefore$ $P.GL+Q.GM+R.GN=0$

or, $P.\dfrac{2}{3}.\dfrac{\triangle}{a}+Q.\dfrac{2}{3}.\dfrac{\triangle}{b}+R.\dfrac{2}{3}.\dfrac{\triangle}{c}=0$

or, $\dfrac{P}{a}+\dfrac{Q}{b}+\dfrac{R}{c}=0$,

or, $\dfrac{P}{2R'\sin A}+\dfrac{Q}{2R'\sin B}+\dfrac{R}{2R'\sin C}=0$



$\left[\because \dfrac{a}{\sin A}=\dfrac{b}{\sin B}=\dfrac{c}{\sin C}=2R'\right.$

Fig. 51

where R′ is the circum-radius of the triangle ABC]

or, $\dfrac{P}{\sin A}+\dfrac{Q}{\sin B}+\dfrac{R}{\sin C}=0$

or, $P \operatorname{cosec} A+Q \operatorname{cosec} B+R \operatorname{cosec} C=0$.

(ii) Draw AL, BM, CN perpendiculars from the vertices A, B, C respectively on the opposite sides. The perpendiculars intersect at the point O. Hence O is the orthocentre of the triangle.

Now, from the $\triangle$OCL, $\dfrac{OL}{LC}=\tan OCL$

or, $OL=LC \tan OCB = AC \cos ACL \tan OCL$

$=b \cos C \tan(90°-B)$ $\left[\because \triangle CNB \text{ is right-angled, } \therefore m\angle NCB + m\angle B = 90°\right]$

$=b \cos C \cot B$

$=b \cos C \dfrac{\cos B}{\sin B} = 2R' \cos C \cos B \left[\because \dfrac{b}{\sin B}=2R'\right]$

$=2R \cos A \cos B \cos C \sec A$

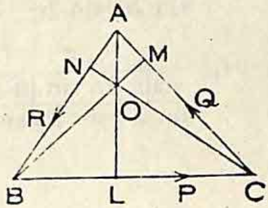


Fig. 52.

Similarly,

$OM=2R' \cos A \cos B \cos C \sec B$

and $ON=2R' \cos A \cos B \cos C \sec C$.

Now, since the resultant of the forces passes through the centre O, so the algebraic sum of the moments of the forces about the point O is zero.

$\therefore \quad P.OL+Q.OM+R.ON=0.$

or, $P.2R' \cos A \cos B \cos C \sec A + Q.2R' \cos A \cos B \cos C \sec B + R.2R' \cos A \cos B \cos C \sec C = 0.$

or, $\quad P.\sec A + Q.\sec B + R.\sec C = 0.$

(iii) As the resultant of the forces passes through both the centroid and the orthocentre of the triangle, so we obtain respectively from (i) and (ii) above,

$P \operatorname{cosec} A + Q \operatorname{cosec} B + R \operatorname{cosec} C = 0 \cdots (1)$

and $P \sec A + Q \sec B + R \sec C = 0 \cdots (2)$

Now, from (1) and (2) by cross-multiplication we obtain,

$$\frac{P}{\operatorname{cosec} B \sec C - \operatorname{cosec} C \sec B}$$

$$=\frac{Q}{\operatorname{cosec} C \sec A - \operatorname{cosec} A \sec C}$$

$$=\frac{R}{\operatorname{cosec} A \sec B - \sec A \operatorname{cosec} B},$$

or, $$\frac{P}{\dfrac{1}{\sin B \cos C} - \dfrac{1}{\sin C \cos B}} = \frac{Q}{\dfrac{1}{\sin C \cos A} - \dfrac{1}{\sin A \cos C}}$$

$$=\frac{R}{\dfrac{1}{\sin A \cos B} - \dfrac{1}{\sin B \cos A}}$$

or, $$\frac{P}{\dfrac{\sin C\cos B-\sin B\cos C}{\sin B\sin C\cos B\cos C}}=\frac{Q}{\dfrac{\sin A\cos C-\sin C\cos A}{\sin C\sin A\cos C\cos A}}$$
$$=\frac{R}{\dfrac{\sin B\cos A-\cos B\sin A}{\sin A\cos A\sin B\cos B}}$$

or, $$\frac{P}{\dfrac{4\sin(C-B)}{\sin 2B\sin 2C}}=\frac{Q}{\dfrac{4\sin(A-C)}{\sin 2A\sin 2C}}=\frac{R}{\dfrac{4\sin(B-A)}{\sin 2B\sin 2A}}$$

or, $$\frac{P}{\dfrac{-\sin 2A\sin(B-C)}{\sin 2A\sin 2B\sin 2C}}=\frac{Q}{\dfrac{-\sin 2B\sin(C-A)}{\sin 2A\sin 2B\sin 2B}}$$
$$=\frac{R}{\dfrac{-\sin 2C\sin(A-B)}{\sin 2A\sin 2B\sin 2C}}$$

or, $$\frac{P}{\sin 2A\sin(B-C)}=\frac{Q}{\sin 2B\sin(C-A)}=\frac{R}{\sin 2C\sin(A-B)}.$$

Ex. 11(*a*). Three forces P, Q, R acting along the three medians $\overrightarrow{AD}$, $\overrightarrow{BE}$ and $\overrightarrow{CF}$ of a triangle ABC are in equilibrium. Prove that the magnitudes of the forces are proportional to the lengths of the three medians of the triangle.

(*b*) If the resultant of the forces $l.\overline{BC}$, $m.\overline{CA}$ and $n.\overline{AB}$ pass through the centroid of the triangle ABC, prove that $l+m+n=0$.

(*a*) From the point A draw perpendiculars on BE and CF and let y and z be the lengths of these perpendiculars.

Now, $BE.y=2\triangle BAE=\triangle ABC$

[as each median bisects the triangle].

Similarly, $CF.z=\triangle ABC$.

$\therefore\quad BE.y=CF.z \quad \cdots \quad \cdots \quad (1)$

Now, as the three forces are in equilibrium, so taking moments about the point A, we obtain $Q.y-R.z=0$

or, $Q.y=R.z \quad \cdots \quad \cdots \quad (2).$

Now, from (1) and (2) we get, $\dfrac{Q}{BE}=\dfrac{R}{CF}$.

Similarly taking moments about the point B, we get $\dfrac{P}{AD}=\dfrac{R}{CF}$.

So, $\frac{P}{AD}=\frac{Q}{BE}=\frac{R}{CF}$ *i.e.*, The magnitudes of the forces are proportional to lengths of the medians.

(*b*) Let G be the centroid of the triangle. Then the distances of G from BC, CA, AB are $\frac{2}{3}.\frac{\Delta}{a}, \frac{2}{3}.\frac{\Delta}{b}, \frac{2}{3}.\frac{\Delta}{c}$ respectively.

[See Ex. 10 (i)]. Now if the resultant of the forces pass through G, then taking moment about G we get,

$$l.BC.\frac{2}{3}.\frac{\Delta}{a}+m.CA.\frac{2}{3}.\frac{\Delta}{b}+n.AB.\frac{2}{3}.\frac{\Delta}{c}=0.$$

or, $\frac{2}{3}\Delta(l+m+n)=0$ $[\because$ $BC=a,\ CA=b,\ AB=c]$

$\therefore$ $l+m+n=0.$

Ex. 12. A man tries to uproot a tree with the help of a rope of length 30 feet, by fastening one extremity at some point of the vertical stem and pulling at the other end from the ground. The least moment about the foot of the tree necessary to uproot it is 1200 ft. lbs. Find the least force that the man has to apply.

Let the tree be AB and A be its lower end. Also let the rope be CD one end of which is tied to the tree at the point C and the man is pulling at the end D. So $CD=30$ ft. From A draw AH perpendicular on CD and let $\angle ADC=\theta$. Let P be the force applied along $\overrightarrow{CD}$.

The moment of the force P about $A=P.AH=P.AD\sin\theta=P.CD\cos\theta\sin\theta=P.30.\frac{1}{2}\sin 2\theta=15\,P\sin 2\theta.$

Now for uprooting the tree the moment required is 1200 ft.lb.

$$\therefore\ 15\,P\sin 2\theta=1200 \quad \text{or,} \quad P=\frac{1200}{15\sin 2\theta}=\frac{80}{\sin 2\theta}$$

Now, P will be least when $\sin 2\theta$ is greatest *i.e.*, 1.

So, $P=80$ lbs.

Hence the least force required is 80 lbs.

Ex. 13. Of four co-planar forces in equilibrium, one is given completely, a second and a third, (which are not parallel), have

their lines of action given, while the fourth has its magnitude only given. Prove that the line of action of the fourth force must touch a fixed circle. [C. U. 1934]

Let the forces P, Q, R, S be in equilibrium. The force P is completely given ; the forces Q and R are not parallel and their lines of action and the magnitude of the force S are known. Let the lines of action of the forces Q and R intersect at the point O. $\therefore$ O is a fixed point.

Since the forces are in equilibrium, so the algebraic sum of the moments of the forces about any point in their plane is zero. Let the perpendicular distances of the point O from the lines of action of the forces P and S be p and s respectively (p is known and s is unknown).

Hence taking moments about the point O we get $P.p+S.s=0$,

or, $s=\dfrac{-P.p}{S}$. Now according to the given condition, P; p and S are of constant magnitude.

$\therefore$ $\dfrac{Pp}{S}$ is constant. Hence the distance of the line of action of the force S from the point O is constant. Hence the line of action of S is a tangent to the circle drawn with centre O and radius $\dfrac{P.p}{S}$.

Ex. 14. ABC is a right-angled triangle, the sides BC, CA, AB being 13, 12 and 5 units of length respectively. The moments of a force F about A, B, C respectively are 0, 25 and 144 units. Find the magnitude, direction and line of action of the force F. [C. U. 1936]

Since the moment of the force F about the point A is zero, so the line of action of the force passes through the point A. Again since the moments of the force about the points B and C are of the same sign (here positive), so the points B and C are on the same side of the line of action of the force F. Draw BL and CM perpendiculars on the line of action of F and let $m\angle BAL=\theta$.

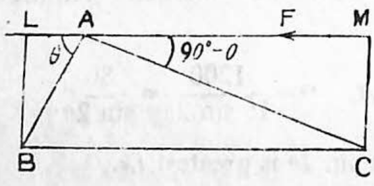


Fig. 53

$\therefore\ m\angle OAM = 90^\circ - \theta$. [$\because\ \angle BAC$ is a right-angle as $BC^2 = CA^2 + AB^2$.]

Now, $BL = AB \sin\theta = 5 \sin\theta$.
and $CM = CA \sin(90^\circ - \theta) = 12 \cos\theta$.
Since the moment of F about the point B is 25 units

$\therefore\ F.BL = 25$, or, $F.5 \sin\theta = 25$, $\therefore\ F = \dfrac{5}{\sin\theta}$.

Again, since the moment of F about the point C is 144,

$\therefore\ F.CM = 144$, or, $F.12 \cos\theta = 144$.

$\therefore\ F = \dfrac{12}{\cos\theta}$

$\therefore\ F = \dfrac{5}{\sin\theta} = \dfrac{12}{\cos\theta} = \sqrt{\dfrac{5^2+12^2}{\cos^2\theta + \sin^2\theta}} = \dfrac{13}{1} = 13.$

$\therefore\ \sin\theta = \dfrac{5}{F} = \dfrac{5}{13} = \sin C$, $\therefore\ \theta = C$.

Hence the line of action of the force F is a tangent to the circum-circle of $\triangle ABC$ at the point A and the magnitude of the force is 13 units.

Ex. 15. $\overrightarrow{OX}$ and $\overrightarrow{OY}$ are two straight lines at right angles, and a force acting in their plane at O has moments G and G′ about the two points whose co-ordinates are (x, y) and (x', y') respectively with repect to the lines $\overrightarrow{OX}$ and $\overrightarrow{OY}$ as axes of co-ordinates. If $(xy' - x'y) \neq 0$, prove that the magnitude R of the force and the angle θ between its line of action and $\overrightarrow{OX}$ are given by $R^2 = \dfrac{(xG' - x'G)^2 + (yG' - y'G)^2}{(xy' - x'y)^2}$ and $\tan\theta = \dfrac{yG' - y'G}{xG' - x'G}$.

[C. U. 1946]

The resolved parts of the force along $\overrightarrow{OX}$ and $\overrightarrow{OY}$ are respectively $R\cos\theta$ and $R\sin\theta$. Now, the algebraic sum of the moments of these two resolved parts about every point of the plane = moment of the force about the same point.

Hence taking moments about the point (x, y) we obtain

$y.R\cos\theta - x.R\sin\theta = G \cdots\cdots (1)$.

Also taking moments about the point (x', y') we get

$y'R\cos\theta - x'R\sin\theta = G' \cdots\cdots (2)$

Solving equations (1) and (2) we get,

$$R \cos\theta = \frac{xG' - x'G}{xy' - x'y} \cdots\cdots(3)$$

and $$R \sin\theta = \frac{yG' - y'G}{xy' - x'y} \cdots\cdots(4)$$

Squaring and adding both sides of equations (3) and (4), we obtain $R^2 = \dfrac{(xG' - x'G)^2 + (yG' - y'G)^2}{(xy' - x'y)^2}$.

Again, dividing equation (4) by equation (3) we get,

$$\tan\theta = \frac{yG' - y'G}{xG' - x'G}.$$

Ex. 16. A uniform plank weighing 200 lbs is 24 ft. long and 8 ft. of it project over the side of a platform. What least weight must be placed on the end of the plank, which is on the platform, so that a man whose weight is 150 lbs may be able to walk to the other end without overturning the plank ?

Let the plank be AB and the portion CB be outside the platform. Also let G be the middle point of the plank. So GC = 4 ft. Let W be the weight placed at the end A. Now as the man starts walking from A, he will reach B without overturning the plank if the algebraic sum of the moments of the forces W acting at A and the weight 200 lbs of the plank acting at G about C be greater than or equal to the moment of the weight 150 lbs of the man when at B.

i.e., if $W \times 16 + 200 \times 4 \geqslant 150 \times 8$

or, $W \times 16 \geqslant 400$ or, $W \geqslant 25$.

So the least value of W is 25.

Hence the weight that is to be placed at the end A is 25 lbs.

Exercise 4

1. In each of the following cases, the magnitude of a force and the perpendicular distance of the point O from the line of action of the force is given. Determine the magnitude of the moment of the force about the point O.

(i) 100 kg., 50 meters ; (ii) 65 lbs. $4\frac{5}{13}$ ft.

2. In the figure below a weightless rod AB has been kept in equilibrium in the horizontal position. C is the mid-point of

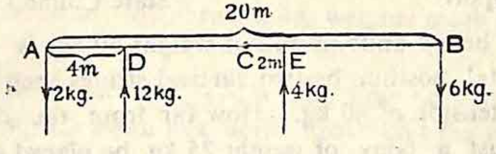


Fig. 54

$\overline{AB}$. Find the magnitude and sign of the moment of each force about each point as shown in the figure.

3. ABC is an equilateral triangle and each side is of length 10 cms. Forces of magnitudes 2, 4 and 8 kg. act along the sides $\overrightarrow{AB}$, $\overrightarrow{BC}$ and $\overrightarrow{CA}$ respectively. Find the moment of each force about the opposite vertex.

4. Show that, if two forces be represented in magnitude, direction and sense by two sides of a triangle taken in order, the algebraic sum of their moments about every point of the base is the same.

5. Masses 1 kg., 2 kgs., 4 kgs., 6 kgs., 8 kgs. and 10 kgs. are suspended from six consecutive points A, C, D, E, F and B respectively of a weightless horizontal rod AB, 10 meters long. The points are at distances of 2 meters, find the point of the rod, the algebraic sum of the moments of the forces about which is zero.

6. A uniform rod 15 meters in length and of weight 30 kg. is supported on two smooth pegs at its ends. A weight 160 kg. is placed on the rod at a point, 8 meters from one end. Find the pressure on the supports.

7. The wire passing round a telegraph post is horizontal and the two portions attached to the pole are inclined at 60° with each other. The post is supported vertically by a wire attached to the post at its middle point by a wire inclined at an angle 60° to the horizon ; show that the tension of this wire is $4\sqrt{3}$ times that of the telegraph wire.

8. A uniform rod 6 meters long and weight 2 kg. is supported at its ends on two smooth pegs. Each peg can bear

a maximum weight of 13 kg. Determine the portion of the rod in which a weight of 16 kg. can be placed without breaking any of the supports. [State Council (W. B.) 1976]

9. A heavy uniform rod of weight 50 kg. is suspended in a horizontal position by two vertical strings each of which can sustain a tension of 40 kg. How far from the centre of the string must a body of weight 25 kg be placed so that one of the strings may just break ?

10. A rod, 16 inches long, rests on two pegs 9 inches apart, with its centre midway between them. The greatest masses that can be suspended in succession from the two ends without disturbing the equilibrium are 4 lbs. and 5 lbs. respectively. Find the weight of the rod and the position of the point at which its weight acts.

11. A uniform rod, 5 meters long and simply supported at its ends is acted on by the following forces ;

(i) a weight 3 tons at a point 2 meters from the left end.

(ii) a weight 2 tons at a point 2 meters from the right end.

(iii) a uniformly distributed load of 1 ton per meter on a length of 3 meters from the right hand end. If the weight of rod be 1 ton, find the reactions of the supports.

12. At what height from the base of a pillar must the end of a rope of given length be fixed so that a man standing on the ground and pulling at its other end with a given force may have the greatest tendency to make the pillar overturn ?

13. The horizontal road way of a bridge 12 meters long and of weight 5 tons is supported on similar supports at its ends. What is the pressure on each support when a truck of weight 3 tons is at a distance 8 meters from one end ?

14. Find the magnitude and direction of the resultant of the force R, 2R, 3R and 4R action along the side $\overrightarrow{AB}$, $\overrightarrow{BC}$, $\overrightarrow{CD}$ and $\overrightarrow{DA}$ of a square ABCD. Find also the point of intersection of the line of action of the resultant with $\overleftrightarrow{AB}$.

15. A man and his son were carrying a load of weight 60 kg. with the help of a uniform rod, 5 meters long and o

weight 15 kg. At what point of the rod, the load must be placed so that the man will bear 2 kg. weights more than his son ?

16. Forces of magnitudes 1, 2, 4 and 5 pounds act along the sides of a square taken in order. Prove that the resultant of the forces is parallel to a diagonal of the square. Find the point at which the line of action of the resultant intersects the line of action of the first force. [C. U. 1937]

17. The lengths of the sides $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ of the right-angled-triangle ABC are 5, 4 and 3 units of length respectively. The moments of a force about the points A, B and C are 0, 25 and 144 units of moment respectively. Find the magnitude, direction and line of action of the force.

18. The moments of a force about the points (0, 0), (8, 0) and (0, 4) are 20, -12 and 32 gramme-centimeters respectively. Find the point at which the line of action of the force intersects the x-axis and also the resolved parts of the forces parallel to the axes.

19. The weights of two heavy uniform rods AB and BC of lengths a and b respectively are proportional to their lengths. The rods are rigidly connected at B and $\angle ABC$ is a right-angle. The connected rods are hung from the point A. Prove that if θ be the inclination of AB with the horizontal, then

$$\cot \theta = \frac{b^2}{a^2 + 2ab}.$$

20. Three forces P, Q, R act in the same sense along the sides $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ of a triangle ABC ; show that if their resultant passes through the in-centre of the triangle, then $P+Q+R=0$.

21. In the previous question if the line of action of the resultant passes,

(i) through the circum-centre of the triangle, instead of the in-centre, show that, $P \cos A + Q \cos B + R \cos C = 0$.

(ii) through both the in-centre and the circum-centre, then show that.

$$\frac{P}{\cos B - \cos C} = \frac{Q}{\cos C - \cos A} = \frac{R}{\cos A - \cos B}$$

22. In question 20, if the line of action of the resultant passes

(a) through both (i) the in-centre and (ii) the centroid of the triangle then prove that, $\dfrac{P}{a(b-c)}=\dfrac{Q}{b(c-a)}=\dfrac{R}{c(a-b)}$;

(b) through both the ortho-centre and the circum-centre, then prove that $\dfrac{P}{(b^2-c^2)\cos A}=\dfrac{Q}{(c^2-a^2)\cos B}=\dfrac{R}{(a^2-b^2)\cos C}$.

§ 5·8. Moment about an axis.

Let $\overleftrightarrow{AB}$ be a fixed axis of a body and P be a force acting on the body.

Let $\overline{ON}$ be the shortest distance between $\overleftrightarrow{AB}$ and the line of action of P. Through N draw a straight line parallel to $\overleftrightarrow{AB}$; Let this parallel straight line is inclined at an angle θ with the line of action of P.

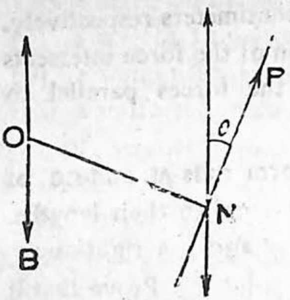


Fig. 55

Now, P.sin θ.ON is said to be the magnitude of the moment of the force P about the fixed axis $\overleftrightarrow{AB}$. The sign of the moment is determined as in the case of moment about a point. Again as in Varignon's theorem, if a number of coplanar forces acting on a body have a resultant, then the algebraic sum of the moments of the forces about any fixed axis of the body is equal to the moment of the resultant force about the same axis.

CHAPTER SIX

COUPLE

§ 6·1. Couple. Two equal and unlike parallel forces constitute a couple if the forces do not act along the same straight line. The perpendicular distance between the lines of action of the forces is said to be the *Arm* of the couple. If P, P be the forces constituting a couple and p be its arm, then generally the couple is indicated as (P, p).

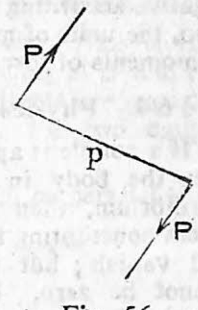


Fig. 56

§ 6·2. The algebraic sum of the moments of the forces constituting a couple, about any point in their plane is constant.

Let (P, p) be a couple and O be a point in their plane. Draw from O a perpendicular on the lines of action of the forces constituting the couple. Let this perpendicular intersect the lines of action of the forces at the points A and B. Hence AB is the arm couple and let AB $= p$.

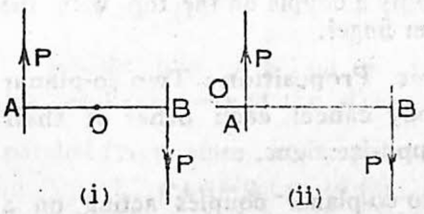


Fig. 57

First let the point O be situated on the same side of the forces [fig. (ii)]. Now the algebraic sum of the moments of the forces about the point O $=$ P.OA $-$ P.OB $=$ P.(OA $-$ OB)

$$= -\text{P.AB} = -\text{P}.p = \text{constant}.$$

Next, let the point O be situated on the opposite sides of the forces. In this case the algebraic sum of the moments of the forces

$$= -\text{P.OA} - \text{P.OB} = -\text{P}(\text{OA} + \text{OB}) = -\text{P.AB} = -\text{P}.p = \text{constant}$$

Cor. Since none of P and p is zero, $\therefore$ $\text{P}.p \neq 0$.

Hence the algebraic sum of the moments of the forces constituting a couple about any point in their plane cannot be zero.

§ 6·3. Moment of a Couple. The product P.p of the magnitude P of one of the two forces constituting a couple and

the arm of the couple is said to be the moment of the couple. Hence from § 6·2 one gets that the algebraic sum of the moments of the two forces constituting a couple about any point in their plane is constant and is equal to the moment of the couple. The moment of the couple is said to be positive or negative according as the algebraic sum is positive or negative. Also, the units of moments of couples are the same as the units of moments of forces.

§ 6·4. Physical concept of moment of a couple.

If a couple is applied on a body, then the couple cannot keep the body in equilibrium. For, if the body remains in equilibrium, then the algebraic sum of the moments of the forces constituting the couple about any point in their plane will vanish ; but according to Cor. of § 6·2 this algebraic sum cannot be zero. If a couple is applied on a body, then the forces of the couple want to rotate the body in opposite directions and as a result the body undergoes a rotation. For example, while winding a watch, the key of the watch is rotated by applying a couple on the key with the help of two fingers. To spin a top we apply a couple on the top with the help of the thumb and another finger.

§ 6·5. Equivalent Couple. Proposition : Two co-planar couples acting on a rigid body cancel each other if their moments be equal but of opposite signs.

Let (P, p) and (Q, q) be two co-planar couples acting on a rigid body and the moments of the couples P.p and Q.q are equal in magnitude but of opposite signs.

First let the forces P and Q be not parallel and let the line of action of a force P of the couple (P, p) intersect the line of action of a force Q of the couple (Q, q) at the point O and the lines of action of the other two forces P and Q intersect at the point O′. Since (P, p) and (Q, q) are two couples, so the lines of action of the forces intersecting at O will not pass through O′ and conversely. Now, draw perpendiculars $\overline{O'M}$ and $\overline{O'N}$ from the point O′ on the lines of action of the forces P and Q acting at O. ∴ O′M $= p$

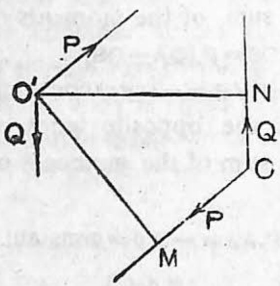


Fig. 58

and $O'N = q$. Now the moments about O of forces acting at O are respectively $P.O'M$ and $Q.O'N$ i.e., $P.p$ and $Q.q$ which are equal in magnitude and the moments are of opposite signs and so the algebraic sum of these moments is zero. Hence the resultant of these two forces passes through O'. Similarly, the resultant of the forces acting at O' passes through the point O. Hence the lines of action and sense of the two resultants are $\overrightarrow{O'O}$ and $\overrightarrow{OO'}$ respectivly. Also as these forces acting at O are equal to the forces acting at O' and they are inclined at equal angles, so the two resultants are equal in magnitude. Hence the two resultants *i.e.*, the two couples cancel each other.

Next let, the force of the couple (P, p) be parallel to the forces of the couple (Q, q) and a straight line perpendicular to these forces intersect their lines of action at A, B, C and D. Now, as the moments of the couples are equal, so $P.AB = Q.CD \cdots (1)$

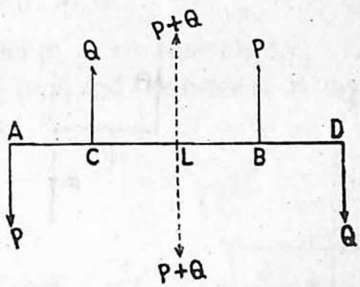


Fig. 59

Let the line of action of the resultant $P+Q$ of the like parallel forces acting at the point B and C intersect $\overleftrightarrow{AB}$ at L.

$\therefore \quad P.LB = Q.LC \cdots\cdots (2)$

Subtracting (2) from (1) we obtain $P.AL = Q.DL$.

Hence the line of action of the resultant $P+Q$ of the like parallel forces P and Q acting at A and D also passes through L. But the senses of these two resultants are opposite and so their lines of action coincide. So, these two resultants and hence the two couples balance each other.

Cor. From the above proposition we get the following important corolary.

If the magnitudes and sign of two coplanar couples are the same, then any one of the couples can be replaced by the other.

Theorem. **The resultant of any number of co-planar couples acting on a body is a couple whose moment is equal to the algebraic sum of the moments of the couple.**

Let (P, p), (Q, q), (R, r), (S, s) etc. be a number of co-planar couples acting on a body.

The moment of the couple (P, p) is P.p, the moment of the couple (Q, q) is $Q.q=\frac{Q.q}{p}.p$.

Hence if the couple $\left(\frac{Q.q}{p}, p\right)$ be such that the lines of action of the forces $\frac{Q.q}{p}$ coincide with the lines of a action of the forces P and their senses be such that the moment of this couple be of the same sign as that of the couple (Q.q) then the couple (Q; q) and $\left(\frac{Q\ q}{q}\right)$ have moments of the same magnitude and sign. So, the couple (Q, q) can be replaced by the couple

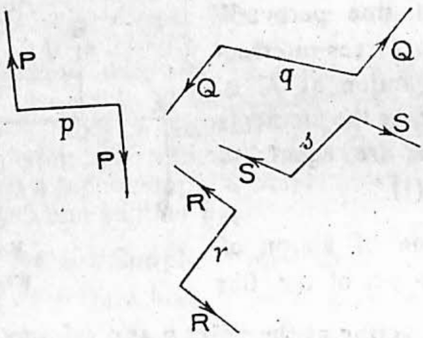


Fig. 60

$\left(\frac{Q.q}{p}, p\right)$. Similarly let us replace the couples (R, r), (S, s) etc, by couples $\left(\frac{Rr}{p}, p\right)\left(\frac{Ss}{p}, p\right)$ etc, the forces of each couple acting along the lines of action of the couple (P, p). Hence the given couples are replaced by a single couple whose arm is p and each force is of magnitude $\left(P+\frac{Qq}{p}+\frac{Rr}{p}+\frac{Ss}{p}+\cdots\right)$

[Here each force is to be taken with its proper sign].

Hence the moment of the resultant couple is

$$\left(P+\frac{Q.q}{p}+\frac{R.r}{p}+\frac{S.s}{p}+\cdots\right)p=P.p+Q.q+R.r+S.s+\cdots$$

=algebraic sum of the moments of the given couples,

Hence the resultant of a number of coplanar couples is a couple and the moment of the resultant couple is equal to the algebraic sum of the moments of the given couples.

§ 6·6. Resultant of a couple and a force :

Theorem : The resultant of a couple and a force acting in the same plane is a single force equal and parallel to the given force.

Let a couple (P, p) and a force F act in the same plane. Now introduce two forces F and F in the same plane one acting along the line of action of the given force F in the opposite sense and the other like parallel to the given force F at a distance $x=\frac{P.p}{F}$ from its line of action so that the couples (F, x) and (P, p) have moments of the same sign.

Now the moment of the couple (F, x) is $F.x=F.\frac{Pp}{F}=P.p$.

Hence the two couples (P, p) and (F, x) are equivalent.

$\therefore$ the resultant of the couple (P,p) and the force F, is the resultant of the couple (F, x) and the force F. Now the two equal and opposite forces (F,F) acting along the line of action of the given force F balance each other and we are left with the force F at a distance $x=\frac{P.p}{F}$ and so this force F is the resultant of the given couple and the given force.

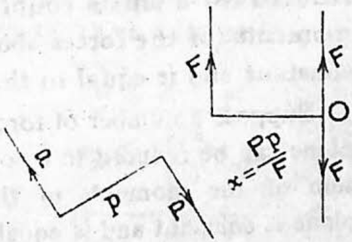


Fig. 61

Cor. 1. A couple and a force cannot produce equilbrium.

Cor. 2. The forces constituting a couple cannot have a resultant. For, if they have a resultant F, then the couple and a force equal and opposite to this resultant will produce equilibrium.

§ 6·7. A force acting at a point of a body can be replaced by an equal and parallel force together with a couple.

Let a force P be acting on a body at a point O. Let O′ be any other point of the body. Now, at O′ introduce two equal and opposite forces each equal and parallel to the force P. Since these two forces balance each other, they do not affect the state of the body.

Now the force P acting at O and the equal and unlike parallel force P acting at O′ constitute a couple. Hence the given force P acting at O and this couple together with the other force P acting at O′ are equivalent *i.e.*, the given force can be replaced by a couple together with an equal, like parallel force.

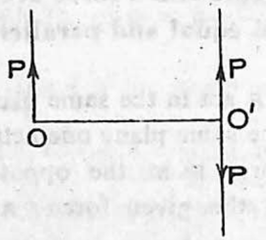


Fig. 62

Also, the moment of the couple is $-P.OO'$ and the moment about O′ of the given force P acting at the point O is $-P.OO'$.

Hence the moment of the couple is equal to the moment of the given force about the point O′.

§ 6·8. **If a number of forces acting in one plane can be reduced to a single couple, then the algebraic sum of the moments of the forces about any point in their plane is constant and is equal to the moment of the couple.**

Suppose a number of forces P_1, P_2, P_3 etc. acting in one plane can be reduced to a couple. To prove that the algebraic sum of the moments of the forces about any point in their plane is constant and is equal to the moment of the couple.

Proof. Let O be any point in the plane of the given forces. Now according to § 6·7, each of the given forces can be replaced by a couple and an equal like parallal force acting at the point O. Also in each case the moment of the couple is constant and equal to the moment of the corresponding given force about the point O.

So, the given forces can be reduced to a number of couples together with a number of forces acting at the point O.

Now the couples can be reduced to a resultant couple whose moment is equal to the algebraic sum of the moments of the couples *i.e.*, constant and is equal to the algebraic sum of the moments of the given forces about the point O.

Again, if the forces acting at the point O be not in equilibrium then they will have a single resultant. In this case the given forces are equivalent to a single couple together with a given force. But according to the given conditions the given

forces are equivalent to a single couple and a single couple and a single force cannot reduce to a couple (§ 6·6).

Hence the forces acting at the point O must be in equilibrium and the given forces will reduce to a single couple whose moment is equal the algebraic sum of the moments of the given forces about the point O.

Thus, the algebraic sum of the moments of the given forces about the point O is equal to the moment of the resultant couple and is constant.

§ 6·9. If three forces acting on a rigid body are represented in magnitude, direction and sense by the three sides of a triangle taken in order, then the forces are equivalent to a couple the magnitude of whose moment is twice the area of the triangle.

Three forces P, Q, R acting on a rigid body are represented in magnitude, direction and sense by the three sides BC, CA, AB of the triangle ABC taken in order. To prove that the forces are equivalent to a couple and the moment of the couple is equal to twice the area of the triangle.

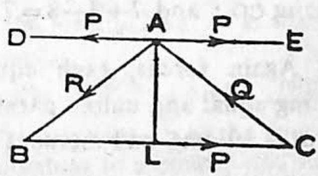


Fig. 63

Through A draw DE parallel to BC and inroduce two equal and opposite forces P, P at A along AD and AE. They will not alter the equilibrium of the given forces as being equal and opposite, these forces cancel each other. Now the three forces P, along $\overrightarrow{AE}$, Q along $\overrightarrow{CA}$ and R along $\overrightarrow{AB}$ acting at the point A, are represented in magnitude direction and sense by the three sides of the triangle ABC taken in order.

So, by the theorem of triangle of forces, the forces are in equilibrium. The remaining forces P along $\overrightarrow{BC}$ and P along $\overrightarrow{AD}$ are equal and unlike parallel forces acting on a rigid body, so they constitute a couple of moment $P \times AL$ [where AL is the perpendicular from A on BC]$= BC \times AL = 2 \times \frac{1}{2} BC \times AL = 2m\triangle ABC$.

Ex. 1. Each side of the square ABCD is of length 10 cms. Forces 5, 7, 20 and 8 kg. weights act along $\vec{AB}$, $\vec{BC}$, $\vec{CD}$ and $\vec{DA}$ and forces $8\sqrt{2}$ and $7\sqrt{2}$ kg. weights act along $\vec{AC}$ and $\vec{DB}$; Determine the resultant of the forces.

The force $8\sqrt{2}$ kg. weight along $\vec{AC}$ is equivalent to two components $8\sqrt{2}\cos 45°$; $8\sqrt{2}\cos 45°$, or, $8\sqrt{2}.\frac{1}{\sqrt{2}}$ and $8\sqrt{2}.\frac{1}{\sqrt{2}}$ i.e., 8 kg. wt. and 8 kg. wt. along $\vec{AB}$ and $\vec{AD}$. Similarly, the force $7\sqrt{2}$ kg. wt. along $\vec{DB}$ is equivalent to components 7 kg. wt. and 7 kg. wt. along $\vec{DC}$ and $\vec{DA}$.

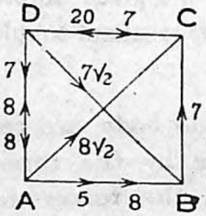


Fig. 64

Hence the given forces can be reduced to four forces $5+8=13$ kg. wt. along $\vec{AB}$; 7 kg. wt. along $\vec{BC}$; $20-7=13$ kg. wt. along $\vec{CD}$; and $7+8-8=7$ kg. wt. along $\vec{DA}$.

Again forces, each equal to 13 kg. wt. along $\vec{AB}$ and $\vec{CD}$, being equal and unlike parallel forces constitute a couple whose arm is 10 cms. and moment is $13.10=130$ kg. cms. Similarly the forces, each equal to 7 kg. wts. along $\vec{BC}$ and $\vec{DA}$, constitute a couple of moment $7.10=70$ kg. cms.

Hence the given forces are equivalent to two couples which in their turn are equivalent to a single couple of moment $130+70=200$ kg. cms.

Thus the given force-system is equivalent to a couple of moment 200 kg. cms.

Ex. 2. Two forces each equal to 10 kg. act along the sides $\overline{AB}$ and $\overline{CD}$ of the square ABCD in the senses from A to B and from C to D respectively. A third force of magnitude 12 kg. wt act along $\vec{CA}$. Find the resultant of the three forces.

The two equal and unlike parallel forces each equal to 10 kg. acting along $\vec{AB}$ and $\vec{CD}$ constitute a couple of moment 10.AD.

Now this couple can be replaced by a couple (12, $\frac{5}{6}$AD) of positive moment [the moment of the couple (10, AD) is positive in our figure] for the moment of the couple (12, $\frac{5}{6}$AD) is $12.\frac{5}{6}$AD $=10.$AD.

Now, this couple is taken in such a way that a force of magnitude 12 kg. act along $\overrightarrow{CE}$, the line of action of the given third force in the opposite sense and the other force is an equal and unlike parallel force of 12 kg. at a distance $\frac{5}{6}$AD from $\overrightarrow{CE}$ on that side of $\overrightarrow{CE}$, so that the moment of the couple (12, $\frac{5}{6}$AD) is positive.

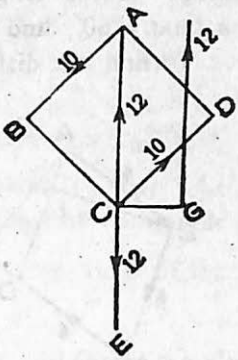


Fig. 65

Hence in place of the given three forces we obtain two forces 12 kg., 12 kg, acting along $\overrightarrow{CA}$ and $\overrightarrow{CE}$ and a third force 12 kg. acting at G, and like parallel to the force acting along $\overrightarrow{CA}$. The two equal and opposite forces acting at C balance each other and we are left with the force 12 kg. at G which is the resultant of the three given forces.

Ex. 3. Five forces of magnitude 1, 2, 3, 4 and $2\sqrt{2}$ act along the side $\overrightarrow{AB}$, $\overrightarrow{BC}$, $\overrightarrow{CD}$, $\overrightarrow{DA}$ and the diagonal $\overrightarrow{AC}$ of the square ABCD. Prove that the forces are equivalent to a couple and find the moment of the couple. [C. U. 1947]

The components along $\overrightarrow{AB}$ and $\overrightarrow{AD}$ of the force $2\sqrt{2}$ along $\overrightarrow{AC}$ are $2\sqrt{2}\cos 45^\circ=2\sqrt{2}.\frac{1}{\sqrt{2}}=2$ and $2\sqrt{2}\sin 45^\circ=2\sqrt{2}.\frac{1}{\sqrt{2}}=2.$

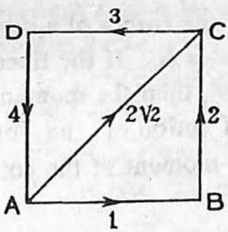


Fig. 66

Hence the given forces can be reduced to forces $1+2=3$ along $\overrightarrow{AB}$, 2 along $\overrightarrow{BC}$, 3 along $\overrightarrow{CD}$ and $4-2=2$ along $\overrightarrow{DA}$. Now these forces constitute two couples (i) 3 and 3 along $\overrightarrow{AB}$ and $\overrightarrow{CD}$ and (ii) 2 and 2 along $\overrightarrow{BC}$ and $\overrightarrow{DA}$. Now the moments of these couples are $3a$ and $2a$ [taking each side of the square as of length a].

Again, the resultant of these two couples is a couple of moment $3a+2a=5a$. Hence the given forces can be reduced to a couple of moment $5a$.

Ex. 4. Forces of magnitude 4, 6, 4 and 6 kg. wts. act respectively along the sides $\overrightarrow{AB}$, $\overrightarrow{BC}$, $\overrightarrow{CD}$ and $\overrightarrow{DA}$ of the rhombus ABCD. If the length of a side of the rhombus be 6 metres and $m\angle BAD=60°$, find the moment of the resultant couple.

To find the distance of $\overline{BC}$ and $\overline{AD}$ draw $\overline{BL}$, perpendicular from B on $\overline{AD}$. Now $BL = AB \sin A$

$=6 \sin 60° = \frac{6\sqrt{3}}{2} = 3\sqrt{3}$ metres.

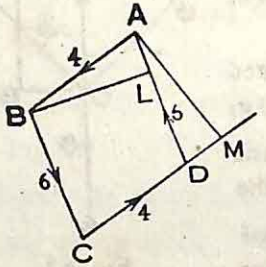


Fig. 67

Again, the distance AM between $\overline{AB}$ and $\overline{CD}$ is $AD \sin \angle ADM = 6 \sin 60°$ $= \frac{6\sqrt{3}}{2} = 3\sqrt{3}$ metres.

Now equal and unlike parallel forces each of magnitude 4 kg. wts. acting along $\overrightarrow{AB}$ and $\overrightarrow{CD}$ constitute a couple of moment $4.AM = 4.3\sqrt{3}$ kg. metres $=12\sqrt{3}$ kg. metres. Also, equal and unlike parallel forces each of magnitude 6 kg. wts. acting along $\overrightarrow{BC}$ and $\overrightarrow{DA}$ constitute a couple of moment $6.BL = 6.3\sqrt{3}$ kg. metres $=18\sqrt{3}$ kg. metres.

Now, the resultant of these couples is a couple of moment $12\sqrt{3}+18\sqrt{3}=30\sqrt{3}$ kg. metres.

Hence the given forces are equivalent to a couple of moment $30\sqrt{3}$ kg. metres.

Ex. 5. The points of application of the forces of a couple are A and B and the moment of the couple is G. If the lines of action of the forces are rotated through 90°, then the moment of the couple becomes H. When the lines of action of the forces are perpendicular to $\overline{AB}$, show that the moment of the couple becomes $\sqrt{G^2+H^2}$.

Let the magnitude of each force is P and at first they are inclined at an angle θ with $\overline{AB}$. In this case the arm of the couple is $AB \sin\theta$ and the moment is $P.AB \sin\theta$,

$\therefore \quad G = P.AB \sin\theta \cdots\cdots(1)$

In the second case the arm is AB $\sin(90^\circ+\theta)=$ AB $\cos\theta$ and the moment of the couple is P.AB $\cos\theta$.

$\therefore$ H$=$P.AB $\cos\theta$ ......(2)

In the third case, the lines of action of the forces being perpendicular to $\overline{AB}$, the arm of the couple is AB and the moment of the couple is P.AB.

Now, from (1) and (2) we obtain,

$G^2+H^2=P^2AB^2(\sin^2\theta+\cos^2\theta)=P^2.AB^2$

$\therefore$ P.AB$=\sqrt{G^2+H^2}$.

Hence the moment of the couple in the third case $=\sqrt{G^2+H^2}$.

Ex. 6. A couple (F, a) is acting in the plane of two like parallel force P and Q. Prove that the resultant of the couple and the parllel forces is at a distance $\dfrac{Fa}{P+Q}$ from that of the resultant of the like parallel forces.

Let the points of application of the like parallel forces P and Q and their resultant P+Q be respectively A, B and C. Now according to § 6·6 the resultant of the forces P+Q and the couple (F, a) is a force P+Q which is like parallel to the force P+Q acting at a point D such that $CD=\dfrac{Fa}{P+Q}$.

Ex. 7. Forces P, Q, R act along the tangents to the circum-circle of the triangle ABC at the vertices A, B, C taken in order. If the forces can be reduced to a couple, show that

$$P:Q:R=\sin 2A:\sin 2B:\sin 2C.$$

Let the tangents to the circum-circle of the triangle ABC at the vertices A, B, C be respectively $\overleftrightarrow{EF}$, $\overrightarrow{FD}$, and $\overrightarrow{DE}$.

Now $m\angle EAC=m\angle B$ (angle of the alternate segment)

$=m\angle ECA$

$\therefore$ from $\triangle EAC$, we obtain, $m\angle E=180^\circ-2m\angle B$.

Similarly, $m\angle F=180^\circ-2m\angle C$ and $m\angle D=180^\circ-2m\angle A$.

Now, let the circum-radius of the triangle DEF be R′ and $\overline{DL}\perp\overleftrightarrow{EF}$.

$\therefore$ DL$=$DE $\sin(180^\circ-2B)=2R'\sin F\sin 2B$.

$\left[\because \text{ In } \triangle DEF,\ \dfrac{EF}{\sin D}=\dfrac{FD}{\sin E}=\dfrac{DE}{\sin F}=2R'\right]$

$=2R'\sin 2C\sin 2B$.

Similarly if $\overline{EM}$ and $\overline{FN}$ be respectively perpendiculars on $\overleftrightarrow{FD}$ and $\overleftrightarrow{DE}$, then $EM = 2R' \sin 2C \sin 2A$ and $FN = 2R' \sin 2A \sin 2B$.

Now as the three given forces can be reduced to a couple, so the algebraic sum of the moments of the forces about each of the points A, B, C are constants. Now the algebraic sum of the moments of the forces about D

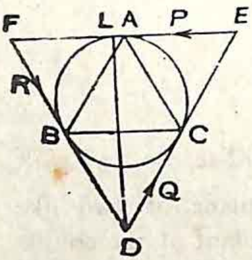


Fig. 68

$$= P.DL + Q.O + R.O$$

$$= P.2R' \sin 2B \sin 2C.$$

Similarly, the algebraic sum of the moments of the forces about E and F are respectively $Q.2R' \sin 2B \sin 2C$ and $R.2R' \sin 2A \sin 2B$.

$$\therefore \quad P.2R' \sin 2B \sin 2C = Q.2R' \sin 2C \sin 2A$$

$$= R.2R' \sin 2A.\sin 2B.$$

or, $$\frac{P}{\sin 2A} = \frac{Q}{\sin 2B} = \frac{R}{\sin 2C}.$$

Ex. 8. In the rectangle ABCD, $AB = CD = l$ and $BC = DA = m$. Forces P, Q, P, Q act along $\overrightarrow{AB}$, $\overrightarrow{BC}$, $\overrightarrow{CD}$ and $\overrightarrow{DA}$. Prove that the perpendicular distance of the lines of action of the resultants of the forces acting at A and C is $\dfrac{Ql + Pm}{\sqrt{P^2 + Q^2}}$.

The resultant of the forces P and Q acting at the point A, is a force $\sqrt{P^2 + Q^2}$ acting at A and that of the forces P and Q acting at the point C, is also a force $\sqrt{P^2 + Q^2}$. Again, the lines of action of these resultants are equally inclined to $\overrightarrow{AB}$ and $\overrightarrow{CD}$. Hence these resultants constitute a couple $(\sqrt{P^2 + Q^2}, d)$ [say].

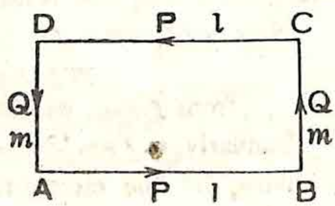


Fig. 69

Again the forces P and P acting long $\overrightarrow{AB}$ and $\overrightarrow{CD}$ is a couple (P, m) and the forces Q and Q acting along $\overrightarrow{BC}$ and $\overrightarrow{DA}$ is a couple

(Q, l). Hence the resultant of the couples (P, m) and (Q, l) is the couple ($\sqrt{P^2+Q^2}$, d)

$$\therefore \quad Pm+Ql=\sqrt{P^2+Q^2}.d.$$

$$\therefore \quad d=\frac{Pm+Ql}{\sqrt{P^2+Q^2}}.$$

Ex. 9. Three forces acting along the three sides BC, CA, AB of a triangle ABC taken in order constitute a couple. Prove that the three forces can be represented by those sides of the triangle.

Let three forces P, Q, R acting along the sides BC, CA, AB of a triangle ABC, taken in order constitute a couple. As the three forces constitute a couple, so the algebraic sum of the moments of the forces about every point in their plane is constant.

Taking moments about A we get P.AD+Q.O+R.O=P.AD (where AD is perpendicular from A on BC)$=P.BC.\frac{AD}{BC}=P.\frac{2\Delta}{a}$ [Δ being the area of the triangle ABC]. Similarly taking moments about B and C we get, the algebraic sum of the moments of these forces about these points are $Q.\frac{2\Delta}{b}$ and $R.\frac{2\Delta}{c}$ respectively.

$$\therefore \quad P.\frac{2\Delta}{a}=Q.\frac{2\Delta}{b}=R.\frac{2\Delta}{c} \quad \text{or,} \quad \frac{P}{a}=\frac{Q}{b}=\frac{R}{c}.$$

Hence the magnitudes of the forces are proportional to the sides BC, CA, AB respectively. Hence the forces can be represented by the three sides of the triangle taken in order.

Exercise 6

1. The lengths of the sides of the square ABCD are 10 cms. each ; forces of magnitudes 1, 2, 8 and 5 kg. weights act along $\overrightarrow{AB}$, $\overrightarrow{BC}$, $\overrightarrow{CD}$ and $\overrightarrow{DA}$ and forces of magnitudes $5\sqrt{2}$, $2\sqrt{2}$ kg. weights act along $\overrightarrow{AC}$ and $\overrightarrow{DB}$ respectively. Prove that the forces are equivalent to a couple and determine the moment of this equivalent couple.

2. Forces of magnitudes 3, 5, 3 and 5 pound weights act along the sides of a square taken in order. Find the resultant of the forces. [C. U. 1932]

3. The moments of two couples are equal in magnitude but are of opposite signs. If the forces of the two couples act along the sides of a parallelogram, prove that the magnitudes of the forces are proportional to the lengths of the sides of the parallelogram.

4. Three forces acting along the sides of a triangle taken in order can be reduced to a single couple. Prove that the magnitudes of the forces are proportional to the lengths of the sides of the triangle.

5. P and Q are two like parallel forces. P+Q is a force acting in the same plane and is unlike parallel to the forces P and Q and its line of action is between the lines of action of P and Q and is at distances a and b from the lines of action of the forces P and Q respectively. Find the moment of the resultant couple.

6. ABCD is rectangle and $AB=CD=l$, $BC=DA=m$. Two forces P, P act along $\overrightarrow{AD}$ and $\overrightarrow{CB}$ and two forces Q, Q act along $\overrightarrow{AB}$ and $\overrightarrow{CD}$. Prove that the distance between the lines of action of the resultant of the forces P, Q acting at A and that of the forces P, Q acting at C is $\dfrac{Pl-QM}{\sqrt{P^2+Q^2}}$.

7. ABCD and EFGH are two parallelograms in the same plane. Four forces act along $\overline{AE}$, $\overline{FB}$, $\overline{CG}$ and $\overline{HD}$ in the plane. Prove that the resulant of the four forces is a couple.

8. Find the resultant of a comple $(4, 2\frac{1}{2})$ and a force of magnitude 5 units acting in the same plane.

9. Six forces are represented in magnitude, direction and sense by the six sides of a regular hexagon taken in order. Prove that the six forces are equivalent to a couple and find the moment of the couple.

10. The lines of action of three forces acting at the vertices A, B, C of a triangle are perpendicular to BC. Prove

that if the lines of action of the forces are turned through the same angle in the same direction, then they reduce to a couple.

11. The sides $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ of the equilateral triangle are divided in the ratio 2 : 1 at the points D, E, F respectively internally. Three forces each of magnitude P act perpendicular to the sides at these points. Prove that the forces constitute a couple of moment $\frac{1}{2}$ P. a where a is the length of each side of the triangle. [H. S. Tech '72]

12. Prove that if the algebraic sum of the moments of two forces about every point in their plane be constant, then the forces constitute a couple.

13. Each forces of a couple is 25 kg. and its arm is 4 cm. Each force of a couple equivalent to the former couple is of magnitude 20 kg. Find the length of this later couple.

14. Each side of the square ABCD is of length 2 units. Forces a, b, c, d act along the sides $\overline{AB}$, $\overline{BC}$; $\overline{CD}$ and $\overline{DA}$ of this square taken in order. Forces $p\sqrt{2}$ and $q\sqrt{2}$ act along $\overline{AC}$ and $\overline{DB}$ respectively. If $p+q=c-a$ and $p-q=d-b$, then show that the forces constitute a couple of moment $a+b+c+d$.

15. ABCD is a rectangle ; forces $p.\overline{AB}$, $q.\overline{BC}$; $r.\overline{CD}$ and $s.\overline{DA}$ act along the sides $\overline{AB}$, $\overline{BC}$; $\overline{CD}$ and $\overline{DA}$ respectively. Show that the necessary and sufficient conditions for the forces to constitute a couple are $p=r$ and $q=s$.

16. At the ends A and B of a weight less uniform rod kept in a horizontal position, act a vertically upward force of 2 lbs. wt. and a vertically downward force 2 lbs. wt. respectively. At a given point C of the rod a force 2 lbs. wt. is applied making with the rod an angle 30° in the downward direction. Find the magnitude, direction and point of application of the force which will keep the forces in equilibrium.

17. Six forces of magnitudes 1, 2, 3, 2, P and Q acting along the sides of a regular hexagon taken in order constitute a couple. Find P and Q and also the moment of the couple.

CHAPTER SEVEN

CENTRE OF GRAVITY

§ 7.1. A rigid body can be conceived of as the collection of particles of matter rigidly connected together. Now, according to Newton's Law of gravitation, every material particle is being attracted towards the centre of the earth ; the force of attraction is proportional to the mass of the particle and is inversely proportional to the square of its distance from the centre of the earth and the force is towards the centre of the earth. This force of attraction is called the **weight** of the material particle. If the size of the particle is very small as compared to that of the earth, then the lines of action of these forces of attraction can be taken as parallel and so the forces can be taken as parallel forces. The point of the body through which the resultant of these forces of attraction passes is called the **centre of gravity** of the body. Hence to determine the Centre of Gravity of a body, one has to determine the resultant of a number of parallel forces. So, in the next article we discuss the method of determination of the resultant of a system of parallel forces.

§ 7·2. **Centre of parallel forces.** Let parallel forces $P_1, P_2, \cdots\cdots, P_n$ act at the points $A_1, A_2, \cdots\cdots A_n$ of a rigid body. We are to find the resultant of these parallel forces.

We know from chapter four that like parallel forces of magnitudes P_1 and P_2 acting at points A_1 and A_2 have as resultant the like parallel force P_1+P_2 acting at a point C_1, dividing the line segment $\overline{A_1A_2}$ internally in the ratio $P_2 : P_1$ *i.e.*, $A_1C_1 : A_2C_1 = P_2 : P_1$. Again the resultant of like parallel forces P_1+P_2 acting at the point C_1 and P_3 acting at the point A_3 is a like parallel force $P_1+P_2+P_3$ acting at a point C_2 of $\overline{C_1A_3}$ where $C_1C_2 : C_2A_3 = P_3 : (P_1+P_2)$. In this way proceeding $(n-1)$

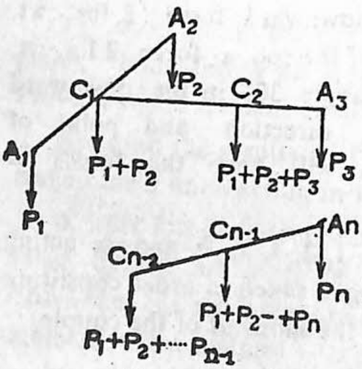


Fig. 70

times we ultimately get a point C_{n-1} of the line segment $\overline{C_{n-2}A_n}$ where the resultant $P_1+P_2+\cdots\cdots+P_n$, a force like parallel to the given forces acts and the point C_{-1} is such that $C_{n-2}C_{n-1} : C_{-1}A_n = P_n : (P_1+P_2+\cdots\cdots+P_{n-1})$.

Let us now discuss the above analytically (*i.e.*, by the method of co-ordinate Geometry).

Let the points $A_1, A_2, \cdots\cdots, A_n$ lie in the same plane and let us take two convenient axes of co-ordinates $\overrightarrow{OX}$ and $\overrightarrow{OY}$ in the plane of the points. Let $(x_1, y_1), (x_2, y_2), \cdots\cdots, (x_n, y_n)$ be the co-ordinates of these points referred to these axes and $(\alpha_1, \beta_1), (\alpha_2, \beta_2), \cdots\cdots, (\alpha_{n-1}, \beta_{n-1})$ be the co-ordinates of points $C_1, C_2 \cdots\cdots, C_{n-1}$. Now, as the point C_1 divides $\overline{A_1A_2}$ internally in the ratio $P_2 : P_1$, so from the formulas of Co-ordinate Geometry we obtain.

$$\alpha_1 = \frac{P_1x_1+P_2x_2}{P_1+P_2},\ \beta_1 = \frac{P_1y_1+P_2y_2}{P_1+P_2}.$$

Again, the point C_2 divides $\overline{C_1A_3}$ in the ratio $P_3 : (P_1+P_2)$.

$$\therefore\ \alpha_2 = \frac{(P_1+P_2)\alpha_1+P_3x_3}{P_1+P_2+P_3}$$

$$= \frac{(P_1+P_2)\dfrac{P_1x_1+P_2x_2}{P_1+P_2}+P_3x_3}{P_1+P_2+P_3} = \frac{P_1x_1+P_2x_2+P_3x_3}{P_1+P_2+P_3}$$

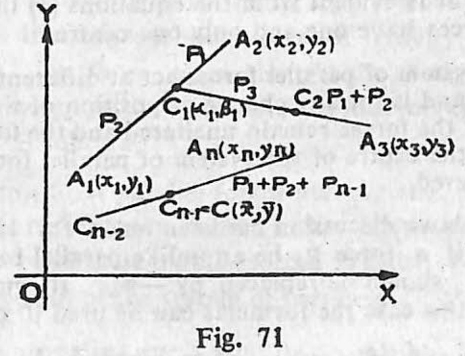


Fig. 71

Similarly $\beta_2 = \dfrac{P_1y_1+P_2y_2+P_3y_3}{P_1+P_2+P_3}$.

Proceeding in this way we ultimately get the point

$C_{n-1}(\alpha_{n-1}, \beta_{n-1})$ as the point of application of the parallel forces $P_1, P_2, \cdots\cdots P_n$ as,

$$\alpha_{n-1} = \frac{P_1x_1 + P_2x_2 + \cdots\cdots + P_nx_n}{P_1 + P_2 + \cdots\cdots + P_n} \text{ and}$$

$$\beta_{n-1} = \frac{P_1y_1 + P_2y_2 + \cdots\cdots + P_ny_n}{P_1 + P_2 + \cdots\cdots + P_n}$$

If we write the point C_{n-1} as the point G and its co-ordinates be $(\bar{x}, \bar{y})$

$$\left.\begin{aligned} \text{then } \bar{x} &= \frac{P_1x_1 + P_2x_2 + \cdots\cdots + P_nx_n}{P_1 + P_2 + \cdots\cdots + P_n} = \frac{\Sigma Px}{\Sigma P} \\ \bar{y} &= \frac{P_1y_1 + P_2y_2 + \cdots\cdots + P_ny_n}{P_1 + P_2 + \cdots\cdots + P_n} = \frac{\Sigma Py}{\Sigma P} \end{aligned}\right\} \cdots(1)$$

Hence the resultant of the parallel forces $P_1, P_2, \cdots\cdots, P_n$ acting at the points $A_1, A_2, \cdots\cdots, A_n$ always passes through the point G $(\bar{x}, \bar{y})$ and it is evident that the position of G does not depend on the direction of the forces $P_1, P_2, \cdots\cdots, P_n$ but on the magnitudes and the points of application of the individual forces.

The above discussion can be summarised as follows :

If the magnitudes and points of application of a system of parallel forces acting at points, rigidly connected together, remain unaltered, then the line of action of the resultant of these parallel forces always passes through a fixed point, whatever be the common direction of the system of forces. This fixed point is called the *centre of the system of parallel forces.*

Note. (1) It is evident from the equations (1) that a system of parallel forces have one and only one centre.

(2) If a system of parallel forces act at different points of a rigid body and if for any change of position of the body, the magnitudes of the forces remain unaltered and the forces remain parallel, then the centre of the system of parallel forces always remains unaltered.

(3) The above discussion has been made for like parallel forces. But if a force P_k be an unlike parallel force, then in formulas-(1) P_k should be replaced by $-P_k$. It must also be noted that in this case the formulas can be used if $\Sigma P \neq 0$.

§ 7·3. Centre of gravity of a rigid body and rigidly connected particles.

At the beginning of this chapter we have said that a rigid body can be conceived of as a collection of rigidly connected material

particles and these particles are being attracted towards the centre of the earth ; the force of attraction is called the weight of the body. Now, if the body be small, then the lines joining the different points of the body with the centre of the earth can be taken as parallel straight lines (as the distance of the earth's surface from the centre of the earth is nearly 4000 miles). Now, the weights of these material particles act along these lines. We can determine the resultant of these parallel forces by the method of § 7·2 and the line of action of these parallel forces passes through a fixed point. The magnitude of the resultant is called the weight of the body and the centre of these parallel forces is called the **centre of gravity** of the body. Since the centre of parallel forces is independent of the directions of the parallel forces, so the centre of gravity of a body remains fixed for different positions of the body. Hence the centre of gravity of a body or a collection of material particles can be defined as follows :

Def. The centre of gravity of a body (or a system of particles) is the point through which the line of action of the weight of the body always passes, whatever be the position of the body.

Note 1. The weight of a body is due to the gravitational attraction on the body. If the body is taken to a place, where there is no attraction of the earth, then the body will have no weight and the concept of centre of gravity will become meaningless. In those cases though the body will have no centre of gravity, yet one can have the idea of another centre of the body as follows :

Definition : If parallel forces act on the particles of a system of particles, proportional to the masses of the particles, then the centre of these parallel forces is called the **centre of mass** or **centroid** of the system of particles.

Since the attraction due to gravity on particles is proportional to the masses of the particles and the centre of a system of parallel forces remains unaltered if each force is increased or decreased in the same proportion, so the centre of gravity and

the centre of mass of a body is the same point. The centre of gravity of a body does not exist in a place where there is no gravitational attraction, but the centre of mass exists everywhere.

(2) Since every system of parallel forces has one and only one centre, so a body can have only one centre of gravity.

(3) If the size of a body be not enough small in comparison with the size of the earth, the forces of attraction acting on the body may not be parallel and so the body may not have a centre of gravity.

§ 7·4. Method of finding the centre of gravity of a continuous body.

To determine the centre of gravity of a body or a collection of rigidly connected material particles, the body or the collection of material particles is generally divided in several parts according to convenience, so that the weight and centre of gravity of the parts can be separately determined. Now the centre of gravity of these separate weights is the centre of gravity of the body.

Let $\omega_1, \omega_2, \omega_3 \cdots$ be the weights of material particles and (x_1, y_1), (x_2, y_2), $(x_3, y_3) \cdots$ be the co-ordinates of the positions of the particles with reference to a suitable chosen pair of axes of co-ordinates. If $(\bar{x}, \bar{y})$ be the co-ordinates of the centre of gravity of the collection of material particles, then according to the formula-(1) of § 7·2,

$$\bar{x} = \frac{\omega_1 x_1 + \omega_2 x_2 + \omega_3 x_3 + \cdots}{\omega_1 + \omega_2 + \omega_3 + \cdots} = \frac{\Sigma \omega x}{\Sigma \omega}$$

and $$\bar{y} = \frac{\omega_1 y_1 + \omega_2 y_2 + \omega_3 y_3 + \cdots}{\omega_1 + \omega_2 + \omega_3 + \cdots} = \frac{\Sigma \omega y}{\Sigma \omega}.$$

Let us now consider a continuous body in a plane. To determine the centre of gravity of the body, divide the body into a number of parts, so that the weight and position of centre of gravity of each part are separately known. Let $\delta\omega$ be the weight of one such part and (ξ, η) be the co-ordinates of the centre of gravity of the part. If $(\bar{x}, \bar{y})$ be the centre of

gravity of the body, then $\bar{x}$ and $\bar{y}$ will be approximately $\frac{\Sigma\xi\delta\omega}{\Sigma\delta\omega}$ and $\frac{\Sigma\eta\delta\omega}{\Sigma\delta\omega}$ where the summation is taken over the whole body. Now let us make the number of parts indefinitely large so that the weight of each part $\delta\omega$ tends to zero. In that case $\Sigma\xi\delta\omega$, $\Sigma\eta\delta\omega$ and $\Sigma\delta\omega$ respectively tend to $\int\xi d\omega$, $\int\eta d\omega$ and $\int d\omega$ and we shall get exact values of $\bar{x}$ and $\bar{y}$ and one can write

$$\bar{x}=\frac{\int\xi d\omega}{\int d\omega} \text{ and } \bar{y}=\frac{\int\eta d\omega}{\int d\omega}\cdots\cdots(1)$$

where (ξ, η) are the co-ordinates of the part of weight $\delta\omega$ and the definite integrals are to be considered over the whole body.

If δm be the mass of the part of weight $\delta\omega$ then $\delta\omega=g\delta m$, g being the acceleration due to gravity. Then the above formulas can be written as,

$$\left.\begin{aligned}\bar{x}&=\frac{\int\xi d\omega}{\int d\omega}=\frac{\int\xi g dm}{\int g dm}=\frac{\int\xi dm}{\int dm}\\ \text{and } \bar{y}&=\frac{\int\eta d\omega}{\int d\omega}=\frac{\int\eta g dm}{\int g dm}=\frac{\int\eta dm}{\int dm}\end{aligned}\right\}\cdots(2)$$

where the definite integrals are taken over the whole area.

Note 1. Formula-(2) is the formula for detemination of centre of mass.

2. From the formulas (1) and (2) it is evident that the centre of gravity and centre of mass are the same point.

§ 7·5. Determination of centre of gravity of elementary bodies.

(A) **Centre of gravity of uniform rod.** Let $\overline{AB}$ be a uniform rod of length l, *i.e.*, $AB=l$. Let us take A as the orign, $\overrightarrow{AB}$ as the x-axis and a straight line perpendicular to $\overrightarrow{AB}$ as the y-axis. Let P be a point of the rod at a distance x from A and dx be the length of an element of mass at P.

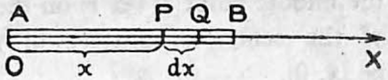



Fig. 72

$\therefore$ $dx=\rho dm$, where ρ is the weight of unit length of the rod. Here the limits of x are from 0 to l.

$$\therefore \quad \bar{x}=\frac{\int xdm}{\int dm}=\frac{\int_0^l xdx}{\int_0^l dx}=\frac{\left[\frac{x^2}{2}\right]_0^l}{[x]_0^l}=\frac{\frac{l^2}{2}}{l}=\frac{l}{2}.$$

Evidently $\bar{y}=0$. Hence the centre of gravity of a uniform rod is at a point of the rod at a distance $\frac{l}{2}$ from A, *i.e.*, the centre of gravity of a uniform rod is the middle point of the rod.

(B) Centre of gravity of a lamina having an axis of symmetry.

A straight line in the plane of a lamina is said to be an axis of symmetry of the lamina, if corresponding to every point of the lamina on one side of the straight line, there exists another point of the lamina on the other side of the straight line at an equal distance. If the lamina is folded along an axis of symmetry, then the two portions of the lamina will completely coincide.

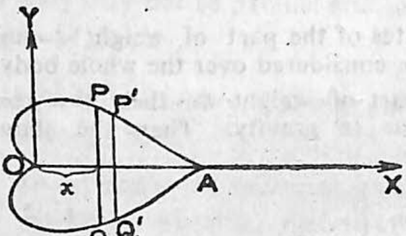


Fig. 73

Let $\overrightarrow{OA}$ be an axis of symmetry of a lamina. Let us take $\overrightarrow{OA}$ as the x-axis and a straight line perpendicular to the axis at the point O as the y-axis. Let PQQ'P' be an element of area of the lamina at a distance x from the y-axis and dx be the width of this element of area. If the co-ordinates of P be (x, y) then the length of this element of area is $PQ = 2y$.

Hence if δm be the mass of unit area of the lamina, then the mass of the element of area $= \delta m = 2y\delta x$.

Now, since the lamina is symmetrical about the x-axis, so the middle point of $\overline{PQ}$ is on the x-axis. Hence the co-ordinates of the centre of gravity of the element of area PQQ'P' will be $(x, 0)$.

$\therefore \quad \eta = 0.$

$$\therefore \quad \bar{y} = \frac{\int \eta dm}{\int dm} = \frac{\int 0.dm}{\int dm} = 0.$$

Hence the centre of gravity of the lamina will be on $\overrightarrow{XO}$. Hence the center of gravity of a lamina having an axis of symmetry lies on the axis of symmetry.

Note : (1) The centre of gravity of a uniform elliptic lamina is the centre of the ellipse. For, the major and minor axes of an ellipse are its two axes of symmetry and the centre of gravity of the lamina will be on each of these axes of symmetry and so is the point of intersection of these two axes.

(2) Similarly the centre of gravity of a uniform lamina in the form of a square is the point of intersection of the diagonals of the square, since the diagonals are two axes of symmetry of a square.

(3) The centre of gravity of a uniform circular lamina or a uniform circular wire is the centre of the circle. For, every diameter of a circle is an axis of symmerty of the circle.

(C) Centre of gravity of a uniform triangular lamina :

Let ABC be a uniform triangular lamina. Divide the lamina into a number of thin strips parallel to $\overline{BC}$. PQQ'P' is one such strip. Let D be the middle point of $\overline{BC}$ and $\overline{AD}$ intersect $\overline{PQ}$ at D_1. Now, as $\overline{PQ} \parallel \overline{BC}$ and D is the middle point of $\overline{BC}$, so D_1 is the middle point of $\overline{PQ}$. The strip $\overline{PQQ'P'}$ can be considered as a thin uniform rod and so, D_1 is the centre of gravity of the strip PQQ'P'. In this way you can see that the centre of gravity of every strip lies on $\overline{AD}$. So the centre of gravity of the lamina will lie on $\overline{AD}$. Similarly by dividing the lamina into a number of thin strips parallel to $\overline{CA}$, it can be shown that the centre of gravity of the lamina will lie on the median $\overline{BE}$ of the triangle. Hence the centre of gravity of the lamina is the point of intersection G of the medians $\overline{AD}$ and $\overline{BE}$, which is the centroid of the traingle ABC. Hence the centre of gravity of a thin uniform lamina is the point of intersection of its medians, i.e. the centroid of the lamina.

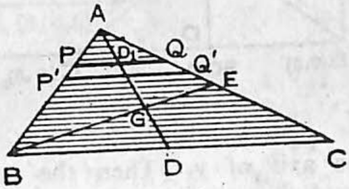


Fig. 74

Note : We know that the point of intersection of the medians divide the medians internally in the ratio 2 : 1.

Corollary 1. If three equal weights are placed at the three vertices of a uniform triangular lamina, then the centre of gravity of the three weights and the lamina will be the same point.

Proof : Let ABC be a uniform triangular lamina and three equal weights be placed at the vertices of the triangle and each weight be w. Now, the resultant of the weights w and w at B and C is a weight $2w$ acting at D, the middle point of $\overline{BC}$. Now, the resultant of the weight w at A and the weight $2w$ at D acts at the point G on $\overline{AD}$ such that $AG : DG = 2 : 1$. Hence G is the centroid of the triangle. Hence the centre of gravity of the three weights coincide with the centre of gravity of the lamina.

Note : In problems relating to centroid of triangular lamina of weight w, the weight w can be replaced by three weights $\frac{w}{3}, \frac{w}{3}, \frac{w}{3}$ acting at the vertices of the triangular lamina.

Ex. 1. The length of the base $\overline{BC}$ of the isosceles triangle ABC is 8 cms. and the other two sides are each of length 5 cms. Weights 3, 4 and 5 grammes are respectively placed at A, B and C. Find the centre of gravity of the three weights.

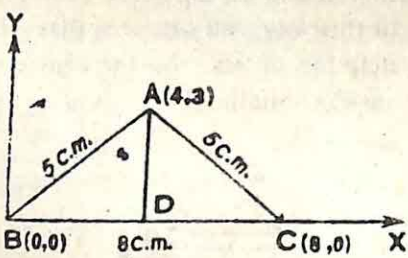


Fig. 75

Let the point B be the origin, $\overrightarrow{BC}$, the x-axis and a straight line perpendicular to $\overline{BC}$ be the axis of y. Then the co-ordinates of the vertices of the triangle are B(0, 0), C(8, 0) and A(4, 3).

[In the figure $AD = \sqrt{AB^2 - BD^2} = \sqrt{5^2 - 4^2} = 3$]

Hence if G $(\bar{x}, \bar{y})$ be the required centroid, then

$$\bar{x} = \frac{3.4 + 4.0 + 5.8}{3 + 4 + 5} = \frac{52}{12} = \frac{13}{3}.$$

$$\bar{y} = \frac{3.3 + 4.0 + 5.0}{3 + 4 + 5} = \frac{9}{12} = \frac{3}{4}.$$

Hence taking $\overrightarrow{BX}$ and $\overrightarrow{BY}$ as the axes of co-ordinates the co-ordinates of the centroid are $(\frac{13}{3}, \frac{3}{4})$, unit length being one centimetre.

Ex. 2. The weight of a uniform square lamina ABCD is 10 pounds ; weights 20 pounds, 30 pounds, 40 pounds and 50 pounds are placed at the vertices A, B, C, D respectively. Find the centre of gravity. [C. U. 1945]

Let the length of each side of the square be a. The weight of a square lamina acts at its centre O. Now let us take A as the origin and $\overrightarrow{AB}$ and $\overrightarrow{AD}$ as the axes of x and y respectively. Hence the co-ordinates of the point A, B, C, D and O are $(0, 0)$, $(a, 0)$, (a, a), $(0, a)$ and $(\frac{a}{2}, \frac{a}{2})$ respectively.

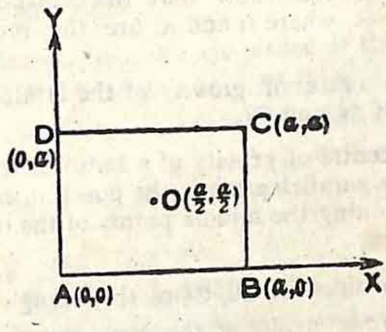


Fig. 76

Let $(\bar{x}, \bar{y})$ be the co-ordinates of the centroid. Then

$$\bar{x}=\frac{20\times 0+30\times a+40\times a+50\times 0+10\times \frac{a}{2}}{20+30+40+50+10}=\frac{75a}{150}=\frac{a}{2}.$$

$$\bar{y}=\frac{20\times 0+30\times 0+40\times a+50\times a+10\times \frac{a}{2}}{20+30+40+50+10}=\frac{95a}{150}=\frac{19}{30}a.$$

Hence the required centroid is at distances $\frac{19}{30}a$ and $\frac{a}{2}$ from $\overline{AB}$ and $\overline{AD}$ respectively and evidently the point is within the square.

Ex. 3. Prove that the centre of gravity of a uniform lamina in the form of a parallelogram is the point of intersection of the straight lines joining the middle points of opposite sides of the parallelogram.

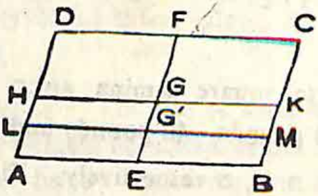


Fig. 78

ABCD is a uniform lamina in the form of a parallelogram. Divide the lamina into a number of thin strips parallel to $\overline{AB}$.

Let $\overline{LM}$ be one such strip and its centre of gravity is its middle point G.

If now E and F be the mid points of $\overline{AB}$ and $\overline{CD}$ respectively, then $\overline{EF}$ bisects all strips parallel to $\overline{AB}$ and so G′ is a point of $\overline{EF}$. So, the centre of gravity of each strip parallel to $\overline{AB}$ lies on $\overline{EF}$ and so the centre of gravity of the lamina lies on $\overline{EF}$.

Similarly if the lamina is divided into thin strips parallel to $\overline{BC}$, then one can show that the centre of gravity of the lamina lies on $\overline{HK}$, where H and K are the mid points of $\overline{AD}$ and $\overline{BC}$.

Hence the centre of gravity of the lamina is the point of intersection G of $\overline{EF}$ and $\overline{HK}$.

Hence the centre of gravity of a uniform plane lamina in the form of a parallelogram is the point of intersection of the straight lines joining the middle points of the opposite sides of the parallelogram.

Ex. 4. The sides $\overline{AB}$, $\overline{BC}$, $\overline{CA}$ of the triangle ABC are three thin uniform rods made of the same material and are of the same cross-section and D, E, F are respectively middle points of $\overline{BC}$, $\overline{CA}$ and $\overline{AB}$. Prove that the centre of gravity of the three rods is the in-centre of the triangle DEF.

The weights of the rods $\overline{AB}$ and $\overline{AC}$ act at the middle points F and E of the rods, vertically downwards and are proportional to the lengths of the rods. The resultant of these two parallel forces acts at a point H of $\overline{EF}$ and

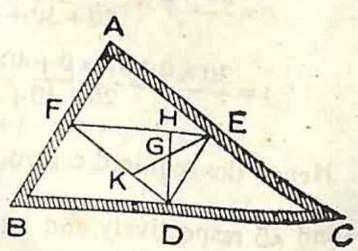


Fig 79

$$\frac{FH}{HE}=\frac{\text{the weight acting at } E}{\text{the weight acting at } F}=\frac{\text{length of the rod } \overline{AC}}{\text{length of the rod } \overline{AB}}$$

$$=\frac{DF}{DE} \quad [\because \ DF=\tfrac{1}{2}\,AC \text{ and } DE=\tfrac{1}{2}\,AB]$$

Hence $\overrightarrow{DH}$ is the bisector of $\angle EDF$.

Again the resultant of the weight of the uniform rod $\overline{BC}$ acting at the point D and the resultant weight of the rods $\overline{AB}$ and $\overline{AC}$ will act at some point of $\overline{DH}$. Hence the centre of of gravity of the three uniform rods is a point of $\overline{DH}$.

Similarly, first considering the resultant weight of the rods $\overline{AB}$ and $\overline{BC}$ and then determining the resultant of the three rods it will be found that the centre of gravity of the three rods is a point of EK, the internal bisector of the angle DEF.

Hence the centre of gravity is the point of intersection of the internal bisectors of the angles EDF and DEF i.e., the incentre of the triangle DEF.

Ex. 5. Weights W_1, W_2, W_3 are placed at the vertices A, B and C respectively of a triangle ABC. Weights m_1, m_2, m_3 are placed at D, E, F, the middle points of $\overline{BC}$, $\overline{CA}$ and $\overline{AB}$ respectively. If the centre of gravity of W_1, W_2 and W_3 be the same point as the centre of gravity of the weights m_1, m_2 and m_3, prove that

$$\frac{W_1}{m_2+m_3}=\frac{W_2}{m_3+m_1}=\frac{W_3}{m_1+m_2}.$$

Let G be the centre of gravity in both cases and $\overline{GL}$ and $\overline{AM}$ be perpendicular on $\overline{BC}$.

Now, the weights m_1, m_2, m_3 at D, E and F respectively are equivalent to weights $\frac{m_1}{2}$ at B, $\frac{m_1}{2}$ at C ; $\frac{m_2}{2}$ at C, $\frac{m_2}{2}$ at A and $\frac{m_3}{2}$ at A, $\frac{m_3}{2}$ at B. *i.e.*, weights $\frac{m_2+m_3}{2}$ at A, $\frac{m_3+m_1}{2}$ at B and $\frac{m_1+m_2}{2}$ at C respectively.

Now, taking moment about $\overline{BC}$ we get

$(W_1+W_2+W_3).GL=W_1.AM.$

and $(m_1+m_2+m_3)\ \mathrm{GL}=\frac{1}{2}(m_2+m_2).\mathrm{AM}$.

$$\therefore \quad \frac{W_1}{\frac{1}{2}(m_2+m_3)}=\frac{W_1+W_2+W_3}{m_1+m_2+m_3}=K \text{ (say)}$$

Similarly, taking moments about $\overline{CA}$ and $\overline{AB}$ one can prove that

$$\frac{W_2}{\frac{1}{2}(m_3+m_1)}=\frac{W_3}{\frac{1}{2}(m_1+m_2)}=K.$$

$$\therefore \quad \frac{W_1}{m_2+m_3}=\frac{W_2}{m_3+m_1}=\frac{W_3}{m_1+m_2}.$$

Ex. 6. A thin uniform wire is bent in the form of a triangle. a, b, c are the lengths of the sides of the triangle and weights $\frac{b+c}{2}$, $\frac{c+a}{2}$, and $\frac{a+b}{2}$ are placed in the corresponding opposite vertices. Show that the centre of gravity of the triangular wire coincides with the centre of gravity of the three weights.

Let ABC be the triangle and w be the weight per unit length of the wire. Hence weights aw, bw and cw act at the middle points D, E, F of $\overline{BC}$, $\overline{CA}$ and $\overline{AB}$ respectively.

Now, the weight aw acting at D, is equivalent to weights $\frac{aw}{2}$ at B and $\frac{aw}{2}$ at C. The weights bw and cw acting at the points E and F are respectively equivalent to weights $\frac{bw}{2}$ and $\frac{bw}{2}$ acting at the points C and A and weights $\frac{cw}{2}, \frac{cw}{2}$ acting at the points A and B. Hence the centre of gravity of the triangular wire is the centre of gravity of weights $\frac{b+c}{2}w$, $\frac{c+a}{2}w$ and $\frac{a+b}{2}w$ acting at the points A, B and C respectively. Again, the centre of gravity of these weights is the centre of gravity of weights $\frac{b+c}{2}$, $\frac{c+a}{2}$, $\frac{a+b}{2}$ acting at the points A, B, C respectively. Hence the centre of the gravity of the triangular wire and the centre of gravity of the weights $\frac{b+c}{2}$; $\frac{c+a}{2}$, $\frac{a+b}{2}$ placed at A; B, C are the same point.

Ex. 7. Find the centre of gravity of a uniform semicircular lamina of radius a.

ABC is a uniform semicircular lamina and its diameter is $\overline{AB}$, centre is O. $\therefore$ OA=OB=a.

Let $\overrightarrow{OY}$ be perpendicular on $\overline{AB}$ at O intersecting the circumference of the lamina at C. Take O as the origin, $\overrightarrow{OA}$ as the positive direction of the x-axis and $\overrightarrow{OY}$ as the positive direction of the y-axis. Hence the equation of the circumference of the lamina is $x^2+y^2=a^2$.

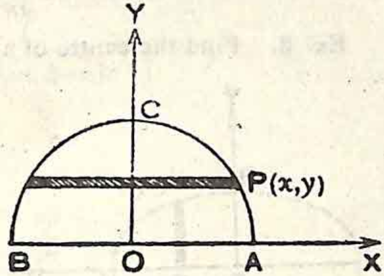


Fig. 80

Let the centre of gravity of the lamina be the point $(\bar{x}, \bar{y})$. Since the semicircle is symmetrical about $\overline{OC}$, *i.e.*, the y-axis, so the centre of gravity is situated on the y-axis. $\therefore$ $\bar{x}=0$. Now divide the lamina into a number of thin strips parallel to $\overline{AB}$. Let P (x, y) be a point on the circumference of the lamina and PQ be the strip through P. Let the width of the thin strip be δy. So its area is $2x\delta y$ and the centre of gravity is its mid point $(0, y)$. Now for the semicircle the limits of y are from 0 to a.

$$\therefore \quad \bar{y}=\frac{\int_0^a y.2xdy}{\int_0^a 2xdy}=\frac{\int_0^a y\sqrt{a^2-y^2}\,dy}{\int_0^a \sqrt{a^2-y^2}\,dy}$$

$$=\frac{a^3\int_0^{\frac{\pi}{2}} \sin\theta\cos^2\theta\,d\theta}{a^2\int_0^{\frac{\pi}{2}} \cos^2\theta\,d\theta} \quad [\text{putting } y=a\sin\theta]$$

$$=a.\frac{\frac{1}{3}}{\frac{1}{2}.\frac{\pi}{2}}=\frac{4a}{3\pi}.$$

Hence the co-ordinates of the centre of gravity of the semi-circular lamina is the point $\left(0, \frac{4a}{3\pi}\right)$ *i.e.*, the centre of gravity is the point on $\overline{OC}$ at a distance $\frac{4a}{3\pi}$ from O.

Ex. 8. Find the centre of a gravity of a lamina in the form of the quadrant of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

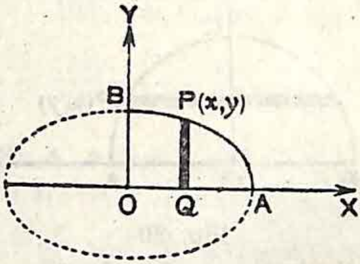


Fig. 81

Let the uniform lamina be the first quadrant AOB of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Divide the uniform lamina into a number of thin strips parallel to the y-axis and let PQ be one such strip and the co-ordinates of P be (x, y). The area of this strip is $y.\delta x$ and its C.G. is its middle point $\left(x, \frac{y}{2}\right)$. The limits of x in this quadrant is from 0 to a.

Hence if the centre of gravity of the lamina be G $(\bar{x}, \bar{y})$, then,

$$\bar{x}=\frac{\int_0^a x.y\, dx\, \rho}{\int_0^a y\, dx\, \rho} \quad [\rho=\text{mass of unit area of the lamina}]$$

$$=\frac{\int_0^a xy\, dx}{\int_0^a y\, dx}=\frac{\int_0^a x.\frac{b}{a}\sqrt{a^2-x^2}\, dx}{\int_0^a \frac{b}{a}\sqrt{a^2-x^2}\, dx}$$

$$=\frac{a^2\int_0^{\frac{\pi}{2}} \sin\theta \cos^2\theta\, d\theta}{a\int_0^{\frac{\pi}{2}} \cos^2\theta\, d\theta} \quad [\text{Putting } x=a\sin\theta]$$

$$=a.\frac{\frac{1}{3}}{\frac{1}{2}\cdot\frac{\pi}{2}}=\frac{4a}{3\pi}.$$

$$\bar{y}=\frac{\int_0^a \frac{y}{2}.y\,dx\,\rho}{\int_0^a y\,dx\,\rho}=\frac{1}{2}\frac{\int_0^a \frac{b^2}{a^2}(a^2-x^2)dx}{\int_0^a \frac{b}{a}\sqrt{a^2-x^2}\,dx}$$

$$=\frac{1}{2}\frac{b}{a}\frac{\left[a^2x-\frac{x^3}{3}\right]_0^a}{\int_0^{\frac{\pi}{2}} a^2\cos^2\theta\,d\theta} \qquad [\text{Let } x=\sin\theta]$$

$$=\frac{1}{2}\frac{b}{a^3}\frac{\frac{2}{3}a^3}{\int_0^{\frac{\pi}{2}}\cos^2\theta\,d\theta}=\frac{b}{3.\frac{\pi}{4}}=\frac{4b}{3\pi}.$$

Ex. 9. Find the centre of gravity of a uniform lamina in the form of a sector of a circle.

Let AOBCA be a uniform lamina in the form of a sector of a circle whose centre is O and radius a. Also let 2α be the measure of the angle subtended by the sector at the centre of the circle, *i.e.*, $m\angle AOB=2\alpha$ (in radians).

Let $\overrightarrow{OC}$ be the bisector of $\angle AOB$. Take O as the origin, $\overrightarrow{OC}$ as the x-axis and the straight line perpendicular to $\overrightarrow{OC}$ at O as the y-axis.

As $\overrightarrow{OC}$ is the bisector of $\angle AOB$, so the sector is symmetrical about $\overrightarrow{OC}$ *i.e.*, the x-axis. So the centre of gravity of the sector is situated on the x-axis.

$\therefore$ If G $(\bar{x}, \bar{y})$ be the centre of gravity of the sector, then $\bar{y}=0$.

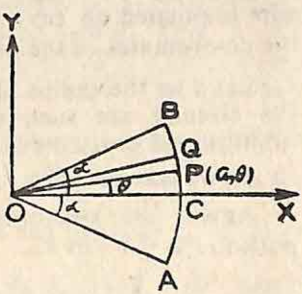


Fig. 82

Now, divide the sector of the circle into a number of thin uniform triangular strips through O. Let POQ be one such strip. Now, if $m\angle POX=\theta$ (in radians), then the area of the strip POQ $=\frac{1}{2}a^2\sin\delta\theta=\frac{1}{2}a^2\delta\theta$ [$\because$ $\delta\theta$ is small, we can take $\sin\delta\theta=\delta\theta$] and the co-ordinates of its centre of gravity is $\left(\frac{2a}{3}\cos\theta, \frac{2a}{3}\sin\theta\right)$.

10. A uniform lamina is in the form of a semi-ellipse, obtained by cutting an ellipse along the minor axis. Find the centre of gravity of the semi-elliptic lamina if the lengths of the major and minor axes of the ellipse be $2a$ and $2b$. [C.U. 1964]

11. Find the centre of gravity of a thin uniform lamina in the form of a parabola bounded by a double ordinate of length 8. [C. U. 1963]

12. Find the centre of gravity of the area bounded by the x-axis, the straight line $x=h$ and the parabola $y^2=4ax$.

13. The length of a uniform rod AB is a and its density at any point is proportional to the square of its distance from the end A. Find the C.G. of the uniform rod.

14. Find the C.G of the area enclosed by the curve $y=\sin x$ and the x-axis.

15. Find the C.G of the area enclosed by the parabola $y^2=x$, the x-axis and the ordinates $x=2$ and $x=3$.

16. A thin uniform wire is bent into an arc of radius 5 cms. If the distance between the ends of the wire be 8 cms, find the position of its C.G.

§ 7.6. Centre of gravity of the join of two bodies.

Let the weights of two bodies ADC and ABC be W_1 and W_2 and their centres of gravity be G_1 and G_2. We are to determine the centre of gravity of the composite body formed by these two bodies. Let G be the centre of gravity of this composite body. Hence G is the point of application of the resultant of the parallel forces W_1 and W_2 acting at the points G_1 and G_2 respectively.

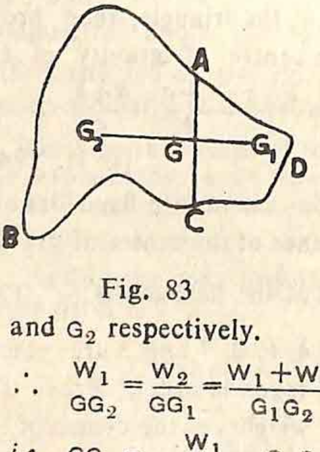


Fig. 83

$$\therefore \quad \frac{W_1}{GG_2}=\frac{W_2}{GG_1}=\frac{W_1+W_2}{G_1G_2}$$

i.e., $GG_2=\dfrac{W_1}{W_1+W_2}G_1G_2$ and $GG_1=\dfrac{W_2}{W_1+W_2}G_1G_2$.

Now, W_1, W_2 and G_1, G_2 are known. Hence the position of G can now be easily determined.

Note. If O be a point on $\overline{G_1G_2}$ such that the distances OG_1 and OG_2 are known, then taking moments about O we get

$$W_1.OG_1+W_2.OG_2=W.OG$$

$$\text{or,}\quad OG=\frac{W_1.OG_1+W_2.OG_2}{W_1+W_2}.$$

If with reference to a suitably chosen axes of reference, (x_1, y_1) (x_2, y_2) and $(\bar{x}, \bar{y})$ be respectively the co-ordinates of G_1, G_2 and G, then,

$$\bar{x}=\frac{W_1.x_1+W_2.x_2}{W_1+W_2};\quad \bar{y}=\frac{W_1.y_1+W_2.y_2}{W_1+W_2}.$$

§ 7·7. Centre of gravity of a cut off body.

Let W be the weight and G be the centre of gravity of a body and a portion of the body of weight W_1 whose centre of gravity is G_1 is cut off from the body. We are to determine the centre of gravity of the remaining portion of the body. Let G_2 be the centre of gravity of this remaining portion. The weight of this remaining portion is $W-W_1$. Hence W is the resultant of the two weights W_1 and $W-W_1$ acting at the points G_1 and G_2 respectively. Hence G is a point on the line segment $\overline{G_1G_2}$, between G_1 and G_2.

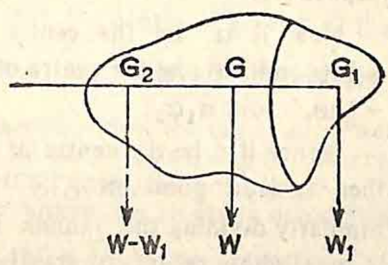


Fig. 84

$$\text{Hence,}\quad \frac{W-W_1}{GG_1}=\frac{W_1}{GG_2}=\frac{W}{G_1G_2},\quad\therefore\quad GG_2=\frac{W}{W-W_1}GG_1.$$

Now, W, W_1 and GG_1 are known. Hence GG_2 and so the position of G_2 can be determined.

Note. 1. If O be a point on $\overline{GG_1}$, then taking moments about O we get

$$W.OG=W_1.OG_1+(W-W_1).OG_2\quad\therefore\quad OG_2=\frac{W.OG-W_1.OG_1}{W-W_1}.$$

If (x_1, y_1), (x_2, y_2) and $(\bar{x}, \bar{y})$ be the co-ordinates of G_1, G and G_2 respectively,

then $\bar{x} = \frac{W.x_2 - W_1 x_1}{W - W_1}$ and $\bar{y} = \frac{W.y_2 - W_1 y_1}{W - W_1}$.

Example. 1. Determine the centre of gravity of uniform lamina in the form of a quadrilateral.

ABCD is a uniform lamina in the form of a quadrilateral. We are to determine the centre of gravity of the lamina.

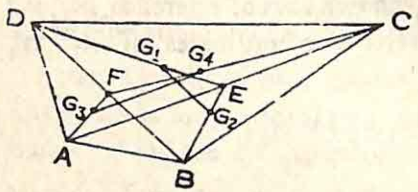


Fig. 85

Join AC and let the mid point of $\overline{AC}$ be E.

Join BE and DE.

Let W_1 and W_2 be the weights of the two triangular laminas $\triangle$ADC and $\triangle$ABC respectively.

Now if G_1 be the centre of gravity of $\triangle$ADC, then $EG_1 = \frac{1}{3}ED$ and if G_2 be the centre of gravity of $\triangle$ABC, then $EG_2 = \frac{1}{3}EB$. Join G_1G_2.

Hence if G be the centre of gravity of the whole lamina, then G is a point on $\overline{G_1G_2}$ such that $G_1G : G_2G = W_2 : W_1$. Similarly dividing the lamina into two triangles $\triangle$ABD and $\triangle$CBD, whose centres of gravity are G_3 and G_4, we can show that the centre of gravity G of the whole lamina lies on $\overline{G_3G_4}$ and $AG_3 = \frac{2}{3}AF$, $CG_4 = \frac{2}{3}CF$, where F is the middle point of $\overline{BD}$.

Hence the centre of gravity is the point of intersection of $\overline{G_1G_2}$ and $\overline{G_3G_4}$ where the positions of the points G_1, G_2, G_3, G_4 are known.

Alternative method. Let the weights of $\triangle$ACD and $\triangle$ACB be W_1 and W_2 respectively.

Now the centre of gravity of $\triangle$ACD is the same as that of weights $\frac{W_1}{3}$, $\frac{W_1}{3}$ and $\frac{W_1}{3}$ placed at the vertices of the triangle. Also the centre of gravity of the $\triangle$ACB is the same as that of weights $\frac{W_2}{3}$, $\frac{W_2}{3}$ and $\frac{W_2}{3}$ placed at the vertices of the triangle.

So, the centre of gravity of the complete lamina is the centre of gravity of weights $\frac{W_1+W_2}{3}$, $\frac{W_1+W_2}{3}$, $\frac{W_1}{3}$ and $\frac{W_2}{3}$ placed at the vertices, A, C, D and B respectively.

Join BD and divide $\overline{DB}$ in the ratio $W_2 : W_1$. So, the weights $\frac{W_1}{3}$ and $\frac{W_2}{3}$ acting at D and B respectively are equivalent

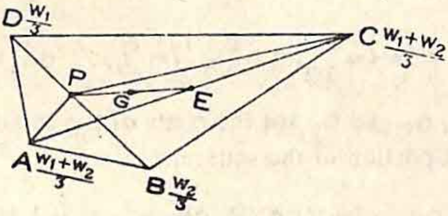


Fig. 86

to the weight $\frac{W_1+W_2}{3}$ at the point P. Hence the total weight of the lamina is equivalent to weights $\frac{W_1+W_2}{3}$ placed at each vertex of the triangle APC. But, the centre of gravity of these equal weights $\frac{W_1+W_2}{3}$ placed at the vertices of the triangle is the centre of gravity of the $\triangle$APC. Now the centre of gravity of $\triangle$APC is the point G on $\overline{PE}$ such that $PG = \frac{2}{3}PE$. Hence P and E (the mid pt. of $\overline{AC}$) being known, the position of G, the centre of gravity of the whole lamina can be easily determined.

Ex. 2. ABCD is a square and E and F are respectively the middle points of $\overline{AB}$ and $\overline{AD}$. If $\triangle$AEF is cut off from the square, determine the position of the centre of gravity of the remaining part.

Let G be the C.G of the whole square. Hence G is the middle point of $\overline{AC}$. Now if $\overline{AC}$ intersects $\overline{EF}$ at the point H, then H is the middle point of $\overline{EF}$ and $\overline{AG}$.

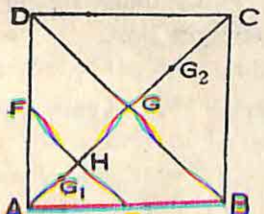


Fig. 87

So, the C.G, G_1, of $\triangle$AEF is such a point on $\overline{AH}$ that $AG_1 = \frac{2}{3}AH$. Hence if G_2 be the centre of gravity of the remaining portion, then G_2 is a point on $\overline{AC}$. Now let W be the weight of the square and l be the length of each diagonal.

Hence the area of the square $=\frac{1}{2}l^2$ and $m\triangle AEF=\frac{1}{2}\,AH.EF$ $=\frac{1}{2}.\frac{1}{2}\,AG.\,\frac{1}{2}\,BD=\frac{1}{8}\,AG.BD=\frac{1}{8}.\frac{1}{2}\,AC.BD\frac{1}{16}l^2$.

Hence, $\dfrac{\text{area of the square ABCD}}{\text{area of }\triangle AEF}=\dfrac{\frac{1}{2}l^2}{\frac{1}{16}l^2}=\dfrac{8}{1}$

$\therefore$ Weight of $\triangle AEF=\dfrac{W}{8}$ and the weight of the remaining portion $=\frac{7}{8}W$.

Now, $AG_1=\dfrac{2}{3}AH=\dfrac{2}{3}\cdot\dfrac{1}{2}AG=\dfrac{2}{3}\cdot\dfrac{1}{2}\cdot\dfrac{1}{2}l=\dfrac{1}{6}l$, $\therefore$ $GG_1=\frac{1}{2}l-\frac{1}{6}l=\frac{1}{3}l$.

Now as G, G_1 and G_2 are the C.G's of the square, $\triangle AEF$ and the remaining portion of the square,

$\therefore$ $\dfrac{7W}{8}\times GG_2=\frac{1}{8}W\times GG_1$ $\therefore$ $GG_2=\frac{1}{7}GG_1=\frac{1}{7}.\frac{1}{3}l=\frac{1}{21}l=\frac{1}{21}AC$.

Hence the C.G of the remaining portion is a point G_2 of $\overline{AC}$ and it is at a distance $\frac{1}{21}AC$ from the centre of gravity of the square.

Ex. 3. ABC is a thin triangular lamina. Determine a point P of the triangle such that if the portion $\triangle PBC$ is removed from the whole triangle, then P will be the C.G of the remaining portion of the triangle.

Let G and G_1 be the C.G's of $\triangle ABC$ and $\triangle PBC$ respectively. Now if D be the middle point of $\overline{BC}$, then G and G_1 are points of $\overline{AD}$ and $\overline{PD}$ respectively.

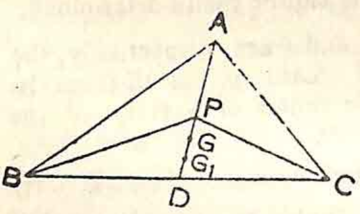


Fig. 88

Now, if P be the centre of gravity of the remaining portion then G, G_1 and P must lie on the same straight line. $\therefore$ P must be a point on $\overline{AD}$.

Now as G_1 is the C.G. of $\triangle PBC$, $\therefore$ $G_1D=\frac{1}{3}PD$, Again $GD=\frac{1}{3}AD$.

Let $PD=x$ and $AD=l$.

Let h and h_1 be the altitudes of $\triangle ABC$ and $\triangle PBC$ and W and W_1 be their respective weights.

Hence the weight of the remaining portion $=W_2=W-W_1$.

Now if w be the weight of unit area of the lamina, then

$W=\frac{1}{2}BC.\,h.\,w$; $W_1=\frac{1}{2}BC.\,h_1.w$ and $W_2=\frac{1}{2}BC(h-h_1)w$.

Now as G, G_1 and P are respectively the C.G's of $\triangle ABC$, $\triangle PBC$ and the remaining portion,

$\therefore\ W_1.G_1G=W_2.PG.$

or, $\dfrac{W_1}{W_2}=\dfrac{PG}{G_1G}$ or, $\dfrac{h_1}{h-h_1}=\dfrac{PG}{GG_1}$

or, $h_1GG_1=(h-h_1)PG$ or $h_1\ (GG_1+PG)=h.PG$

or, $\dfrac{PG}{PG}=\dfrac{h}{h_1}$; Now, $\dfrac{AD}{PD}=\dfrac{h}{h_1}=\dfrac{PG_1}{PG}=\dfrac{\frac{2}{3}PD}{PD-\frac{1}{3}AD}=\dfrac{2PD}{3PD-AD}$,

or, $\dfrac{l}{x}=\dfrac{2x}{3x-l}$ or, $3xl-l^2=2x^2$,

or, $2x^2-3xl+l^2=0$, or, $(2x-l)(x-l)=0$

$\therefore\ x=\frac{1}{2}l$ or, l. But $x\neq l$.

$\therefore\ x=\frac{1}{2}l$ *i.e.*, the point P is the mid point of $\overline{AD}$.

Ex. 4. The radius of a uniform circular lamina is r and from it a circle drawn on a radius as diameter is removed. Determine the C. G. of the remaining portion.

The radius of the portion removed is $\frac{r}{2}$.

Hence the area of the given circle is πr^2, the area of the removed portion is $\frac{\pi r^2}{4}$ and that of the remaining portion is $\frac{3}{4}\pi r^2$. Hence if W be the weight of the whole circular lamina, then $\frac{W}{4}$ and $\frac{3}{4}W$ are respectively the weights of the removed portion and the remaining portion.

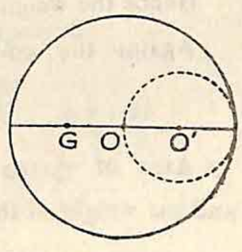


Fig. 89

The centres of gravity of the whole circular lamina and the removed portion are their centres O and O′ respectively. Now if G be the centre of gravity of the remaining portion, then G, O and O′ will lie in the same straight line and $\frac{3}{4}$ W. OG$=\frac{1}{4}$ W. OO′$=\frac{1}{4}$ W. $\frac{r}{2}$. $\therefore$ OG$=\frac{r}{6}$.

Ex. 5. Find the eentre of gravity of a uniform lamina in the form of a segment of a circle.

Let r be the radius of a circle whose centre is O. To determine the centre of gravity of a uniform lamina in the form of a segment ACBLA of this circle.

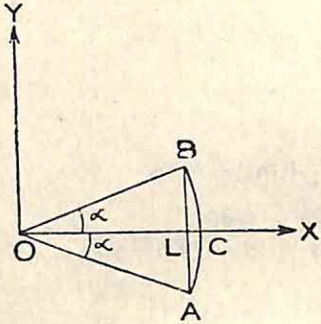


Fig. 90

The uniform lamina can be regarded as obtained by removing the uniform lamina in the form of $\triangle$AOB from the uniform sector of the circular lamina ACBOA.

Let now $m\angle AOB = 2\alpha$ and the weight of unit area of the uniform lamina be W. Let $\overline{OL}$ be perpendicular on AB and let OL intersect the arc of the segment at the point C. Take the centre O of the circle as the origin, $\overrightarrow{OC}$ as the x-axis and the perpendicular $\overrightarrow{OY}$ to the x-axis at the point O as the y-axis. As the segment of the circle is symmetrical about $\overline{OC}$, the y-axis, so the y-co-ordinate of the C.G of the segment of the circle is 0.

Now area of the sector OACBO $=\pi r^2 \dfrac{2\alpha}{2\pi} = r^2\alpha$.

Hence the weight of the sector $= r^2\alpha.W$.

Again, the co-ordinates of the C.G of the sector is

$$\left(\frac{2}{3}r\frac{\sin\alpha}{\alpha}, 0\right) \qquad \text{[See Ex. 7. § 6·5]}$$

Area of $\triangle OAB = \frac{1}{2}OL.AB = \frac{1}{2}r\cos\alpha.2r\sin\alpha = r^2\sin\alpha\cos\alpha$ and the weight of this portion $= r^2\sin\alpha\cos\alpha.W$.

Again the co-ordinates of the C.G of $\triangle$OAB are $\left(\frac{2}{3}r\cos\alpha, 0\right)$

Hence if $(\bar{x}, \bar{y})$ be the C.G of the segment of the circle, then

$$\bar{x} = \frac{r^2\alpha.\frac{2}{3}.r\frac{\sin\alpha}{\alpha}.W - r^2\sin\alpha\cos\alpha.\frac{2}{3}r\cos\alpha.W}{r^2\alpha.W - r^2\sin\alpha\cos\alpha.W}$$

$$=\frac{2}{3}r\frac{\sin\alpha-\sin\alpha\cos^2\alpha}{\alpha-\sin\alpha\cos\alpha}=\frac{2}{3}r\frac{\sin\alpha(1-\cos^2\alpha)}{\alpha-\sin\alpha\cos\alpha}$$

$$=\frac{2}{3}r\frac{\sin^3\alpha}{\alpha-\sin\alpha\cos\alpha}.$$

Again, it is evident that $\bar{y}=0$.

Hence the required co-ordinates of the C.G are

$$\left(\frac{2}{3}r\frac{\sin^3\alpha}{\alpha-\sin\alpha\cos\alpha}, 0\right);$$

Ex. 6. The lengths of the parallel sides of a trapezium are a and b and h is their distance. Show that the distance of the C.G of the trapezium from the side of length a is $\frac{h}{3}\cdot\frac{a+2b}{a+b}$.

Let ABCD be a given trapezium with $\overline{AB} \parallel \overline{DC}$, AB$=a$; CD$=b$ and the distance between $\overline{AB}$ and $\overline{DC}$ be h.

Now trapezium

ABCD $=\triangle$ABC $+ \triangle$ACD.

Let w be the weight of unit area of the trapezium.

Now, $m\triangle$ABC$=\frac{1}{2}$AB.$h=\frac{1}{2}ah$.

and $m\triangle$ACD$=\frac{1}{2}$CD.$h=\frac{1}{2}.bh$.

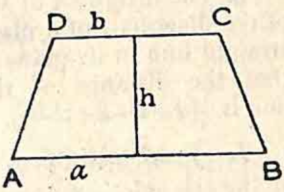


Fig. 91

Hence the weights of the two triangles are $\frac{1}{2}ah.\ w$ and $\frac{1}{2}bh.w$ respectively. Again, the distances from $\overline{AB}$ of the centres of gravity of $\triangle$ABC and $\triangle$ADC are $\frac{h}{3}$ and $\frac{2h}{3}$ respectively. Hence the distance from $\overline{AB}$ of the C.G of the trapezium

$$=\frac{\frac{1}{2}ah\times\frac{h}{3}w+\frac{1}{2}bh\times\frac{2h}{3}w}{\frac{1}{2}ahw+\frac{1}{2}bhw}=\frac{\frac{h^2w}{6}\ (a+2b)}{\frac{hw}{2}\ (a+b)}=\frac{h}{3}\cdot\frac{a+2b}{a+b}.$$

Exercise 7B

1. The diagonals of the square ABCD intersect at the point O. If $\triangle$AOB is removed from the square, then determine the position of the C.G of the remaining portion.

2. Two isosceles triangles ABC and DBC are on the opposite sides of the same base $\overline{BC}$. The heights of the triangle are h_1 and h_2 respectively. Determine the centre of gravity of the quadrilateral ABCD.

3. Remove a circular portion of radius 12 cm. from a unifrom circular lamina of radius 36 cm., so that the C.G of the remaining portion is at a distance 2 cms. from the centre of gravity of the circle.

4. A square hole is punched out of a circular lamina, the diagonal of the square being a radius of the circle. Show that the centre of gravity of the remainder is at a distance $\frac{a}{8\pi-4}$ from the centre of the circle, where a is its diameter.

[H. S. 1967]

5. AB and AC are two uniform rods of lengths $2a$ and $2b$ respectively. If $m\angle BAC=\theta$, prove that the distance from A of the C.G. of the two rods is

$$\frac{(a^4+2a^2b^2\cos\theta+b^4)^{\frac{1}{2}}}{a+b}$$ [C. U. 1939]

6. The distances of the vertices and the point of intersection of the diagonals of a plane uniform quadrilateral lamina from a straight line in its plane are a, b, c, d and e respectively. Show that the distance of the C.G. of the lamina from the straight line is $\frac{1}{3}(a+b+c+d-e)$.

7. From a uniform triangular lamina one-fourth portion of the triangle is removed by a straight line parallel to the base of the triangle. Determine the centre of gravity of the remaining portion.

8. From a uniform triangular lamina two triangular portions at the vertices B and C each of $\frac{1}{n}$th area of the whole triangle are removed by straight lines parallel to the opposite sides. Determine the centre of gravity of the remaining portion.

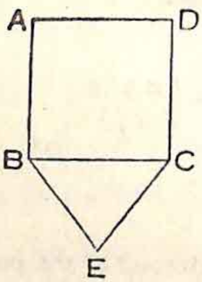


Fig. 92

9. In fig. 92. ABCD is a square, BCE is an isosceles triangle. Each side of the square is 12 cms. in length and the height of the triangle is 6 cms. Find the distance of the C.G of the whole area ABECDA from the straight line $\overline{AD}$.

10. The diagonals of the parallelogram ABCD intersect at O. ΔBOC is removed from the parallelogram. Find the centre of gravity of the remaining portion.

Miscellaneous Examples on Centre of Gravity

Ex. 1. Find the distances of the centre of gravity of a uniform lamina in the form of a right angled triangle from the sides containing the right angle.

Let ABC be the right angled triangle of weight W and B be the right angle. Now, the C.G of the lamina is the same as the C.G of three equal weight $\frac{W}{3}$, $\frac{W}{3}$, $\frac{W}{3}$ placed at the vertices. So, if G be the C.G of the lamina, and GL be its distance from BC, then taking moments about BC, we get $W.GL=\frac{W}{3}.AB$

$$\therefore \quad GL=\frac{AB}{3}.$$

So, the distance of the C.G from BC is $\frac{AB}{3}$.

Similarly the distance of the C.G from AB is $\frac{BC}{3}$.

Ex. 2. From a uniform circular lamina a circular portion of diameter $\frac{1}{3}r$ is punched out. If the circumference of this portion passes through the centre of the lamia, find the C.G of the remaining portion of the lamina.

The area of the whole lamina is πr^2 and that of the punched out portion is $\pi\left(\frac{r}{6}\right)^2=\frac{\pi r^2}{36}$.

So if the weight of the whole lamia be W, then the weight of the punched out portion is $\frac{W}{36}$ and that of the remaining portion is $\frac{35W}{36}$.

The C.G of the whole lamina is its centre O. If G be the C.G of the remaining portion, then G will lie on the line joining O with the centre of the punched out portion on the opposite side of O in which the centre of the punched out portion lies.

$$\therefore \quad \frac{35}{36}W.OG=\frac{W}{36}.\frac{r}{6} \quad \therefore \quad OG=\frac{r}{6\times 35}=\frac{r}{210}.$$

Ex. 3. From a uniform circular lamina of radius 3 ft. a circular portion of radius 1 ft. is to be punched out so that the

C.G of the remaining portion will be at a distance of 2 inches from the centre of the bigger lamina. Find the centre of the portion punched out.

Let O, be the centre of the given lamina, C that of the punched out portion and G be the C.G of the remaining portion. According to the problem, $OG = 2$ inches $= \frac{1}{6}$ ft. Now if W be the weight of the whole lamina, then that of the punched out portion $= \frac{\pi 1^2}{\pi 3^2} W = \frac{1}{9} W$. So, the weight of the remaining portion is $W - \frac{1}{9} W = \frac{8}{9} W$.

So taking moments about O, we get $\frac{8}{9} W.OG = \frac{1}{9} W.OC$

or, $8.OG = OC$

$\therefore$ $OC = 8.OG = 8.\frac{1}{6}$ ft. $= \frac{4}{3}$ ft.

$\therefore$ So, the centre of the punched out portion is at a distance $1\frac{1}{3}$ ft. from the centre of the whole lamina.

Ex. 4. ABCDEF is a regular hexagon. Five particles of equal weights have been placed at the angular points A, B, C, D and E. Find the C.G of the particles.

Let the weight of each particle be ω. Due to symmetry the C.G of the particles placed at A, B, D and E is situated at the centre O of the hexagon. Now if G be the C.G of the five particles then the resultant of the weights 4ω at O and ω at C act at G. So, $4\omega OG = \omega.GC = \omega(OC - OG)$, or, $5.OG = OC$

$\therefore$ $OG = \frac{1}{5} OC$.

Ex. 5. A thin uniform wire is bent into two coplanar circular rings of radii r and r', touching each other externally. Find the distance of its centre of gravity from the point of contact it $r' > r$. [H. S. 1978]

Let A and B be the centres of the co-planar circular rings of radii r and r' respectively touching each other externally at O. Now the C.G of the ring of radius r is its centre A and its weight is $2\pi r.\rho$, where ρ is the weight of unit length of the wire. The C.G of the ring of radius r' is its centre B and its weight is $2\pi r'.\rho$.

As $r' > r$, so the second ring is heavier than the first. So, the C.G. G of the composite wire is nearer to B. So, the resultant $2\pi(r + r')\rho g$ of the two parallel forces acting at A and

B respectively passes through G. So, taking moments about O we get, $2\pi . r\rho . g . OA - 2\pi r'\rho . g . OB = -2\pi(r+r')\rho . OG$

$$\therefore \quad OG = \frac{2\pi r'^2 - 2\pi r^2}{2\pi(r'+r)} \quad [\because \quad OA = r \text{ and } OB = r']$$

$$= \frac{r'^2 - r^2}{r'+r} = r' - r.$$

So, the C.G of the two rings is at a distance $(r'-r)$ from O towards B.

Ex. 6. A uniform lamina is made of a rectangle ABCD and an isosceles triangle CDE outside the rectangle. Find the distance of the C.G of the lamina from the side AB, where AB = 10 cms., AD = 4 cms. and CE = DE = 13 cms.

Let L and M be respectively the middle points of CD and AB respectively. Due to symmetry, the C.G of the lamina is situated on the line ELM. Let G_1 and G_2 respectively be the C.G's of the rectangle and the isosceles triangle.

$\therefore \quad MG_1 = 2$ cms. and $EL = \sqrt{13^2 - 5^2} = 12$ cms.

$\therefore \quad MG_2 = ML + \frac{1}{3}EL = 4 + 4 = 8$ cms.

Let $\bar{x}$ be the distance of the C.G of the lamina from AB.

Now the weight of the rectangle $= 10 \times 4$ sq. cms. $= 40$ sq.cms.

The weight of the isosceles triangle $= \frac{1}{2} \times 10 \times 12$ sq. cms.
$= 60$ sq. cms.

Hence if ρ be the weight of unit area of the lamina,

$$x = \frac{40.\rho \times 2 + 60.\rho \times 8}{40\rho + 60\rho} = \frac{80+480}{100} \text{ cms} = \frac{560}{100} \text{ cms} = 5{\cdot}6 \text{ cms.}$$

Ex. 7. Squares are constructed on each side of an isosceles right angled triangle. Find the C.G of the figure thus formed.

Let ABC be the isosceles right angled triagle and A be the right angle.

Let $AB = AC = a$, so $BC^2 = a^2 + a^2 = 2a^2$.

The squares constructed on each of AB and AC are equal and let area of each be W. Then the weight of the square on BC is 2W and that of the triangle ABC $= \frac{1}{2}$W [as $\triangle ABC = \frac{1}{2}a^2$]. The C.G's G_1, G_2 of the squares on AB and AC are equidistant from A and so the total weight 2W of these squares act at A. The C.G. of the triangle is a point G_3 of the median AD and the

C.G of the square on BC is a point G_4 on this median produced. Let AD produced intersect the square on BC at E. Hence the C.G. of the whole figure is on the straight line ADE. Let $\bar{x}$ be the distance of the C.G from A.

$$\therefore \quad \bar{x}=\frac{2W\times 0+\frac{1}{2}W.\frac{2}{3}AD+2W\left(AD+\frac{DE}{2}\right)}{2W+\frac{1}{2}W+2W}$$

$$=\frac{\frac{1}{3}AD+2AD+2AD}{\frac{9}{2}}=\frac{26}{27}AD.$$

Ex. 8. From a thin uniform triangular lamina ABC, the portion constituting the inscribed circle is removed. Prove that the C.G of the remaining portion from the side BC is $\frac{\Delta}{3as}\cdot\frac{2s^3-3\pi a\Delta}{s^2-\pi\Delta}$. Δ being the area, and s the semi-perimeter of the lamina. [U. P. 1953 ; C. H. 1965]

Let ω be the weight of unit area of the lamina.

$\therefore$ The weight of the lamina $W=\Delta\omega$ and that of the inscribed circle $=\pi r^2\omega=\pi\left(\frac{\Delta}{s}\right)^2.\omega$. [From trigonometry]

Let p be the length of the perpendicular from A on BC. So if G_1 be the C.G of the whole lamina, then the distance of G_1 from BC $=\frac{p}{3}$.

Now if $\bar{x}$ be the distance of the C.G of the remaining portion from BC, then

$$\bar{x}=\frac{\Delta\omega.\frac{p}{3}-\pi\left(\frac{\Delta}{s}\right)^2.\omega\frac{\Delta}{s}}{\Delta\omega-\pi\left(\frac{\Delta}{s}\right)^2.\omega}=\frac{\frac{p\Delta}{3}-\pi\frac{\Delta^3}{s^3}}{\Delta-\pi\frac{\Delta^2}{s^2}}$$

$$=\frac{\frac{2\Delta}{a}.\frac{\Delta}{3}-\pi\frac{\Delta^3}{s^3}}{\Delta-\pi\frac{\Delta^2}{s^2}} \quad [\because \ \tfrac{1}{2}ap=\Delta]$$

$$=\frac{\Delta^2\left(\frac{2}{3a}-\frac{\pi\Delta}{s^3}\right)}{\Delta\left(1-\pi\frac{\Delta}{s^2}\right)}=\frac{\Delta\left(\frac{2s^3-3\pi a\Delta}{3as^3}\right)}{\frac{s^2-\pi\Delta}{s^2}}=\frac{\Delta}{3as}\cdot\frac{2s^3-3\pi a\Delta}{s^2-\pi\Delta}.$$

Ex. 9. DE is parallel to the side BC of a triangle ABC. If from the triangle ADE is removed and a and b be the distances of BC and DE from A, then the distance of the C.G of the remainder from BC is $\dfrac{a^2+ab-2b^2}{3(a+b)}$. [C. U. 1938]

Let, y be the distance of the required C.G from BC ; A the area of the whole triangle, A_1 that of $\triangle$ADE and AZ and AM be perpendiculars from A on DE and BC respectively.

Now, $\dfrac{A_1}{A}=\dfrac{DE^2}{BC^2}=\dfrac{AZ^2}{AM^2}=\dfrac{b^2}{a^2}$; $\therefore\ A_1=\dfrac{b^2}{a^2}A$

$$\therefore\ \bar{y}=\frac{A.\frac{1}{3}a-\frac{b^2}{a^2}\left\{\left(a-b\right)+\frac{b}{3}\right\}}{A\left(1-\frac{b^2}{a^2}\right)}=\frac{\frac{1}{3}a-\frac{b^2}{3a^2}\left(3a-2b\right)}{\frac{a^2-b^2}{a^2}}$$

$$=\frac{1}{3}\,\frac{a^3-3ab^2+2b^3}{a^2-b^2}=\frac{1}{3}\,\frac{(a-b)(a^2+ab-2b^2)}{a^2-b^2}$$

$$=\frac{1}{3}\,\frac{a^2+ab-2b^2}{a+b}.$$

CHAPTER EIGHT

CONDITIONS OF EQUILIBRIUM OF COPLANAR FORCES

§ 8·1. **Theorem.** If a finite number of co-planar forces acting on a rigid body be not in equilibrium, then they can be reduced either to a single force or to a single couple.

Let n coplanar forces P_1, P_2, P_3,·····P_n acting on a rigid body be not in equilibrium. To prove that the forces can be reduced either to a single force or to a single couple. Before proving this theorem we shall prove the following proposition.

Proposition ; If three co-planar forces act on a rigid body, then they can be reduced to two forces.

Let the three forces be P, Q, R. Now if the forces P, Q be intersecting then (shifting the points of application of the forces to the point of intersection), the forces can be reduced to a single resultant force F. Hence we obtain two forces F and R in place of the three forces P, Q, R.

Let now the forces P, Q be like parallel or unequal and unlike parallel. In each of these cases the forces P and Q will have a resultant F and so in every case we get two forces F and R. If P and Q constitute a couple, then the resultant F_1 of P and R and the force Q are the two forces. If P and R also constitute a couple, then Q and R will be like parallel forces and will have a resultant force $Q+R$. This resultant force $Q+R$ and P are the two forces in this case. Hence we find that in all cases the three forces can be reduced to two forces.

Now, according to the above proportion, the three forces P_1, P_2, P_3 can be reduced to two forces Q_1 and Q_2. Again the three forces Q_1, Q_2 and P_4 can be reduced to two forces R_1 and R_2. The three forces R_1, R_2 and P_5 in their turn can be reduced to two forces S_1, S_2. In this way the given n forces can be ultimatly reduced to two forces T_1 and $T_2 \equiv P_n$. Now the two forces T_1 and T_2 will have either a single resultant or they will constitute a couple. Hence the theorem is completely proved.

§ 8·2. **Theorem.** A finite number of coplanar forces acting on a rigid body can ultimately be reduced to a single force acting at a given point together with a couple. The resolved part of the single force in any direction is equal to the algebraic sum of the resolved parts of the forces in the same direction. Also the algebraic sum of the moments of the forces about the given point will be equal to the moment of the couple.

Let $P_1, P_2, \cdots, P_n$ be n coplanar forces acting on a rigid body and O be a point of the rigid body. Now, at O introduce two equal and opposite forces equal each to P_1, (hence one is like parallel and the other unlike parallel to the given force P_1). These forces P_1 and P_1 being equal and opposite, cancel each other and do not alter the state of the body. Now, the given force P_1 and the unlike parallel force at O constiute a couple. Hence corresponding to the given force P_1, we obtain a couple and an equal, like parallel force acting at O. Similarly corresponding to each forces $P_2, P_3, \cdots, P_n$ we shall get forces $P_2, P_3, \cdots P_n$ acting at O and like parallel to the respective forces and $(n-1)$ couples. Hence in place of the given forces $P_1, P_2, \cdots, P_n$ we obtain n-couples (P_1, p_1), $(P_2, p_2) \cdots (P_n, p_n)$ [$p_1, p_2, \cdots, p_n$ are the lengths of the prependiculars from O on the lines of action of the forces $P_1, P_2, \cdots, P_n$ respectively] and n forces $P_1, P_2, P_3, \cdots, P_n$ acting at O and respectively like parallel to the given forces $P_1, P_2, \cdots, P_n$. Now the n couples can be reduced to a single couple and the n forces acting at the point O can be reduced to a single force. Hence the given forces can ultimately be reduced to a single force. Hence the given forces can ultimately be reduced to a single force F and a single couple (P, p).

Now as the single force acting at O is the resultant of the forces $P_1, P_2, \cdots\cdots, P_n$ acting at O which are respectively equal and parallel to the given forces $P_1, P_2, \cdots\cdots, P_n$ so the resolved part of F along any direction is equal to the algebraic sum of the resolved parts of the given forces $P_1, P_2, \cdots\cdots, P_n$ along the same direction. Also the resultant couple (P, p) being the resultant of the couples (P_1, p_1), $(P_2, p_2), \cdots\cdots (P_n, p_n)$, its moment

is equal to the algebraic sum of the moments of the couples $(P_1, p_1), (P_2, p_2), \cdots\cdots (P_n, p_n)$

$$= \pm P_1 . p_1 \pm P_2 . p_2 \pm \cdots\cdots \pm P_n . p_n$$

= algebraic sum of the moments of the given forces $P_1, P_2, \cdots, P_n$ about the given point O.

Example 1. Three forces P, Q, R act along the sides $\overrightarrow{BC}$ $\overrightarrow{CA}$, $\overrightarrow{AB}$ of the triangle taken in order.

Prove that the resultant of the three forces is

$$\sqrt{P^2+Q^2+R^2-2QR\cos\alpha-2RP\cos\beta-2PQ\cos\gamma}$$

$[\alpha = m\angle BAC,\ \beta = m\angle ABC,\ \gamma = m\angle BCA]$.

Let F be the resultant of the three forces and its line of action intersect $\overrightarrow{BC}$ at an angle θ.

Now, resolving the three given forces and their resultant F along and perpendicular to $\overrightarrow{BC}$ we obtain

$$F\cos\theta = P - Q\cos BCA - R\cos ABC$$
$$= P - Q\cos\gamma - R\cos\beta \cdots\cdots (1)$$

and $F\sin\theta = Q\sin BCA - R\sin ABC$

$$= Q\sin\gamma - R\sin\beta \cdots\cdots (2)$$

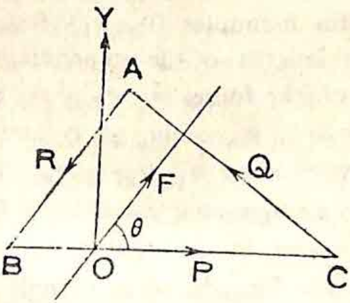


Fig. 93

Squaring and adding (1) and (2) we obtain,

$$F^2(\cos^2\theta+\sin^2\theta) = (P-Q\cos\gamma-R\cos\beta)^2 + (Q\sin\gamma-R\sin\beta)^2$$

$$= P^2+Q^2\cos^2\gamma+R^2\cos^2\beta+2QR\cos\beta\cos\gamma-2RP\cos\beta - 2PQ\cos\gamma+Q^2\sin^2\gamma+R^2\sin^2\beta-2QR\sin\beta\sin\gamma$$

or, $F^2 = P^2+Q^2(\cos^2\gamma+\sin^2\gamma)+R^2(\cos^2\beta+\sin^2\beta) + 2QR(\cos\beta\cos\gamma-\sin\beta\sin\gamma) - 2RP\cos\beta-2PQ\cos\gamma$

$$=P^2+Q^2+R^2+2QR\cos(\beta+\gamma)-2RP\cos\beta-2PQ\cos\gamma$$
$$=P^2+Q^2+R^2+2QR\cos(180^\circ-\alpha)$$
$$-2RP\cos\beta-2PQ\cos\gamma \quad [\because \ \alpha+\beta+\gamma=180^\circ]$$
$$=P^2+Q^2+R^2-2QR\cos\alpha-2RP\cos\beta-2PQ\cos\gamma.$$
$$\therefore \quad F=\sqrt{P^2+Q^2+R^2-2QR\cos\alpha-2RP\cos\beta-2PQ\cos\gamma}.$$

Ex. 2. Four forces of magnitudes 7, 6, 9 and 10 kg. wts. act respectively along the sides $\overrightarrow{AB}$, $\overrightarrow{BC}$, $\overrightarrow{CD}$ and $\overrightarrow{DA}$ of a square ABCD. Prove that if the resultant of the forces intersect AB at O, then $OA=\frac{15}{4}AB$.

The resultant of the unlike parallel forces 7 kg. wts. and 9 kg. wts. is a parallel force 2 kg. wts. and like parallel to the force 9 kg. wts. Let $\overrightarrow{EF}$ be the line of action of this force. Again, the resultant of the unlike parallel forces 10 kg. wts. and 6 kg. wts. is a force 4 kg. wts. and like parallel to the force 10 kg. wts.

Let $\overrightarrow{HK}$ be the line of action this force.

Hence the resultant of the given forces is the resultant of the two forces 2 kg. wts, and 4 kg. wts. acting along $\overrightarrow{EF}$ and $\overrightarrow{HK}$. So ultimately the resultant force is $\sqrt{2^2+4^2}=2\sqrt{5}$ kg. wts.

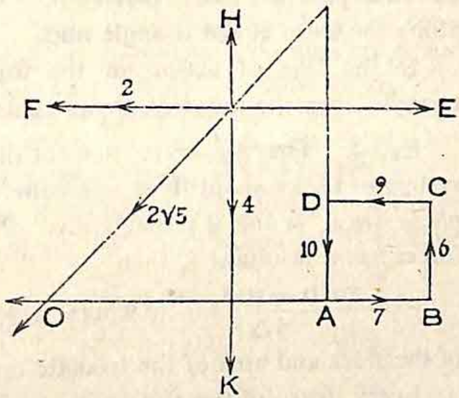


Fig. 94

Let now the line of action of this resultant intersect $\overleftarrow{AB}$ at O. Now the algebraic sum of the moments of the given forces about O = moment of the resultant about O.

$$\therefore \quad 7.0+6.OB+9.AD-10.OA=0.$$
$$[\because \ \text{the resultant passes through O}]$$
$$\text{or,} \quad 6(OA+AB)+9.AB-10.OA=0 \qquad [\because \ AB=AD]$$
$$\text{or,} \quad 15AB=4.OA \quad \therefore \quad OA=\tfrac{15}{4}AB.$$

Ex. 3. Prove that a force acting in the plane of a triangle can be resolved into three components acting along the sides of the triangle.

Let F be a given force acting in the plane of the triangle ABC and the line of action of F intersect $\vec{BC}$ at D. Join AD. Now, resolve the force F into two components along $\overleftrightarrow{DC}$ and $\overleftrightarrow{DA}$. Again, the component along $\vec{DA}$ can be resolved into two components along $\vec{AB}$ and $\vec{AC}$. Hence ultimately the given force F is resolved into three components acting along the sides of the triangle ABC.

If the line of action of the given force F be parallel to $\vec{BC}$, then let it intersect $\vec{CA}$ at the point E and now the force can be, as in the previous case, resolved into three components acting along the sides of the triangle ABC.

If the line of action of the force F be any side of the triangle, then the required components are F, 0, 0.

Ex. 4. The algebraic sum of the moments of a coplanar system of forces about three non-collinear points A, B, C in their plane are L, M and N respectively. Prove that if the system of forces has a resultant F, then

$F^2 = \dfrac{\Sigma a^2(L-M)(L-N)}{4\Delta^2}$, where a, b, c and Δ are the lengths of the sides and area of the triangle respectively.

Every force of the given system of forces can be reduced to three forces acting along the sides of the triangle (see Ex. 3 above). Hence the given system of forces can ultimately be reduced to three forces P, Q, R acting along the sides of the triangle.

Hence the algebraic sum of the moments of the forces about A = L = moment of the force P acting along $\overleftrightarrow{BC}$ about A [as the other two forces Q and R pass through A]

$\therefore\ P.\dfrac{2\Delta}{a} = L$ [For, the distance of A from $\vec{BC} = \dfrac{2\Delta}{a}$

see Ex. 9(i) chapter 5]

$\therefore\ P = \dfrac{aL}{2\Delta}$. Similarly $Q = \dfrac{bM}{2\Delta}$ and $R = \dfrac{cN}{2\Delta}$.

Again, as in Ex. 1 above, F being the resultant of the forces P, Q, R, $F^2 = P^2 + Q^2 + R^2 - 2QR \cos A - 2RP \cos B - 2PQ \cos C$

or, $F^2 = \frac{a^2L^2}{4\triangle^2} + \frac{b^2M^2}{4\triangle^2} + \frac{c^2N^2}{4\triangle^2}$

$$-\frac{2bcMN}{4\triangle^2}\cos A - \frac{2caNL}{4\triangle^2}\cos B - \frac{2abLM}{4\triangle^2}\cos C$$

or, $F^2 = \frac{1}{4\triangle^2}\left\{\Sigma a^2L^2 - \Sigma 2bcMN\left(\frac{b^2+c^2-a^2}{2bc}\right)\right\}$

[Putting the values of cos A, cos B, cos C]

$$= \frac{1}{4\triangle^2}\{\Sigma a^2L^2 - \Sigma MN(b^2+c^2-a^2)\}$$

$$= \frac{1}{4\triangle^2}\{\Sigma a^2(L^2+MN-NL-LM)\}$$

$$= \frac{1}{4\triangle^2}\Sigma a^2(L-M)(L-N).$$

Ex. 5. A system of coplanar forces are equivalent to a couple of moment G_1. If the lines of action of every force is rotated about its point of application through 90°, then the forces become equivalent to a couple of moment G_2. Prove that if the line of action of every force is rotated through an angle θ about its point of application, then the forces can be reduced to a couple of moment $G = G_1 \cos\theta + G_2 \sin\theta$.

Let the co-ordinates of the points of application of the forces $P_1, P_2 \cdots\cdots P_n$ with respect to a convenient set of rectangular

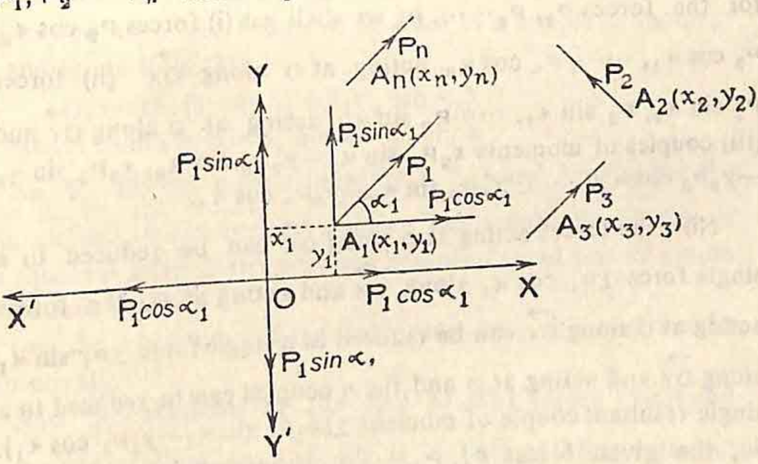


Fig. 95

axes be $(x_1, y_1), (x_2, y_2) \cdots\cdots (x_n, y_n)$ respectively. Let also

$\alpha_1, \alpha_2, \cdots\cdots \alpha_n$ be the inclinations of the lines of action of the forces $P_1, P_2 \cdots\cdots, P_n$ respectively with x-axis.

Now, resolve the forces into components $P_1 \cos \alpha_1$, $P_2 \cos \alpha_2 \cdots\cdots P_n \cos \alpha_n$ and $P_1 \sin \alpha_1$, $P_2 \sin \alpha_2, \cdots\cdots, P_n \sin \alpha_n$ parallel to $\overrightarrow{OX}$ and $\overrightarrow{OY}$. Now introduce at O along $\overleftarrow{OX}$ and $\overleftrightarrow{OY}$ equal and opposite pairs of forces equal and parallel to these components. Every pair of forces thus introduced being equal and opposite cancel each other and their application do not alter the state of the body.

Now the force $P_1 \cos \alpha_1$ acting at (x_1, y_1) and the equal unlike parallel force $P_1 \cos \alpha_1$ acting at O constitute a couple of moment $-y_1 P_1 \cos \alpha_1$; Hence the component $P_1 \cos \alpha_1$ of the given force P_1 acting at (x_1, y_1) is equivalent to a couple of moment $-y_1 P_1 \cos \alpha_1$ and an equal and like parallel force $P_1 \cos \alpha_1$ acting at O.

Similarly the component $P_1 \sin \alpha_1$ of the given force P_1 acting at the point (x_1, y_1) is equivalent to a couple of moment $x_1 P_1 \sin \alpha_1$ and an equal, like parallel force $P_1 \sin \alpha_1$ acting at O.

Hence the given force P_1 acting at the point (x_1, y_1) is equivalent to forces $P_1 \cos \alpha_1$, $P_2 \cos \alpha_2$ along the x and y axis, and a couple of moment $x_1 P_1 \sin \alpha_1 - y_1 P_1 \cos \alpha_1$. Similarly for the forces $P_2, P_3 \cdots\cdots, P_n$ we shall get (i) forces $P_2 \cos \alpha_2$, $P_3 \cos \alpha_3, \cdots\cdots, P_n \cos \alpha_n$. acting at O along $\overrightarrow{OX}$ (ii) forces $P_2 \sin \alpha_2$, $P_3 \sin \alpha_3, \cdots\cdots P_n \sin \alpha_n$ acting at O along $\overrightarrow{OY}$ and (iii) couples of moments $x_2 P_2 \sin \alpha_2 - y_2 P_2 \cos \alpha_2$, $x_3 P_3 \sin \alpha_3 - y_3 P_3 \cos \alpha_3, \cdots\cdots, x_n P_n \sin \alpha_n - y_n P_n \cos \alpha_n$.

Now the forces acting at O along $\overrightarrow{OX}$ can be reduced to a single force $\Sigma P_1 \cos \alpha_1$ along $\overrightarrow{OX}$ and acting at O, the forces acting at O along $\overrightarrow{OY}$ can be reduced to a single force $\Sigma P_1 \sin \alpha_1$ along $\overrightarrow{OY}$ and acting at O and the n couples can be reduced to a single resultant couple of moment $\Sigma(x_1 P_1 \sin \alpha_1 - y_1 P_1 \cos \alpha_1)$. So, the given forces $P_1, P_2, \cdots\cdots, P_n$ are equivalent to forces $\Sigma P_1 \cos \alpha_1$ and $\Sigma P_1 \sin \alpha_1$ both acting at O along $\overrightarrow{OX}$ and $\overrightarrow{OY}$ respectively and a couple of moment $\Sigma(x_1 P_1 \sin \alpha_1 - y_1 P_1 \cos \alpha_1)$.

Now, as the forces are equivalent to a single couple of moment G_1, so $\Sigma P_1 \cos \alpha_1 = 0 = \Sigma P_1 \sin \alpha_1 \cdots\cdots(1)$

and $G_1 = \Sigma(x_1 P_1 \sin \alpha_1 - y_1 P_1 \cos \alpha_1) \cdots\cdots(2)$.

Now, when the line of action of each force is rotated through an angle θ, then the algebraic sum of the resolved parts of the forces along $\overrightarrow{OX}$ and $\overrightarrow{OY}$ are respectively $\Sigma P_1 \cos(\alpha_1+\theta)$ and $\Sigma P_1 \sin(\alpha_1+\theta)$.

Now, $\Sigma P_1 \cos(\alpha_1+\theta) = \Sigma P_1(\cos \alpha_1 \cos\theta - \sin\alpha_1 \sin\theta)$

$= \cos\theta \Sigma P_1 \cos\alpha_1 - \sin\theta \Sigma P_1 \sin\alpha_1$

$= \cos\theta.0 - \sin\theta.0$ [from (1)] $= 0$.

Again, $\Sigma P_1 \sin(\alpha_1+\theta) = \Sigma P_1(\sin\alpha_1 \cos\theta + \cos\alpha_1 \sin\theta)$

$= \cos\theta \Sigma P_1 \sin\alpha_1 + \sin\theta \Sigma P_1 \cos\alpha_1$

$= \cos\theta.0 + \sin\theta.0$ [from (1)] $= 0$.

Hence in this case also the force system will reduce to a couple and the moment of this couple is

$$G = \Sigma\{x_1 P_1 \sin(\alpha_1+\theta) - y_1 P_1 \cos(\alpha_1+\theta)\}$$

$$= \Sigma\{x_1 P_1 (\sin\alpha_1 \cos\theta + \cos\alpha_1 \sin\theta) - y_1 P_1(\cos\alpha_1 \cos\theta - \sin\alpha_1 \sin\theta)\}$$

$$= \cos\theta\{\Sigma(x_1 P_1 \sin\alpha_1 - y_1 P_1 \cos\alpha_1) + \sin\theta \Sigma(x_1 P_1 \cos\alpha_1 + y_1 P_1 \sin\alpha_1)\} \cdots\cdots(3)$$

Now if $\theta = 90°$, i.e., if the lines of action of the given forces be rotated through a right angle about their respective points of application then the forces will reduce to a couple of moment G_2 and hence from (3),

$$G_2 = \Sigma(x_1 P_1 \cos\alpha_1 + y_1 P_1 \sin\alpha_1)$$

Hence from (3), $G = G_1 \cos\theta + G_2 \sin\theta$.

Ex. 6. Forces proportional to 1, 2, 4 and 3 act along the sides $\overrightarrow{AB}$, $\overrightarrow{BC}$, $\overrightarrow{AD}$ and $\overrightarrow{DC}$ of a square ; the length of each side of the square is 2 ft. Determine the magnitude and line of action of the four forces.

Let the magnitudes of the four forces be 1, 2, 4 and 3 units respectively.

Now the resultant of the like parallel forces 1 unit and 3 units acting along $\overrightarrow{AB}$ and $\overrightarrow{DC}$ is a like parallel force of magnitude 4 units. Also the resultant of the like parallel forces 2 units and 4 units is a like parallel force of magnitude 6 units.

Now as ABCD is a square, so these two resultants are inclined at a right angle.

Hence the resultant of these two resultants, i.e., the resultant of the four given forces is $\sqrt{4^2+6^2}=2\sqrt{13}$ units and the line of action of this resultant force is inclined with $\overrightarrow{AD}$ at an angle $\tan^{-1}\frac{2}{3}$.

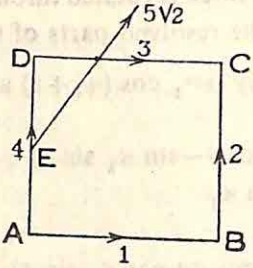


Fig. 96

Let the line of action of the resultant force intersect $\overrightarrow{AD}$ at E.

Now, the algebraic sum of the moments of the given forces about E = the moment of their resultant $2\sqrt{13}$ about E = 0.

$\therefore$ 1.AE + 2.AB − 3.DE = 0 or, AE + 2 × 2 − 3(2 − AE) = 0.

$\therefore$ AE = $\frac{1}{2}$ $\therefore$ ED = $2-\frac{1}{2}=\frac{3}{2}$ $\therefore$ AE : ED = 1 : 3.

Exercise 8A

1. Prove that if a system of co-planar forces be not in equilibrium, then they can be reduced to two forces acting at two given points.

2. Prove that if a system of co-planar forces be not in equilibrium, then they can be reduced to two forces, one acting at a given point and the other acting along a given straight line.

3. The algebraic sums of the moments of a system of co-planar forces, not in equilibrium, about two given points in their plane are zero. Prove that the algebraic sum of the resolved parts of the forces along a straight line perpendicular to the straight line joining those points is zero.

4. Three forces P, Q, R act along the sides $\overrightarrow{BC}$, $\overrightarrow{CA}$, $\overrightarrow{AB}$ of a triangle ABC taken in order and have a resultant F. The moments of the force F about the points A, B, C are respectively G, H and K. Prove that

$$\frac{P}{aG}=\frac{Q}{bH}=\frac{R}{cK}.$$

5. Two systems of forces P, Q, R and P′, Q′, R′ act along the sides $\overrightarrow{BC}$, $\overrightarrow{CA}$, $\overrightarrow{AB}$ of a triangle ABC; prove that their resultants will be parallel if

$$(QR'-Q'R)\sin A+(RP'-R'P)\sin B+(PQ'-P'Q)\sin C=0.$$

[Lucknow, 1929]

6. Three equal forces each equal to P act along the sides $\overrightarrow{BC}$, $\overrightarrow{CA}$, $\overrightarrow{AB}$ of a triangle ABC. Prove that the magnitude of the resultant of the three forces is

$$P\sqrt{\left(1-8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right)}.$$

7. ABC is an equilateral triangle; forces 4, 2 and 1 lb wt. act along the sides AB, BC, CA respectively, in the senses indicated by the order of the letters. Find the magnitude, direction and the line of action of the resultant.

8. Forces proportional to AB, BC and 2CA act along the sides of a triangle ABC taken in order; show that the resultant is represented in magnitude and direction by $\overline{CA}$ and that its line of action cuts $\overrightarrow{BC}$ produced at a point D, where CD is equal to BC.

9. The moments of a system of co-planar forces (not in equilibrium) about three collinear points A, B, C in their plane are G_1, G_2, G_3, prove that (with due regard to the sign)

$$G_1.BC+G_2.CA+G_3.AB=0.$$

[P. U. 1939]

10. If six forces of relative magnitude 1, 2, 3, 4, 5 and 6 act along the sides of a regular hexagon, taken in order show that the single equivalent force is of relative magnitude 6 and that it acts along a line parallel to the force 5, at a distance from the centre of the hexagon $3\frac{1}{2}$ times the distance of a side from the centre. [M. T. 1908]

11. The algebraic sum of the moments of a system of forces, acting in the plane of a lamina in the form of a square ABCD, about the points A, B and C are respectively G_1, G_2 and G_3. Prove that the algebraic sum of the moments of the system of forces about D is $G_1-G_2+G_3$. [Burdwan 1969]

12. Six co-planar forces act along the six sides $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DE}$, $\overline{EF}$ and $\overline{FA}$ of a regular hexagon ABCDEF; AB = 1 ft. and the

magnitudes of the forces are respectively 10, 20, 30, 40, P and Q lbs wt. If the forces are equivalent to a couple, determine the values of P and Q. [P. U. 1930]

§ 8·3. Condition of equilibrium of three co-planar forces.

Theorem : If three co-planar forces acting on a rigid body be in equilibrium, then they are either concurrent or are parallel to one another.

Three forces P, Q, R acting on a rigid body are in equilibrium. To prove that the forces are either concurrent or are parallel to one another.

Proof. The two forces P and Q are either (i) intersecting or (ii) parallel.

First let the forces P and Q intersect at the point O. So, by the parallelogram law of forces, their resultant R′ can be determined. Now, as the three forces P, Q, R are in equilibrium, so R and R′ must balance each other and so they must be equal and act in opposite senses along the same line.

So, R must pass through O and the three forces P, Q, R are concurrent at O.

Next let P and Q be parallel.

Now, P and Q cannot constitute a couple. For, in that case a couple and the force R will be in equilibrium which is not possible. So, the resultant R′ of P and Q can be determined and the force R′ will be parallel to the forces P and Q.

Now as the forces P, Q, R are in equilibrium, so R and R′ must balance each other and so R and R′ must be equal and act in opposite senses along the same line. Hence the force R must be parallel to the forces P, Q.

Hence the three forces in this case are parallel.

Note. If the forces be concurrent, then in solving different problems one can use any one of the following methods; (1) Resolution of the forces in any two mutually perpendicular directions. (2) Converse of triangle of forces. (3) Use of

Lami's theorem. (4) Determination of moments about a point.

Example 1. A heavy uniform rod is in equilibrium with one end resting against a smooth vertical wall and the other against a smooth plane inclined to the wall at an angle θ. Prove that if α be the inclination of the rod to the horizon, then $\tan\alpha = \frac{1}{2}\tan\theta$. [C. U. 1963]

Let $\overline{AB}$ be the rod, $\overline{CA}$ the smooth vertical wall and $\overline{CB}$ be the smooth plane. Now the forces acting on the rod are (i) the weight W of the rod acting at its centre of gravity G vertically downwards; (ii) the reaction R of the wall in the horizontal direction and (iii) the reaction S of the inclined plane acting at the point B perpendicular to $\overline{CB}$. As R and W are not parallel and the forces are in equilibrium, so the lines of action of the three forces are concurrent at a point O. Now, the three forces W, R and S acting at the point O are in equilibrium. Hence by Lami's theorem

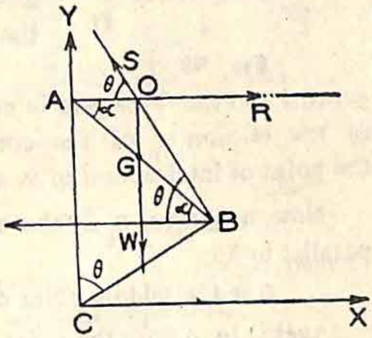


Fig. 97

$$\frac{W}{\sin(180^\circ-\theta)} = \frac{R}{\sin(90^\circ+\theta)} = \frac{S}{\sin 90^\circ}.$$

$$\text{or, } \frac{W}{\sin\theta} = \frac{R}{\cos\theta} \quad \therefore \frac{W}{R} = \tan\theta \cdots\cdots(i)$$

Again taking moment of the forces about the point B, we get $Wl\cos\alpha = R.2l\sin\alpha$ (taking the length of the rod as $2l$).

$$\therefore \frac{W}{R} = 2\tan\alpha \cdots\cdots(ii)$$

Now, combining (i) and (ii) we get

$$\tan\theta = 2\tan\alpha \quad \text{or,} \quad \tan\alpha = \tfrac{1}{2}\tan\theta$$

Ex. 2. A heavy uniform rod $\overline{AB}$ rests on a smooth wall at the end A and the rod is kept in equilibrium by a cord BC; the point C is vertically above A. If $AB = \sqrt{2}AC$, then

determine the inclination of the rod with the wall and the reaction of the wall. [H. S. 1963]

Let the rod be AB and it rest against the smooth vertical wall CL at the end A. Now, the forces acting on the rod are (1) the weight W of the rod acting vertically downwards at its middle point G; (2) the reaction R of the smooth wall at the point A in the horizontal direction and (3) the tension T of the cord.

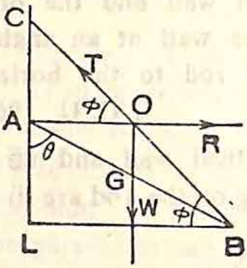


Fig. 98

Since the forces W and R are not parallel and the forces are in equilibrium, so the line of action of the tension T of the cord must pass through the point O, the point of intersection of W and R.

Now, in $\triangle BAC$; G is the middle point of $\overline{BA}$ and $\overline{GO}$ is parallel to $\overline{AC}$.

$\therefore$ O is the middle point of $\overline{BC}$.

Again, in $\triangle BCL$, the point O is the middle point of $\overline{BC}$ and $\overline{OA}$ is parallel to $\overline{BL}$.

$\therefore$ A is the middle point of $\overline{CL}$ i.e., $AL = AC$.

Now if θ be the inclination of the rod with the vertical,

$$\cos\theta = \frac{AL}{AB} = \frac{AC}{AB} = \frac{AC}{\sqrt{2}AC} = \frac{1}{\sqrt{2}} = \cos 45^\circ$$

$[\because\ AB = \sqrt{2}.AC$ (given) $]$

$\therefore\ \theta = 45^\circ$.

Hence $m\angle ABL = 45^\circ$ and $BL = AL = \frac{1}{2}CL$.

Now, let $m\angle AOC = m\angle LBO = \phi$

$$\therefore\ \tan\phi = \frac{CL}{BL} = \frac{2BL}{BL} = 2.$$

Now the three forces W, R and T acting at the point O are in equilibrium. Hence by Lami's theorem

$$\frac{W}{\sin(180^\circ - \phi)} = \frac{R}{\sin(90^\circ + \phi)} \quad \text{or,} \quad \frac{W}{\sin\phi} = \frac{R}{\cos\phi}$$

$$\therefore\ R = W\cot\phi = W.\tfrac{1}{2} = \frac{W}{2}.$$

Ex. 3. One end of a heavy uniform rod rests on a smooth horizontal plane and a string tied to the other end of the rod is fastened to a fixed point above the plane ; find the tension of the string.

Let $\overline{AB}$ be the rod of weight W acting at the middle point G of the rod vertically downwards. Again the reaction R of the smooth horizontal plane act vertically upwards. Let T be the tension of the string. Hence the three forces R, W and T are in equilibrium. Now the force R and W are parallel. Hence the tension T must be parallel to R and W. Hence the tension T of the string is also acting vertically upwards. So the resultant of R and T must be equal and opposite to W i.e., the resullant of R and T is a force W acting at G vertically upwards.

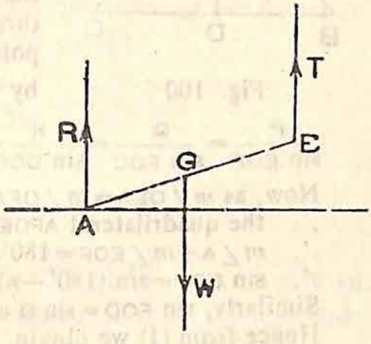


Fig. 99

$\therefore \quad R+T=W$ and $\dfrac{R}{T}=\dfrac{BG}{AG}=1 \quad \therefore \quad R=T.$

$\therefore \quad W=2T$ i.e., $T=\dfrac{W}{2}$.

Ex. 4. Prove that a heavy rod cannot rest in equilibrium with its ends on two smooth planes, one of which is horizontal and the other inclined to the horizontal plane at any angle.

Here the reaction of the smooth horizontal plane and the weight of the rod both acting in the vertical direction are parallel. Again since the two smooth planes are intersecting and their reactions on the rod act in directions perpendicular to the planes, so the reactions are also intersecting. Hence of the three forces acting on the rod two are intersecting and two are parallel ; hence the three forces cannot be concurrent or parallel to one another. So the forces cannot remain in equilibrium i.e., the rod cannot remain in equilibrium.

Ex. 5. Three forces acting along the perpendiculars of the sides of the triangle ABC (all inwards or all outwards) are in equilibrium. Prove that the forces are proportional to sin A, sin B and sin C.

Let three forces P, Q, R acting along the perpendiculars at D, E, F of the sides $\overline{BC}$, $\overline{AC}$, $\overline{AB}$ respectively of the triangle ABC (all inwards or all outwards) are in equilibrium. Hence no two of the forces are parallel and so the three forces are concurrent, and let O be the point of concurrence. Now the three forces P, Q, R acting at the point O are in equilibrium. Hence by Lami's theorem.

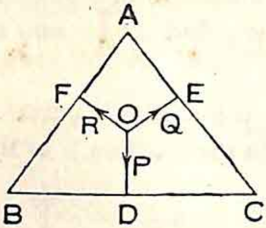


Fig. 100

$$\frac{P}{\sin EOF}=\frac{Q}{\sin FOC}=\frac{R}{\sin DOE}\cdots(1)$$

Now, as $m\angle OEA = m\angle OFA = 90°$,

$\therefore$ the quadrilateral AFOE is cyclic.

$\therefore$ $m\angle A + m\angle EOF = 180°$.

$\therefore$ $\sin EOF = \sin(180° - A) = \sin A$.

Similarly, $\sin FOD = \sin B$ and $\sin DOE = \sin C$.

Hence from (1) we obtain.

$$\frac{P}{\sin A}=\frac{Q}{\sin B}=\frac{R}{\sin C}.$$

Ex. 6. A heavy smooth sphere in contact with a smooth vertical wall is supported by a string fastened to a point on the surface of the sphere, the other end of the string is attached to a point on the wall. The length of the string is equal to the radius of the sphere. Find (i) the inclination of the string with the vertical ; (ii) the tension of the string and (iii) the reaction of the wall.

Let the radius of the sphere be r ;

$\therefore$ length of the string is r. Now the forces acting on the sphere are (i) the weight W of the sphere acting at the centre C of the sphere vertically downwards ; (ii) the tension T of the string at the point B and (iii) the reaction R of the smooth vertical wall.

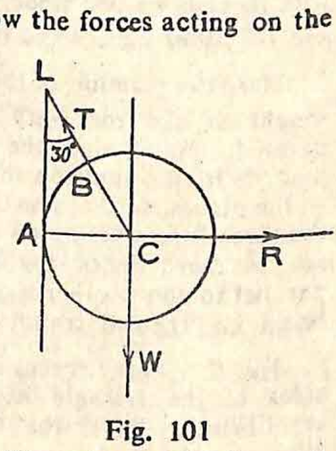


Fig. 101

Now, as the sphere has touched the wall at the point A so its radius $\overline{CA}$ is perpencular to the wall and hence the line of action of the reaction R passes through the centre C of the sphere. Hence two of the forces R

and W intersect at C. So, considering the equilibrium of the sphere, the line of action of the third force i.e., the tension T will also pass through C. $\therefore$ $CL = CT + TL = r + r = 2r$.

Now $\sin ALC = \dfrac{CA}{CL} = \dfrac{r}{2r} = \frac{1}{2}$ $\therefore$ $m\angle ALC = 30°$.

Hence the inclination of the string with the vertical is 30° Now as the forces are in equilibrium, so resolving the forces in the horizontal and vertical directions we get

$$T \cos ALC = W \quad \text{or,} \quad T = \frac{W}{\cos 30°} = \frac{W}{\frac{\sqrt{3}}{2}} = \frac{2W}{\sqrt{3}}.$$

$$\text{and } R = T \sin 30° = \frac{2W}{\sqrt{3}} \cdot \frac{1}{2} = \frac{W}{\sqrt{3}}.$$

Ex 7. The weight of a gate is 60 kg. and it is supported by two hinges at a distance of 90 cms in the same vertical line. This vertical line is at a distance of 120 cms. from the C.G of the gate. If the upper hinge shares the whole load, find the reactions of the hinges.

Let the hinges be A and B and $AB = 90$ cms. ; the weight 60 kg. wts. of the gate act at the C.G. G of the gate vertically downwards. Since the lower hinge does not share any load, its reaction R at the point A acts in the horizontal direction. Hence the lines of action of the forces 60 kg. and R will intersect. Let the point of intersection be O. Hence as the gate is in equilibrium, the reaction S of the upper hinge will pass through O and its line of action is $\overrightarrow{OB}$.

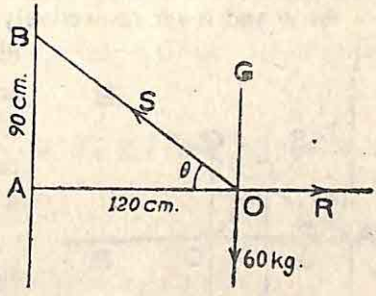


Fig. 102

Now, for the equilibrium of the forces, taking moment about the point B we get,

$$R.90 = 60.120 \quad \therefore \quad R = \frac{60 \times 120}{90} = 80 \text{ kg. wts.}$$

Again resolving the forces in the vertical direction we get,
$S.\sin\theta = 60$ kg.wts$\cdots$(1), where $m\angle AOB = \theta$ (say).
Now, $OB = \sqrt{OA^2+AB^2} = \sqrt{120^2+90^2} = 150$ cms.

$\therefore\quad \sin\theta = \frac{AB}{OB} = \frac{90}{150} = \frac{3}{5}$.

Now from (1) we get, $S = \frac{60\text{ kg. wts.}}{\sin\theta} = \frac{60\text{ kg. wts.}}{\frac{3}{5}}$

$= 100$ kg. wts.

Ex. 8. A uniform beam of length $2a$, rests in equilibrium, with one end resting against a smooth vertical wall and with a point of its length resting upon a smooth vertical rod which is parallel to the wall and at a distance b from it; show that the inclination of the beam to the vertical is, $\sin^{-1}\left(\frac{b}{a}\right)^{\frac{1}{3}}$.

Let the rod be $\overline{AB}$, G its mid point or C.G. and θ be the inclination of the rod with the vertical. Now the forces acting on the rod are (i) the weight W of the rod acting at the point G vertically downwards, (ii) the reaction S of the smooth rod at the point C, normal to the rod and (iii) the reaction R of the smooth wall acting in the horizontal direction.

As W and R act respectively in the vertical and horizontal directions, hence their lines of action will intersect at a point O. Hence, since the forces are in equilibrium, the line of action of S will pass through O.

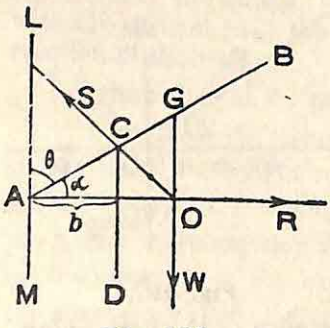


Fig. 103

Now resolving the forces in the vertical direction, in consideration of the equilibrium of the rod we get, $S\sin\theta - W = 0$.

or, $\quad S\sin\theta = W\cdots(1)$

Again, taking moments of the forces about A we get,

$S.b\operatorname{cosec}\theta - W.a\sin\theta = 0$.

$\therefore\quad S = \frac{Wa\sin^2\theta}{b}\quad$ or, $\quad \frac{W}{\sin\theta} = \frac{Wa\sin^2\theta}{b}$ [From (1)]

$\therefore\quad \sin^3\theta = \frac{b}{a},\quad$ or, $\quad \sin\theta = \left(\frac{b}{a}\right)^{\frac{1}{3}}$.

Ex. 9. A beam whose centre of gravity divides it into two portions a and b is placed inside a smooth sphere; show that, if θ be its inclination to the horizon in the position of equilibrium and 2α be the angle subtended by the beam at the centre of the sphere, then $\tan\theta = \dfrac{b-a}{b+a}\tan\alpha$.

Let the rod be $\overline{AB}$ and G be its C.G. Now the rod is in equilibrium under the action of the following forces.

(i) The reaction R of the sphere at the point A.

(ii) The reaction S of the sphere at the point B.

(iii) The weight W of the rod acting at its C.G. vertically downwards.

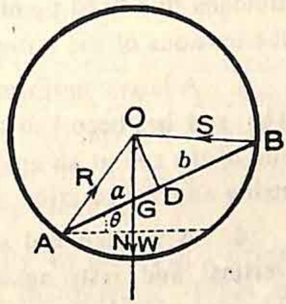


Fig. 104

Now, the lines of actions of the reactions R and S pass through the centre O of the sphere, so the line of action of W will also pass through O. Hence G and O will be in the same vertical line, G below O. Let $\overrightarrow{OG}$ intersects a horizontal line through A at N.

$\therefore\ m\angle GAN = \theta$ and $m\angle GNA = 90°$. Let $\overline{OD}$ be perpendicular on the rod.

$\therefore\ m\angle AOD = m\angle BOD = \alpha$.

and $m\angle DOG = 90° - m\angle DGO = 90° - m\angle AGN = \theta$

$$\therefore\ \frac{a}{b} = \frac{AG}{GB} = \frac{AD - GD}{BD + GD} = \frac{OD\tan AOD - OD\tan GOD}{OD\tan BOD + OD\tan GOD}$$

$$= \frac{\tan\alpha - \tan\theta}{\tan\alpha + \tan\theta}.$$

Now, by componendo and dividendo we obtain,

$$\frac{b-a}{b+a} = \frac{\tan\theta}{\tan\alpha} \qquad \therefore\ \tan\theta = \frac{b-a}{b+a}\tan\alpha.$$

Exercise 8B

1. A heavy uniform rod of length 'a' rests with one end against a smooth vertical wall, the other end being tied to a point of the wall by a string of length l. Prove that the rod may remain in equilibrium at an angle θ to the wall, given by,

$$\cos^2\theta = \frac{l^2 - a^2}{3a^2}.$$ [C. U. 1941]

2. A uniform rod is supported by means of two strings attached to a fixed point and to the ends of the rod. Show that the tensions of the strings are proportional to their lenghs.

3. A heavy uniform rod AB can turn about its extremity A. The rod has been kept horizontal by a string tied at the other end of the rod at an angle of 45°. Prove that the tension of the string and the reaction of the rod at the point A are equal.

4. A uniform rod AB is inclined at an angle 60° with the vertical and rests against a smooth vertical wall at the end A. The rod is tied at a point at a distance 1 ft. from the end B with a ring by a string. The length of the rod is 4 ft. and the ring and the point A are in the same vertical line. Find (i) the position of the ring (ii) the tension of the string and (iii) the reaction of the wall.

5. A uniform square lamina rests in equilibrium under gravity in a vertical plane with two of its sides in contact with two smooth pegs in the same horizontal line at a distance c apart. Show that the angle θ made by a side of the square with the horizontal in a non-symmetrical position of equilibrium is given by, $c(\sin\theta + \cos\theta) = a$, where $2a$ is the length of a side of the square. [C. U. 1946]

6. A uniform rod can turn freely about one of its ends, and is pulled aside from the vertical by a horizontal force acting at the other end of the rod and equal to half its weight. Show that the rod rests at an inclination of 45° with the vertical. [C. U. 1945]

7. A heavy uniform rod 40 cms. long and of weight 50 kg. is suspended from a nail by strings fastened to its ends, the lengths of the string being 32 cms. and 24 cms.; find the tensions of the strings.

8. A rod whose centre of gravity divides it into two portions whose lengths are a and b, has a string of length l, tied to its two ends and the string is slung over a small smooth peg ; find the position of equilibrium of the rod, in which it is not vertical. Also find the tension of the string in this position.

9. A heavy uniform rod 150 cms. long is suspended from a fixed point by string fastended to its ends, their lengths being 90 cms. and 120 cms. ; if θ be the angle at which the rod is inclined to the vertical, show that, $25 \sin \theta = 24$.

10. A weightless rod 3 metres long is supported horizontally, one end being hinged to a vertical wall and the other attached by a string to a point 4 metres above the hinge ; a weight of 180 kg. is hung from the end supported by the string. Calculate the tension in the string and the pressure along the rod.

11. A picture hangs symmetrically by means of a string over a nail and attached to two rings in the picture. What is the tension of the string when the picture weighs 16 kg., the string is 36 cms. long and the nail 12 inches above the horizontal line joining the rings.

12. A picture of 10 kg. wts. is hung with its upper and lower edges horizontal by a cord fastened to the two upper corners and passing over a nail, so that the parts of the cord at the two sides of the nail are inclined to each other at an angle of 120°. Find the tension of the cord.

13. A uniform rod AB is in equilibrium at an angle α with the horizontal, with its upper end A resting against a smooth peg, and its lower end B attached to a light cord, which is fastened to a point C on the same level as A. If the cord is inclined to the horizontal at an angle β, then

$$\tan \beta = 2 \tan \alpha + \cot \alpha.$$

14. A uniform rod of weight W rests with its ends in contact with two smooth planes, inclined at angles α and β respectively to the horizontal and intersecting in a horizontal line. If θ be the inclination of the rod to the vertical, show that $2 \cot \theta = \cot \beta - \cot \alpha$.

Also find the reactions at the ends of the rod. [P. U. 1933]

15. A uniform rod of length $2l$ rests with its lower end in contact with a smooth vertical wall. It is supported by string of length a, one end of which is fastened to a point in the wall and the other end to a point in the rod at a distance b from its lower end. If the inclination of the string to the vertical be θ,

show that, $\cos^2\theta = \dfrac{b^2(a^2-b^2)}{a^2 l(2b-l)}$. [C. U. 1940]

16. A uniform bar of length a rests suspended by two strings of lengths l and l' fastened to the ends of the bar and to two fixed points in the same horizontal line at a distance b apart. If the directions of the strings produced meet at right angles, and if T_1 and T_2 be the tensions of the strings, then

$$\frac{T_1}{T_2} = \frac{al + bl'}{al' + bl}.$$

Short Answer Type Questions

In the Higher Secondary Examinations short answer type questions are set in Mechanics. Below is a list of such questions on Statics. These questions or their answers are in the main body of the text in some forms or others. So, without giving complete answers of those questions, we have referred relevant sections, worked out examples or exercises of the text in order to avoid repetitions. In some cases hints have also been given.

Correct or justify the following statements with reasons.

1. (*a*) The greatest and least resultants of two forces of magnitudes 12 kg. and 8 kg. acting at a point are 20 kg. and 4 kg. respectively. [*Ans.* True. See § 2·3 Cor. 1 & Cor. 2]

(*b*) The greatest and least resultants of two forces acting at a point are 17 kg. and 7 kg. respectively. When the forces are are inclined at an angle 90°, their resultant is 12 kg.

[*Ans.* False. correct statement 13 kg. See § 2·2 Cor. 3]

2. Two forces P and Q acting at a point are inclined at angles θ and ϕ respectively with the line of action of their resultant. $P>Q$ and so $\theta>\phi$.

[*Ans.* not correct ; Correct Ans. $\theta<\phi$; See § 2 3 Cor. 5]

3. The resultant of two equal forces each equal to P and acting at a point and inclined at an angle θ with each other is $2P\cos\theta$. [*Ans.* not correct ; correct statement : $2P\cos\frac{\theta}{2}$; See § 2 3 Cor. 4]

4. The resultant of two unequal forces acting at a point has the least value 4 units and if the angle between the forces is 90°, their resultant will be 20 units. The resultant of the forces has the maximum value 28 units.

[*Ans.* Correct. See § 2·3, Cor. 1, 2, 3. See Example 7, worked out after § 2·3 (*a*)]

5. Two equal forces act at a point and are inclined at an angle 120° with each other. The resultant is also a third equal force P.

[*Ans.* Correct. See § 2·3 Cor. 4. Compare Ex. 1, after § 2·3 (*a*)]

6. A weightless wire of length 6 ft. is suspended between two points A and B on the same level. A weight 10 lbs is hung at the middle point of the wire and causes it to drop 1·5 ft. below the middle point. The tension of the string is then 10 lbs.

[*Ans.* Correct]

[Hints : If T be the tension of each portion of the string, then $2T \cos \alpha = 10$ lbs where 2α is the angle between the two portions of the string. Here $\cos \alpha = \frac{1 \cdot 5 \text{ ft.}}{3 \text{ ft.}} = \frac{1}{2}$].

$\therefore \; 2T.\frac{1}{2} = 10$ lbs. or, $T = 10$ lbs.

7. The magnitude of the resultant of two given concurrent forces can be made as large as possible by proper adjustment of their lines of action.

[*Ans.* Incorrect. See § 2·3 Cor. 1]

8. The greater is the angle between two forces acting at a point, the greater is their resultant.

[*Ans.* Incorrect. Correct statement ; the less is the resultant. See § 2·3, Cor. 6]

9. The line of action of the resultant of two equal forces acting at a point bisects the angle between the forces.

[*Ans.* Correct. See § 2·3, Ex. 2 worked out]

10. A weight is suspended by two cords of equal length which are attached to two points on the same horizontal line. If the lengths of the cords are equally increased, then their tensions will also increase.

[*Ans.* Statement not correct. Correct statement : will also decrease. See Ex. 10 after § 2·3 (*a*)]

11. If the resultant of two forces P and Q acting at a point bisects the angle between the forces, then $P = Q$.

[*Ans.* Correct statement]

[Hints : In fig. 4, $\angle BAC = \angle ACB$. So $AB = BC$ or, $P = Q$.]

12. The component of a force 100 kg. in a direction making an angle 60° with the direction of the force, is 50 kg.

[*Ans.* Not true. Correct statement resolved part.]

13. A system of co-planar concurrent forces will be in equilibrium if the algebraic sum of the resolved parts of all the

forces along any direction is equal to the algebraic sum of the resolved parts along the direction perpendicular to the previous direction. [*Ans.* Incorrect. See § 2·7]

14. The magnitudes of two forces acting at a point are in the ratio $\sqrt{3} : 2$. Show that the resultant of the forces cannot be inclined at an angle greater than $\frac{\pi}{3}$ with the line of action of the greater force. [*Ans.* Statement correct. Use § 2·4]

15. R is the resultant of two forces P and Q acting at a point and so, the line of action of R bisects the angle between P and Q.

[*Ans.* Statement not correct. For justification of falsity of the statement assume $P \neq Q$.]

16. The resultant of two forces $\overline{OA}$ and $\overline{OB}$ is represented by $2\overline{OC}$, where C in the middle point of AB.

[*Ans.* Correct. See cor. § 2·8]

17. The resultant of three forces represented by $\overline{OA}$, $\overline{OB}$, $\overline{OC}$ is $3\overline{OI}$, where I is the incentre of the triangle ABC.

[*Ans.* Incorrect statement. See Example 1 after § 2·8]

18. Forces $\overline{PA}$, $\overline{PB}$, $\overline{PC}$, $\overline{PD}$ keep a particle placed at P at rest, if P is the middle point of the line segment joining the middle points of the diagonals of the quadrilateral ABCD.

[*Ans.* Correct statement. See Ex. 4 after § 2·8]

19. AB and CD denote two equal and parallel chords of a circle. P is a point on the circumference of the circle equidistant from A and B. Show that the resultant of forces acting at P and represented by $\overline{PA}$, $\overline{PB}$, $\overline{PC}$, $\overline{PD}$ is constant.

[*Ans.* Correct statement]

20. The resultant of two forces represented by $\overline{BC}$ and $\overline{CA}$ is represented by $\overline{AB}$. [H· S. 1979]

[*Ans.* Incorrect statement. See § 2·3(*a*)]

21. If three forces acting at a point act along the sides $\overrightarrow{BC}$, $\overrightarrow{CA}$ $\overrightarrow{AB}$ of a triangle ABC, taken in order, then the forces are in equilibrium. [*Ans.* Incorrect statement. See § 3·1]

22 If three forces are represented in magnitude, direction and sense by the three sides of a triangle taken in order then the forces are in equilibrium.

[*Ans.* Incorrect statement. See § 3·1]

23. ABCDE and A′B′C′D′E′ are two polygons with corresponding sides parallel. Forces acting along sides of the polygon ABCDE taken in order and acting at a point are in equilibrium. The forces can be represented by the sides of the polygons A′B′C′D′E, taken in order.

[*Ans.* The statement is not correct. See Note § 3·4]

24. Three forces P, Q, R acting at a point are in equilibrium ; they will continue to be in equilibrium if each force is increased by the same amount. [H. S. 1978]

[*Ans.* Incorrect statement]

25. Two forces P, Q acting at a point have a resultant R ; their resultant will remain unaltered if each force is decreased by the same amount. [*Ans.* Incorrect]

26. (*a*) Three concurrent forces acting in a plane can produce equilibrium for some arrangement of the force if the sum of the magnitudes of two of them be less than that of the third force.

[*Ans.* Incorrect statement. See Example 1 after § 3·2]

(*b*) Three forces acting at a point whose magnitudes are in the ratio 5 : 13 : 7 cannot produce equilibrium [H. S. 1980].

[*Ans.* Correct statement. See Example 1 (ii) after § 3·2]

(*c*) Three forces acting at a point whose magnitudes are in the ratio 5 : 12 : 7 cannot produce equilibrium.

[*Ans.* Incorrect statement. See. Ex. 1 (ii) after § 3·2]

27. Forces of magnitudes proportional to $2 : \sqrt{2} : \sqrt{3}+1$ act at a point and are in equilibrium. The angle between the first and third forces is 30°.

[*Ans.* Incorrect statement. See. Ex. 2 worked out after § 3·2]

28. ABC is a triangle ; D, E, F are the middle points of the sides BC, CA, AB respectively. The forces acting on a particle and represented by the straight lines AD, BE, CF are in equilibrium.

[*Ans.* Correct statement. See. Ex. 11 exercise 3A]

29. P is a point within a parallelogram ABCD. The forces $\overline{PA}$, $\overline{PC}$, $\overline{BP}$, $\overline{DP}$ produce equilibrium for every position of P.

[*Ans.* Correct statement. See. Ex. 8 Exercise 3A]

30. ABCD is a parallelogram. Show that the resultant of the forces $\overline{AC}$ and $\overline{BD}$ is $2\overline{BC}$.

[*Ans.* Correct statement. See. Ex. 9 Exercise 3A]

31. If three equal forces acting at a point are in equilibrium, then the forces are equally inclined to one another.

[*Ans.* Correct statement. Apply Lami's theorem.]

32. If three equal forces acting at a point are equally inclined with one another, then the forces are in equilibrium.

[*Ans.* Correct statement. Apply converse of Lami's theorem.]

33. If three forces acting at a point be in equilibrium and are equally inclined with one another, then the forces are equal in magnitude.

[*Ans.* Correct statement. Apply Lami's theorem.]

34. If three forces acting at a point along the sides of a triangle taken in order, be in equilibrium, then the forces are proportional to the lengths of the corresponding sides.

[*Ans.* Correct statement. Converse of theorem of triangle of forces.]

35. Three equal forces acting at a point parallel to the sides of an equilateral triangle taken in order are in equilibrium.

[*Ans.* Correct statement. Theorem of Triangle of forces.]

36. A cord cannot bear a weight more than 15 lbs. A weight of 30 lbs. is tied with the cord on a smooth inclined plane of inclination 40°. It is not possible for the cord to produce equilibrium.

[*Ans.* Correct statement. Compare Ex. 11 worked out after § 3·6]

37. If forces $(m-n)\,\overline{OP}$, $(n-l)\,\overline{OQ}$ and $(l-m)\,\overline{OR}$ be in equilibrium, then P, Q, R are collinear.

[*Ans.* Correct statement. See, Ex. 12. Exercise 2C]

38. Two parallel forces P and Q act at point A and B respectively. The resultant of the forces act at a point C on AB. If $P > Q$ then $AC > BC$.

[*Ans.* Incorrect statement. See, Note 1 § 4·2 and Note 1 § 4·3.]

39. The magnitude and point of application of two like (or unequal and unlike) parallel forces depend on the direction of the forces. [*Ans.* Incorrect statement. See, Note 4. § 4·2]

40. (*a*) The magnitude of the resultant of two like parallel forces will remain unaltered if each force is increased by the same amount.

(*b*) The magnitude of the resultant of two unequal and unlike parallel forces will remain unaltered if each force is increased by the same amount.

[*Ans.* (*a*) Incorrect statement. (*b*) Correct statement.

Hints : (*a*) $(P+R)+(Q+R) \neq P+Q$.

(*b*) $(P+R)-(Q+R)=P-Q$.]

41. The point of application of the resultant of two like (or unequal and unlike) parallel forces remains unaltered if each force is decreased by the same amount.

[*Ans.* Incorrect statement. See, Example 3 worked out after § 4·4.]

42. A man carries a bundle at the end of a stick, placed horizontally over his shoulder. In order to decrease the pressure on his shoulder, he will decrease the distance between his hand and shoulder.

[*Ans.* Incorrect statement. See, Ex. 7. Exercise 4.]

43. (*a*) The resultant of three like parallel forces acting at the angular points of a triangle passes through the incentre of the triangle.

(*b*) The centre of gravity of three equal weights placed at the angular points of a triangle is the incentre of the triangle.

(*c*) The centre of gravity of three equal weights placed at the middle points of the sides of a triangle is the incentre of the triangle. [*Ans.* All the three statements are incorrect.]

Hints. In each case replace the term 'incentre' by the term 'centroid'. See. Ex. 17. Exercise 4. Cor. 1 and Cor. 2. § 7·5(*c*).]

44. The centre of gravity of a uniform triangular lamina is the same as that of three equal weights $\frac{\omega}{3}$ placed at (i) the

angular points or (ii) at the middle points of the sides of the triangle, where ω is the weight of the lamina.

[*Ans.* Correct statement. See, Cor. 1, Cor. 2 of § 7·5(*c*)]

45. Three like parallel forces P, Q, R act at the angular points of a triangle. If their resultant passes through the centroid of the triangle whatever be the common direction of the forces, then $P=Q=R$.

[*Ans.* Correct statement. See Ex. 18. Exercise 4.]

46. Three like parallel forces proportional to the lengths of the sides BC, CA, AB of a triangle ABC act respectively at the vertices A, B, C of the triangle. The resultant of the forces passes through the centroid of the triangle.

[*Ans.* Incorrect statement. Replace the term 'centroid' by the term 'Incentre'. See Ex. 22 Exercise 4]

47. Two like parallel forces have a resultant R. If one of the forces undergoes a displacement, then the resultant will also undergo the same displacement.

[*Ans.* Incorrect statement. See Ex. 19 Exercise 4]

48. The algebraic sum of the resolved parts of a pair of parallel forces (not forming a couple), along any line in their plane is equal to the resolved part of their resultant along the same line. [*Ans.* Correct statement. See Ex. 16. Exercise 4]

49. A body can have more than one centre of gravity.

[*Ans.* Incorrect statement. See § 7·3. Note (2)]

50. The centre of mass and the centre of gravity of a body are the same point.

[*Ans.* Correct statement. The statement may be modified as "The centre of mass and the centre of gravity (when it exists) of a body are the same point. See § 7·3, Note (2)].

51. A body always possesses a centre of gravity.

[*Ans.* Incorrect statement. See Note (3) § 7·3]

52. The centre of a gravity is necessarily a point of the body. [H. S. 1978] [*Ans.* Incorrect statement]

Hints : The centre of gravity of a circular wire is the centre of the circle which is not a point of the wirc.

53. If a number of like parallel forces act at points of a given straight line, then the centre of the parallel forces is oollinear with the points of application of the parallel forces.

[*Ans*. The statement is correct. See § 4·4]

54. A finite number of parallel forces will always have a resultant force. [*Ans.* Incorrect statement. See § 4·4]

55. The position of the centre of a system of parallel forces (when it exists) depends upon the direction of the forces.

[*Ans*. Incorrect statement. See Note § 4·4]

56. P and Q are two equal and unlike parallel forces, acting at A and B respectively. AH and BK represent respectively, the resultants of the two forces P and a force F acting at A along $\vec{AB}$ and the two forces Q and a force F acting at B along $\vec{BA}$. AH and BK are produced. They will intersect at a point.

[*Ans*. Incorrect statement. See Note 4. § 4·3]

57. The moment of a couple cannot be zero.

[*Ans.* Correct statement. See § 6·2 and its corolary]

58. The forces constituting a couple can have a resultant.

[*Ans.* Incorrect statement. See Note 4. § 4·3, and cor. 2. § 6·6]

59. A couple and a force cannot produce equilibium.

[*Ans*. Correct statement. See Note § 6·6]

60. The resultant of a couple and a force is a single force equal and parallel to the given force.

[*Ans*. Correct statement. See § 6·6]

61. If three forces acting on a rigid body are represented in magnitude and direction by the three sides of a triangle taken in order, then the forces are in equilibrium.

[*Ans*. Incorrect statemeut. See § 6·9]

62. The moment of a couple formed by two equal and unlike parallel forces each of given magnitude is independent of the direction of the forces.

[*Ans*. Incorrect statement. See Ex. 5 worked out, chapter VI]

63. If three forces acting along the three sides of a triangle taken in order reduce to a couple, then the forces can be represented by the sides of the triangle.

64. A couple acting on a body cannot produce any motion of translation of the body. [*Ans.* Correct statement]

65. If the algebraic sum of the moments of a finite number of coplanar forces not constituting a couple about a fixed point of their plane be zero, then the forces are in equilibrium.
[*Ans.* Incorrect statement. See Cor. 3. § 5·6]

66. If a finite number of co-planar forces possess a resultant, then the algebraic sum of the moments of the forces about every point of the line of action of the resultant is zero.
[*Ans.* Correct statement. See Cor. 1. § 5 6]

67. If the algebraic sum of the moments of a finite number co-planar forces about every point in their plane be zero, then the forces are in equilibrium.
[*Ans.* Correct statement. See Cor. 3. § 5·6]

68. If a finite number of co-planar forces be in equilibrium, then the algebraic sum of the moments of the forces about every point in their plane is zero.
[*Ans.* Correct statement. See § 5·6 and its corolaries]

69. A heavy uniform sphere touching a smooth vertical wall, is kept in equilibrium by a string whose one extremity is attached to the wall at a point L and the other to a point B of the sphere. The centre of the sphere is collinear with the points L and B. [*Ans.* Correct statement. See Ex. 6 after § 8·3]

70. It is impossible for a heavy rod to rest in equilibrium with its ends in two smooth planes, one of which is horizontal and the other is inclined to the horizontal at an angle other than 90°. [*Ans.* Correct statement. See Ex. 4, worked out after § 8·3]

ANSWERS

Exercise 1

1. 1 cm. = 5 kg. wts. 3. (ii) is true. 5. No.

Exercise 2A

1. (i) 16 kg. wts. (ii) 23·7 kg. wts. (iii) 60° (iv) 45 kg. wts, 2. (*a*) 60° (c) 90° (f) 3 lbs. 1 lbs.
3. 48 kg. wts. and 14 kg. wts.
4. 79 kg. wts. and 21 kg. wts.
5. 12 kg. wts. and 5 kg. wts. 6. 120° ; 8. 90°.
13. $P+Q$; $(S \neq P-Q)$.

Exercise 2B

1. The resultant is 2 kg. wts. and it is inclined at an angle 120° with the sense of the force 2 kg. wts.
2. 14·64 kg. wts. and 10·35 kg. wts.
3. $150\sqrt{5}$ kg. wts. and it makes an angle $\tan^{-1}\frac{1}{2}$ with the sense of the given 300 kg. wts. 4. 50 lbs. wt.
5. Each component is $\frac{100}{3}\sqrt{3}$ gm. wts. Each resolved part is $50\sqrt{3}$ gm. wts; 6. 65 kg. wts. inclined at an angle $\tan^{-1}(\frac{5}{12})$ with the eastern direction.
7. The magnitude of the other force is $10\sqrt{5}$ lbs. making an angle $\tan^{-1}\frac{1}{2}$ with the vertical.
8. 1 ; 30° with AD. 9. 10 lbs. ; 60° with AB.
10. $P\sqrt{3}$ lbs. wt. making an angle 210° with the force P.
11. $\sqrt{3}\,S$, making an angle 90° with R.
12. $F\sqrt{2}$, 135°. 17. $\sqrt{277}a$, $\tan^{-1}\frac{19\sqrt{3}}{5}$ with AB.

Exercise 3A

1. (i) ; (ii). 2. Equilibrium is possible if the three forces act along the same straight line and the smaller two forces act in the opposite sense of the biggest force.
4. 48 kg. wts. ; 64 kg. wts. 4. $133\frac{1}{3}$ kg. wts.

6. Reaction $\frac{20}{3}\sqrt{3}$ kg. wts. and tension $\frac{10}{3}\sqrt{3}$ kg. wts.

7. $Q = 12 \cdot 25$ kg. wts. (nearly) ; $R = 13 \cdot 7$ kg. wts. (nearly).

Exercise 3B

11. 60 kg. wts & 25 kg. wts. 17. 48 lbs. & 64 lbs.

Exercise 4

1. (i) 10 kg. wts. ; at a distance of 18 cms, from the point of application of the 4 kg. wt. force.

(ii) 800 gm. wts. at a distance of 20 cms. from the point of application of the 600 gm. wt. force.

(iii) 14 lbs. wt. ; at a distance of 44 inches from the point of application of the 3 lbs. wt. force.

2. (i) 6 kg. wts ; at a distance of 24 cms. from the $7\frac{1}{2}$ kg. wt. force and the nearer to the $7\frac{1}{2}$ kg. wt. force than the $1\frac{1}{2}$ kg. wt. force.

(ii) 12 kg. wts. ; at a distance of 30 cms. from the 16 kg. wt. force and nearer to this force than the other.

(iii) 200 lbs. wt. ; at a distance of 112·5 ft. from the 1000 lbs. wt. force and nearer to it than the other.

3. (*a*) 24 kg. wts. and at a distance of 2 metres from the 8 kg. wt. force.

(*b*) 8 kg. wts. and at the other end of the rod.

4. Like parallel to the bigger force and the line of action is at a distance of 72 cms. and 90 cms. respectively from the bigger and smaller force.

5. The measure of the larger force is 20 dynes and its line of action is at a distance of $7\frac{1}{3}$ cms. from the smaller force.

6. At a distance of 13 cms. from the larger force.

7. The pressure is inversely proportional to the distance between his hand and shoulder.

8. 96 kg. wts. 9. $8\frac{1}{3}$ kg. wts. and $3\frac{1}{3}$ kg. wts.

10. $\frac{2}{3}$ metres and $\frac{4}{3}$ metres.

11. At a distance of 4 ft. from the weaker man.

12. Pressure on one support will decrease by 20 kg. wts. and on the other increase by 20 kg. wts.

15. 20 kg. wts. ; $\frac{1}{3}$m., $\frac{2}{3}$m. 26. $R\left(1-\frac{Q}{P}\right)$

Exercise 5

1. (i) 5000 kg. metres (ii) 285 lbs ft.

2.	Force	About A	About B	About C
At A	2 kg. wts.	0	+ 40 kg. m.	+20 kg. m.
At D	12 kg. wts.	+ 48 kg. m.	−192 kg. m.	−72 kg. m.
At E	4 kg. wts.	+ 48 kg. m.	− 32 kg. m.	+ 8 kg. m.
At B	6 kg. wts.	−120 kg. m.	0	−60 kg. m.

3. $10\sqrt{3}$ kg. cm., $20\sqrt{3}$ kg. cm., $40\sqrt{3}$ kg. cm.
(All of the same sign)

5. About a point at a distance $7\frac{3}{31}$ metres from the point A.

6. $100\frac{1}{3}$ kg. wts., $89\frac{2}{3}$ kg. wts.

8. Within a distance of $1\frac{1}{2}$ ft. from the middle point.

9. Within a distance of $\frac{1}{10}$ th of its length from the middle point of the rod on bothside of the middle point.

10. The weight is $3\frac{1}{2}$ lbs and the point is at a distance of $\frac{1}{2}$ inch from the middle point.

11. 5 tons wt. and 4 tons wt.

12. If the length of the cord be l, then it is to be attached at a height $\frac{1}{2}l\sqrt{2}$.

13. $4\frac{1}{2}$ tons wt. on the pillar nearer to the lorry and $3\frac{1}{2}$ tons wt. on the other pillar.

14. $2\sqrt{2}$ R, parallel to $\overline{CA}$ and at a distance $-\frac{5}{2}l$ from A
[l is the length of the side of a square].

15. $2\frac{5}{12}$ meters from the father.

16. Divides in the ratio 2 : 3.

17. 37 unit force (nearly), the line of action of the force is tangent to the circumcircle of the triangle ABC at A and its sense makes an angle $\sin^{-1}\frac{3}{25}$ with $\overrightarrow{AB}$.

18. (20, 0), 3 gms. and 1 gm.

Exercise 6

1. 80 kg. cm.

2. If a be the length of a side of the square, then a couple of moment $8a$. 5. $Pa \sim Qb$.

8. 5 units of force at a distance of 2 units from the given force.

9. If the the magnitude of a force be P and the length of each side of the regular hexagon be a, then the moment of the couple is $3\sqrt{3}a.P$. 13. 5 cms. 16. 700 kg. cms.

17. $P=1$, $Q=4$. If a be the length of each side of the hexagon, then the moment of the couple is $\frac{13\sqrt{3}a}{2}$.

Exercise 7A

1. $\frac{1}{3}(z_1+z_2+z_3)$.

2. (i) Point of intersection of the medians. (ii) Let one unlike parallel force be applied at C and F the mid. point of $\overline{AB}$; the point G on $\overrightarrow{CF}$ produced is the required C.G. where $CF=FG$.

3. E and F are points on $\overline{AB}$ and $\overline{CD}$ such that $BE=\frac{1}{3}AB$ and $DF=\frac{4}{9}CD$. The centre of gravity G is a point on $\overline{EF}$ such that $\frac{EG}{FG}=\frac{3}{1}$.

4. Each one carries a load $\frac{1}{3}W$.

10. The C.G is a point on the major axis of the ellipse at a distance $\frac{4a}{3\pi}$ from the centre of the ellipse.

11. If the length of the latus-rectum of the parabola be $4a$, the C.G is a point on the axis of the parabola at a distance $\frac{12}{5a}$ from the vertex of the parabola.

12. The co-ordinates of C.G is $(\frac{3}{4}h, \frac{3}{4}\sqrt{ah})$.

13. The C.G is a point on the rod at distance $\frac{2}{3}a$ from the point A.

14. The co-ordinates of the C.G are $\left(\frac{\pi}{2}, \frac{\pi}{8}\right)$.

15. Co-ordinates of the C.G are $\left(\frac{3}{5}\cdot\frac{9\sqrt{3}-4\sqrt{2}}{3\sqrt{3}-2\sqrt{2}}, \frac{15}{8(3\sqrt{3}-2\sqrt{2})}\right)$

16. The C.G is a point on the line segment joining the centre of the arc and the middle point of the arc at a distance $\frac{4}{\sin^{-1}\frac{4}{5}}$ cms. from the centre.

Exercise 7B

1. If the perpendicular $\overline{CE}$ on $\overline{CD}$ intersects $\overline{CD}$ at E, the C.G.G is a point on $\overline{OE}$ such that $OG=\frac{1}{5}CD$.

2. on $\overline{AD}$ at a distance $\frac{2h_1+h_2}{3}$ from A.

3. The centre of the portion to be removed is at a distance 16 cms. from the centre of the circle.

7. If D be the mid point of the base $\overline{BC}$ of triangle ABC, then C.G is a point G on AD such that $DG=\frac{2}{5}$ AD.

8. If G be the C.G of $\triangle ABC$, then the C.G of the remaining portion will divide $\overline{GA}$ in the ratio $(\sqrt{n}-1):(n\sqrt{n}-3\sqrt{n}+1)$.

9. $7\frac{3}{5}$ cms.

10. If G be the C.G of the remaining portion, then G lies on the line through O parallel to $\overline{AB}$ and $GO=\frac{1}{5}AB$.

Exercise 8A

7. A force of $3\sqrt{3}$ lbs. wt. along a st. line perpendicular to $\overline{BC}$ at a point dividing $\overline{BC}$ in the ratio 1 : 2.

12. $P=10$ lbs. wt., $Q=60$ lbs. wt.

Exercise 8B

4. Tension $=W.\frac{2}{\sqrt{3}}$; Reaction $=W\ \frac{1}{\sqrt{3}}$ (W is the weight of the rod.) If D be the position of the ring, then $AD=3$ ft.

7. 40 kg. wts. and 30 kg. wts.

8. If the weight of the rod be W and the rod subtends an angle 2α at the peg, and also θ be the required slope of the rod, then $\cos\theta=\frac{L\sin\alpha}{a+b}$ and required tension $=\frac{W}{2}\sec\alpha$.

10. 225 and 135 kg. wts. 11. 12 kg. wts. 12. 10 kg. wts.

14. $\frac{W.\sin\beta}{\sin(\alpha+\beta)}$ and $\frac{W.\sin\alpha}{\sin(\alpha+\beta)}$ respectively.

Higher Secondary Examination, 1978

Mathematics—Second Paper

GROUP A

1. (*a*) A function $f(x)$ is defined as follows :

$f(x) = -x$ when $x \leqslant 0$;
$\quad = x$ when $0 < x < 1$.

Draw the graph of $f(x)$.

From this graph find the value of $f(x)$ at $x = -\frac{1}{2}$ and state whether $f(x)$ is continuous at $x = 0$.

(*b*) Find the limit of *any one* of the following :

(i) $\underset{x \to 0}{Lt} \dfrac{1-\cos x}{x^2}$; (ii) $\underset{x \to 0}{Lt} \dfrac{1}{x}\left\{\sqrt{1+x} - \sqrt{1-x}\right\}$.

2. (*a*) Find $\dfrac{dy}{dx}$, if $y = \dfrac{1}{x}$.

For what value of x it is not possible to find $\dfrac{dy}{dx}$ in the above ?

(*b*) Find $\dfrac{dy}{dx}$ in *any two* of the following :

(i) $y = e^x \sec x$; (ii) $y = \dfrac{\sin x}{\log x}$;

(iii) $y = \dfrac{x+2}{(x-1)(x+5)}$; (iv) $y = x^5 + \dfrac{6}{x} - \tan x^2$.

3. (*a*) If $y = x^6$, find the value of $\dfrac{dy}{dx}$ from the definition.

(*b*) If $y = 4\cos 5x$, prove that $\dfrac{d^2y}{dx^2} = -25y$.

(*c*) If $x > \frac{1}{2}$, show that $x(4x^2 - 3)$ is steadily increasing.

GROUP—B

4. Evaluate *any two* of the following :

(i) $\displaystyle\int \frac{dx}{e^x - e^{-x}}$; (ii) $\int x^2 \cos x^3\, dx$;

(iii) $\displaystyle\int \frac{x\,dx}{\sqrt{(3x^2+1)}}$; (iv) $\displaystyle\int \frac{dx}{\sin^2 x \cos^2 x}$.

5. Evalutate *any two* of the following :

(i) $\int x^2 e^x \, dx$; (ii) $\int_0^{\frac{\pi}{4}} x \cos x \, dx$;

(iii) $\int_1^2 \log x \, dx$; (iv) $\int_0^{\frac{\pi}{3}} \frac{\cos x \, dx}{1+\sin^2 x}$.

6. (*a*) Define definite integral as the limit of a sum.

(*b*) If $c=\int e^x \cos x \, dx$ and $s=\int e^x \sin x \, dx$, prove that
$$c+s=e^x \sin x.$$

(*c*) A plane area is bounded by $y=x^2$, $y=0$ and $x=1$; find its area.

7. Solve *any two* of the following questions -

(i) $y(1+x)dx+x(1+y)dy=0$;

(ii) $\frac{dy}{dx}=\frac{x+y+1}{3x+3y+1}$; (iii) $\frac{dy}{dx}=\sqrt{y-x}$;

(iv) $\frac{dy}{dx}=-2y$ (under the condition that $y=2$ when $x=0$).

GROUP C

8. Correct or justify giving reasons *any two* of the following statements :

(i) Three forces P, Q and R acting at a point are in equilibrium ; they will continue to be in equilibrium if each force is increased by the same amount.

(ii) C.G. of a body is bound to lie within the body.

(iii) A thief jumping from the roof of a building with a heavy box on his head will not feel the weight of the box while he is in air.

(iv) While a train moves at a constant velocity, the pull of the engine balances the resisting forces.

9. (*a*) Two like parallel forces P and R act at the points A and B respectively of a rigid body ; find the resultant completely.

(*b*) The resultant of two forces 3P and 2P acting at a point is R ; if the first force is doubled in magnitude, the resultant is also doubled in magnitude. Find the angle between the forces.

10. (*a*) Three forces P, Q and R acting on a rigid body keep it in equilibrium. If Q and R are known to meet at a point C, prove that the line of action of P must also pass through C.

(*b*) A thin uniform wire is bent into two coplanar circular rings of radii r and r', touching each other externally. If $r > r'$, find the distance of the centre of gravity of the system from the point of contact.

11. (*a*) State Newton's Second Law of Motion, and deduce the formula $P = mf$.

(*b*) Two particles start together from the same point along a given straight line, the first with a constant velocity u and the second with a uniform acceleration f starting from rest. Prove that before the second particle catches the first, the greatest distance between them is $u^2/2f$ at time u/f from the start.

12. (*a*) A particle is projected horizontally from the top of a tower ; show that the path of the projectile will be a parabola.

(*b*) A particle starts from rest at a distance a from a fixed point O, and moves under the action of a force which is always directed towards O, and is proportional to the distance from O. Find the distance of the particle from O after a time t from the start.

Higher Secondary Examination, 1979

MATHEMATICS—Second Paper

GROUP A

1. (*a*) A function $f(x)$ is defined as follows :

$$\begin{aligned} f(x) &= x-1, \text{ when } x>0 \\ &= -\tfrac{1}{2}, \text{ when } x=0, \\ &= x+1 \text{ when } x<0. \end{aligned}$$

Draw the graph of $f(x)$. From the graph find the value of $f(x)$ at $x = -\frac{1}{2}$ and state whether $f(x)$ is continuous at $x=0$.

(*b*) Find the limit of *any one* of the following :

(i) $\displaystyle \operatorname*{Lt}_{h\to 0} \frac{\sin(x+h)-\sin x}{h}$;

(ii) $\underset{x \to 1}{\text{Lt}} \dfrac{x^2-1}{\sqrt{3x+1}-\sqrt{5x-1}}$.

2. (*a*) Define the derivative of a function and apply it to obtain the derivative of $-2x$.

(*b*) Find $\dfrac{dy}{dx}$ in *any two* of the following cases :

(i) $x=a \cos \dfrac{\theta}{2},\ y=b \sin \dfrac{\theta}{2}$. (ii) $y=x^{\sin x/2}$;

(iii) $y=3 \log (x+\sqrt{x^2-a^2})$. (iv) $y=\dfrac{1}{2} \sin^{-1} \dfrac{2x}{1+x^2}$.

3. (*a*) If $x=\dfrac{1}{z}$ and $y=f(x)$, show that

$$\frac{d^2f}{dx^2}=2z^3\frac{dy}{dz}+z^4\frac{d^2y}{dz^2}.$$

(*b*) If $x>0$, show that $\log(1+x)<x$.

(*c*) Are the two functions x and $\dfrac{x^2}{x}$ the same ? Give reasons for the answer.

GROUP B

4. (*a*) State the formula for *integration by parts.*

(*b*) Evaluate *any two* of the following :

(i) $\int x^2 \sin x \, dx$; (ii) $\int \log x^2 dx$; (iii) $\int e^x \left(\dfrac{1}{x}-\dfrac{1}{x^2}\right) dx$.

5. Evaluate *any two* of the following :

(i) $\int \dfrac{2x+1}{x(x+3)} dx$. (ii) $\int \dfrac{x \sin^{-1} x}{\sqrt{1-x^2}} dx$.

(iii) $\int_0^{\frac{1}{2}} \dfrac{dx}{\sqrt{3-2x}}$. (iv) $\int_0^{\pi/2} \sin^4 x \cos^3 x \, dx$.

6. (*a*) Obtain $\int \dfrac{dx}{\sqrt{(x-1)(2-x)}}$ by the substitution. $x=\cos^2\theta+2\sin^2\theta$.

Hence deduce the value of $\int_1^2 \dfrac{dx}{\sqrt{(x-1)(2-x)}}$.

(*b*) A plane area is bounded by the curve $y=x(4-x)$ and the x-axis ; find its area.

(*c*) Is the following statement correct ?

"If $\int_0^1 f(x)\,dx = \int_0^1 \phi(x)\,dx$ then $f(x) = \phi(x)$". Examine by a suitable example.

7. Solve *any two* of the following :

(i) $x\sqrt{y^2-1}\,dx - y\sqrt{x^2-1}\,dy = 0.$

(ii) $(6x+9y-7)dx = (2x+3y-6)dy.$

(iii) $dx - dy + ydx + xdy = 0.$

(iv) $\frac{dy}{dx} = \frac{2}{y}$, under the condition that $y=0$ when $x=0$.

GROUP C

8. Correct or justify giving reason *any two* of the following statements :—

(i) If two forces P and Q be represented in magnitude and direction by the sides BC and CA of a triangle, then their resultant is represented in magnitude and direction by the side AB.

(ii) A force and a couple cannot produce equilibrium.

(iii) If a ball is let fall from the hand of a passenger in a moving train the ball hits the floor of the train at exactly the same spot where it would fall if the train were at rest.

(iv) In a ball-throwing competition, a competitor should throw the ball at an angle of 60° to the horizontal direction to get the best result.

9. Obtain the magnitude, direction and point of application of the resultant of two unlike and unequal parallel forces P and Q acting at the points A and B respectively of a rigid body.

If P and Q(P>Q) are both increased by S, show that the point of application of the resultant will shift through a distance $d = \frac{S}{P-Q}AB$.

10. (*a*) Forces 1, 2, 3, 4 and $2\sqrt{2}$ lb. wt. act along the sides AB, BC, CD, DA and along the diagonal AC of a square ABCD. Show that these forces are equivalent to a couple and find the moment of the couple.

(*b*) Find the C. G. of a uniform triangular lamina.

11. (*a*) Prove the formula $s=ut+\frac{1}{2}ft^2$, for a particle moving with uniform acceleration along a straight line.

(*b*) A steamer is moving with a velocity of 15 km. per hour towards the north-east. To a passenger on board the steamer wind appears to blow from the north with a velocity of 15 $\sqrt{2}$ km per hour. Find the magnitude and direction of the true velocity of the wind.

12. (*a*) Show that for a projectile, H the greatest height attained, R the horizontal range and T, the time of flight are given by

$$H=\frac{u^2}{2g}\sin^2\alpha,\quad R=\frac{u^2}{g}\sin 2\alpha \text{ and } T=\frac{2u}{g}\sin\alpha;$$

where u is the velocity of projection and α is the angle of projection.

(*b*) A particle is projected vertically upwards, the greatest height attained by it is 144 feet. Find when it will be at a height of 80 feet after projection.

Higher Secondary Examination, 1980

MATHEMATICS—Second Paper

GROUP A (Answer any *two* questions)

1. (*a*) A function is defined as follows :

$$\begin{aligned} f(x) &= x, \text{ when } x<1 \\ &= 1+x, \text{ when } x>1 \\ &= \tfrac{3}{2}, \text{ when } x=1. \end{aligned}$$

Draw the graph of $f(x)$ and examine whether $f(x)$ is continuous at $x=\frac{1}{2}$ and at $x=1$.

(*b*) Find the limit of any *one* of the following :

(i) $\underset{x\to 3}{Lt}\ \dfrac{x-3}{\sqrt{x-2}-\sqrt{4-x}}$ (ii) $\underset{h\to 0}{Lt}\ \dfrac{\tan(x+h)-\tan x}{h}$.

2. (*a*) Find, from definition, the derivative of x^2.

(*b*) Find $\frac{dy}{dx}$ (any *two*) :—

(i) $y=\frac{x^2-4}{x^2(x+4)}$ (iii) $y=\sin^{-1}\frac{x}{1+x}$

(ii) $y=\sqrt{\frac{x^2+1}{x^2-1}}$ (iv) $y=(\sin x)^{\log x}$.

3. (*a*) $x=a\cos\theta, y=b\sin\theta$, find $\frac{dy}{dx}$.

(*b*) Find $\frac{dy}{dx}$, when $x^2+y^2=4$.

(*c*) $y=x\sin x$, prove that

$$x^2\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+(2+x^2)y=0.$$

GROUP B (Answer Q. No. 7 and any *two* of the rest)

4. Evaluate any *two* of the following :

(i) $\int\frac{xdx}{\sqrt{3x^2+4}}$ (iii) $\int x^3\sin x\,dx$

(ii) $\int\frac{dx}{x^2+x+1}$ (iv) $\int\cos^{-1}x\,dx$.

5. (*a*) Define definite integral as the limit of a sum.

(*b*) Evaluate any *two* of the following :

(i) $\int_0^1\frac{1-x}{1+x}dx$ (iii) $\int_1^2 x.e^x\,dx$

(ii) $\int_0^\pi\sin^3x\cos^3x\,dx$ (iv) $\int_1^2 x\log x\,dx$.

6. Draw the sketch graphs of the functions $y=x^2$ and $y=x^3$ and shade the areas of $\int_0^1 x^2\,dx$ and $\int_0^1 x^3\,dx$.

What will be the value of the area enclosed by these two curves ?

7. Solve (any *two*) :

(i) $\tan x\frac{dy}{dx}=1+y^2$, when $x=\frac{\pi}{2}$ and $y=1$

(ii) $(x+y)\frac{dy}{dx}=x-y$ (iii) $x^2\frac{dy}{dx}=y^2-5y+6$

(iv) $y(1+x^2)\,dy = x(1+y^2)\,dx$

(v) $\dfrac{dy}{dx} = \dfrac{3x^2+1}{4y+2}$, when $x=1$ and $y=1$.

GROUP C (Answer Q. No. 8 and any *two* of the rest)

8. Correct or justify giving reasons any *two* :

(*a*) Three forces acting at a point whose magnitudes are in the ratio of 5 : 13 : 7 cannot produce equilibrium.

(*b*) The resultant of three equal and like parallel forces acting at the vertices of any triangle passes through its centroid.

(*c*) A stone is let fall from the roof of a building. It reaches the ground in 2 seconds. The height of the building is not less than 20 metres.

(*d*) If the angle between two velocities acting simultaneously on a body is increased, the magnitude of the resultant velocity will be increased.

9. (*a*) State and prove the theorem of triangle of forces.

(*b*) Two forces P and Q acting at a point have got a resultant R. If Q is doubled, R is doubled. Again, if Q is reversed in direction, then also R is doubled. Show that

$$P : Q : R = \sqrt{2} : \sqrt{3} : \sqrt{2}.$$

10. (*a*) Find the resultant of two equal and like parallel forces acting on a rigid body.

(*b*) A square hole is punched out of a circular lamina, the diagonal of the square being a radius of the circle. Show that the centre of gravity of the remainder is at a distance $\dfrac{a}{8\pi-4}$ from the centre of the circle, where a is its diameter.

11. (*a*) Define velocity and acceleration.

Prove, with usual notations, the formula $v^2 = u^2 + 2fs$.

(*b*) A stone is dropped into a well and its splash is heard after 2·5 seconds. If the velocity of sound be 330 metres per second, find the depth of the well.

12. (*a*) A bullet moving with a velocity of 20 metres per second penetrates a body through a distance of 16 cms against a resistance of 4×10^6 dynes. Find the mass of the bullet.

(*b*) Prove that the equation of the path of a projcetile *in vacuo* may be written in the form $y = x \tan \alpha \left(1 - \frac{x}{R}\right)$ where α is the angle of projection and R the horizontal range.

Higher Secondary Examination, 1981

Mathematics—Second Paper

GROUP A : (Answer any *two* questions)

1. (*a*) A function $f(x)$ is defined as follows :

$$f(x) = x, \text{ when } x \geqslant 0$$
$$= -x, \text{ when } x < 0.$$

Draw a graph of $f(x)$. From the graph examine whether $f(x)$ is continuous at $x = 0$.

(*b*) Find the limit of any *two* of the following :

(i) $\underset{x \to 0}{Lt} \dfrac{1 - \sqrt{1 - x^2}}{x^2}$, (ii) $\underset{h \to 0}{Lt} \dfrac{\cos(x+h) - \cos x}{h}$,

(iii) $\underset{x \to \frac{\pi}{2}}{Lt} \dfrac{2x - \pi}{\cos x}$.

2. (*a*) Find, from first principles, the differential coefficient of x^3.

(*b*) While a train is travelling from rest to the next station, its distance x km from the start in t hours is given by $x = 90t^2 - 45t^3$. Find its velocity and acceleration after 6 minutes.

$$\left(\frac{dx}{dt} = \text{velocity and } \frac{d^2x}{dt^2} = \text{acceleration}\right)$$

(*c*) Find $\dfrac{dy}{dx}$ and its numerical value under given conditions (any *one*) :

(i) $x = a \sec^2\theta, y = a \tan^3\theta$; when $\theta = \frac{\pi}{4}$.

(ii) $\log_e(xy) = x^2 + y^2$; when $x = 1, y = 1$.

3. (*a*) If $\frac{d}{dx}\{f(x)\}=\frac{d}{dx}\{g(x)\}$, will $f(x)$ be equal to $g(x)$? Give reasons for your answer.

(*b*) Find $\frac{dy}{dx}$ and reduce it to the simplest form (any *one*) :

(i) $y=\frac{\cos x-\cos 2x}{1-\cos x}$ (ii) $y=\tan^{-1}\frac{\cos x}{1+\sin x}$

(iii) $e^{xy}-4xy=4$.

(*c*) If $y=ae^{mx}+b\cos mx$, show that

$$\frac{d^2y}{dx^2}+m^2y=2am^2e^{mx}$$

GROUP B (Answer Question No. 7 and any *two* from the rest)

4. (*a*) Write down the formula for the integration by parts of the product of two functions.

(*b*) Evaluate any *two* of the following :

(i) $\int(x+1)^2 \log x \, dx$ (ii) $\int\frac{dx}{\sqrt{1-x-x^2}}$

(iii) $\int \cos^5 x \, dx$.

5. (*a*) Evaluate $\int_0^1 x \, dx$ from the definition of definite integral.

(*b*) Evaluate any *two* of the following :

(i) $\int_0^{\frac{\pi}{2}} \sin^2 x \, dx$ (iii) $\int_1^2 \log x \, dx$

(ii) $\int_0^{\frac{1}{2}} \frac{dx}{\sqrt{4-x^2}}$ (iv) $\int_0^{\frac{\pi}{2}} x^2 \sin x \, dx$.

6. (*a*) Find the value of : $\int\sqrt{\frac{1+x}{1-x}}dx$.

(*b*) Draw a graph of the curve $y=3x^2+2x+4$. Shade the area enclosed by the curve, the x-axis and the lines $x=-1$ and $x=3$. Find the area of the shaded region by the method of integration.

7. Solve (any *two*) :

(i) $(1-x^2)\frac{dy}{dx}=2y$; when $x=2$, $y=1$.

(ii) $\tan x\frac{dy}{dx}=\tan y$; when $x=\frac{\pi}{6}$ and $y=\frac{\pi}{3}$.

(iii) $x\frac{dy}{dx}=y+x\tan\left(\frac{y}{x}\right)$. (iv) $\frac{dy}{dx}=e^{x-y}+1$.

GROUP C

(Answer *Question No.* 8 and any *two* from the rest)

8. Correct or justify, giving reasons, any *two* of the following :

(*a*) If two forces P and Q (P>Q) inclined at a certain angle act on a particle, the angle that their resultant makes with Q is less than that it makes with P.

(*b*) If three particles of equal weight are placed at the vertices of a triangle, then the centre of gravity of the three particles and the centre of gravity of the triangle will be identical.

(*c*) A swimmer will have to swim perpendicularly to the direction of the current in a flowing river in order to reach the other bank in shortest time.

(*d*) A particle is projected vertically upwards from the surface of the earth with a given velocity. Its time of rise will equal to its time of fall to the surface of the earth.

9. (*a*) A particle moves from rest with an initial velocity and a uniform acceleration. Its distance after 4 seconds from start is 56 cms. If it goes through a distance of 88 cms. in the next 4 seconds, find its initial velocity and acceleration.

(*b*) What is a simple harmonic motion ? Prove, under usual notations, that the equation of simple harmonic **motion** can be written in the form :

$$\frac{dx}{dt}=-\sqrt{\mu(a^2-x^2)}.$$

10. (*a*) A particle is projected with an initial velocity v at an angle α with the horizontal. Find expressions for the horizontal range (R) and the maximum height (H) in terms of v and α. Prove that : $16gH^2-8v^2H+gR^2=0$.

(*b*) A body acted upon by a force moves through 1 metre in 10 seconds from rest. Find the ratio of the force to the weight of the body. (Take $g=981$ cms./sec^2).

(*c*) A bullet fired into a target with a given velocity loses half its velocity after penetrating 3 cms. What further distance will it penetrate till it comes to rest ?

11. (*a*) Prove that if three forces acting at a point be in equilibrium, then each force is proportional to the sine of the angle between the other two.

(*b*) Three equal forces acting at a point are in equilibrium. How are they inclined to one another ?

(*c*) If the position of the resultant of two like parallel forces R and S is unaltered when the positions of R and S are interchanged, show that R = S.

12. (*a*) Define the following : moment of a force about a point, a couple, moment of a couple.

Find the resultant of a force and a couple in magnitude and direction.

(*b*) From a circular disc of radius r is cut out a circle which passes through the centre and whose diameter is $r/3$. Find the C.G. of the remainder.

ANSWERS TO H. S. QUESTIONS

1978

Differential Calculus

1. (*a*) See Chapter II Ex. 3, Graph of $|x|$. At $x=-\frac{1}{2}$, $f(x)=\frac{1}{2}$, continuous at $x=0$.

(*b*) (i) $\frac{1}{2}$, (ii) 1. See Ex. 12 (i) w. out after § 3·6.

2. (*a*) $-\dfrac{1}{x^2}$; $x=0$. (*b*) (i) $e^x \sec x\,(\tan x+1)$;

(ii) $\dfrac{x\cos x\log x-\sin x}{x(\log x)^2}$; (iii) $-\dfrac{x^2+4x+13}{(x-1)^2(x+5)^2}$;

(iv) $5x^4-\dfrac{6}{x^2}-2\sec^2 x^2$. 3. (*a*) $6x^5$.

Integral Calculus

4. (i) $\tan^{-1}(e^x)+c$ [Ex. 9. Ex. IID]

(ii) $\frac{1}{3}\sin x^3+c$. [Ex. 8. Ex. IIC] (iii) $\frac{1}{3}\sqrt{3x^2+1}+c$.

(iv) $\tan x-\cot x+c$, [See Ex. 9. w. out Examples I]

5. (i) $e-2$, (ii) $\dfrac{1}{4\sqrt{2}}(\pi+4)-1$, (iii) $\log\dfrac{4}{c}$,

(iv) $\dfrac{\pi}{4}$, [Ex. 2. Ex. IVB]. 6. (*c*) $\frac{1}{3}$. 7. (i) $xy\,e^{x+y}=c$.

(ii) $x+c=\frac{3}{4}(x+y)-\frac{1}{8}\log(4x+4y+2)$ [See Ex. 5. Ex. VE]

(iii) $x+c=2\sqrt{y-x}+2\log(\sqrt{y-x}-1)$ [See Ex. 2. Ex. VC]

(iv) $y=2e^{-2x}$.

Mechanics

8. (i) Incorrect [See Statics Short Answer type question 24]

(ii) Incorrect [See Statics Short Answer type question 52]

(iii) Correct. (iv) Correct.

9. (*a*) See Statics § 4·2, (*b*) 120°, See Ex. 6. Exercise IIA

10. (*a*) See Statics § 8·3, (*b*) $(r'-r)$; See Statics Ex. 5 w. out page 148. 11. (*a*) See Dynamics § 5·4, § 5·5. (*b*) See Differential Calculus Miscellaneous Ex. 4. Ex. 23 w. out 12. (*a*) See Dynamics § 7·5. (*b*) See Dynamics § 8·2.

1979

Differential Calculus

1. (*a*) When $x=-\frac{1}{2}$, $f(x)=\frac{1}{2}$. Discontinuous at $x=0$.

(*b*) (i) $\cos x$ [See Ex. 3. Ex. 8 w. out].

(ii) -4. [See Exercise 3D, Q. 2(iii)].

2. (*a*) (i) -2. [See Exercise 4B, Q. 1(i)]

(*b*) (i) $-\frac{b}{a}\cot\frac{\theta}{2}$. [See § 4·8 Ex. 1 w. out]

(ii) $x \sin\frac{x}{2}\left[\frac{1}{2}\cos\frac{x}{2}\log x+\frac{1}{x}\sin\frac{x}{2}\right]$

(iii) $\frac{3}{\sqrt{x^2-a^2}}$. [See Miscellaneous Example 2(V) w. out]

(iii) $\frac{1}{1+x^2}$. 3. (*a*) [Exercise 4. Q. 13(i)]

(*b*) See Miscellaneous Examples Chapter IV, Ex. 15.

(*c*) Different. [Sec § 3·3. Note after Ex. 5]

Integral Calculus

4. (*a*) See §3·1, (*b*) (i) $-x^2\cos x+2x\sin x+2\cos x+c$.

(ii) $2x(\log x-1)+c$. [See § 3·2 Ex. 1 w. out. Cor.]

(iii) $e^x.\frac{1}{x}+c$. 5. (i) $\frac{1}{3}\{\log x+5\log(x+3)\}+c$.

(ii) $-\sqrt{1-x^2}\sin^{-1}x+x+c$. (iii) $\sqrt{3}-\sqrt{2}$, (iv) $\frac{2}{35}$.

6. (*a*) π. (*b*) $\frac{32}{3}$ (*c*) Not correct.

7. (i) $\sqrt{x^2-1}=\sqrt{y^2-1}+c$ [Exercise VB. 6]

(ii) $x+c=\frac{1}{11}(2x+3y)-\frac{3}{11}\log(2x+3y-3)$.

(iii) $(1+y)dx-(1-x)dy=0$. [§ 5·6 Ex. 2 w. out]

(iv) $y^2=4x$ [See Ex. VB, 10].

Mechanics

8. (i) Incorrect [Statics Short Answer Type Q. No. 20]

(ii) Correct [See Statics § 6·6, Cor. 1]

(iii) Correct [See Dynamics, Short Answer Type Question]

(iv) Incorrect. Ans. 45° [See Dynamics § 7·2 Note (ii)]

9. (*a*) See Statics § 4·3. (*b*) See Statics Ex. 3, Page 69.

10. (*a*) See Statics Ex. 3, Page 113. (*b*) See Statics § 7·5 (c)

11. (*a*) Dynamics § 4·4. (*b*) 15 km/hr in S. W. direction.

12. (*a*) See Dynamics § 7·2. (*b*) 2 secs or 3 secs.

1980

Differential Calculus

1. (*a*) Continuous at $x=\frac{1}{2}$; discontinuous at $x=1$.

(*b*) (i) 1. (ii) $\sec^2 x$ [Ex. 8 (ii) Exercise 3]

2. (*a*) $2x$ [See Ex. 9(i) Exercise 3].

(*b*) (i) $y\left[\frac{2x}{x^2-4}-\frac{2}{x}-\frac{1}{x+4}\right]$. (ii) $\frac{-2xy}{x^4-1}$

(iii) $\frac{1}{(1+x)\sqrt{2x+1}}$ (iv) $y\left[\frac{\log \sin x}{x}+\cot x \log x\right]$.

3. (*a*) $-\frac{b}{a}\cot\theta$ [See Ex. 1 § 4·8] (*b*) $-\frac{x}{y}$ [See Ex. 1 § 4·7]

Integral Calculus

4. (i) $\frac{1}{3}\sqrt{3x^2+4}+c$.

(ii) $-x^3\cos x+3x^2\sin x+6x\cos x-6\sin x+c$.

(iii) $\frac{2}{\sqrt{3}}\tan^{-1}\frac{2x+1}{\sqrt{3}}+c$ [See Ex. 2. w. out § 2·8].

(iv) $x\cos^{-1}x-\sqrt{1-x^2}+c$ [See Ex. 1 w. out cor. § 3·3].

5. (*b*) (i) $\log 4-1$ [See Ex. 1 w. out Examples 4].

(ii) 0. (iii) e^2. (iv) $\log 4-\frac{3}{4}$. 6. $\frac{1}{12}$ sq. units.

7. (i) $\tan^{-1}y-\log\sin x=\frac{\pi}{4}$. (ii) $y^2+2xy-x^2=c$.

(iii) $\frac{y-2}{y-3}=c\,e^{\frac{1}{x}}$. (iv) $(1+x^2)=c(1+y^2)$ [Ex. 5. Ex. VB]

(v) $2y^2+2y=x^3+x+2$.

Mechanics

8. (*a*) Correct. See. Ex. 1. w. out Statics Page 41.

(*b*) Correct. See Ex. 17 Statics Exercise 4.

(*c*) Incorrect (less then 20 metres).

(*d*) Incorrect (will be decreased) See Short answer Type questions Statics Ex. 8. Replace Force by velocity.

9. (*a*) See Statics § 3·1. (*b*) See Statics Ex. 11 Exercise 2A.

10. (*a*) See Statics § 4·2 and Note 2.

(*b*) Exercise 7B. Ex. 4. Statics.

11. (*a*) See § 4·4 Dynamics. (*b*) 28·6 metres (nearly).

12. (*a*) 32 gms. (*b*) See Ex 8 w. out Chapter VII.

1981

Differential Calculus

1. (*a*) Continuous. See Ex. 3. w. out § 2·5.

(*b*) (i) $\frac{1}{2}$ [See Ex. 12 (iii) § 3·6].

(ii) $-\sin x$ [See Ex. 8(ii) Exercise 3]. (iii) -2.

2. (*a*) $3x^2$. (*b*) $16\frac{13}{20}$ km./hour ; 153 km./hour^2.
(*c*) (i) $\frac{3}{2}$. (ii) -1. 3. (*a*) No. See Ex. 8 Page 155.
(*b*) (i) $-2 \sin x$. (ii) $-\frac{1}{2}$ [See Ex. 4. Ex. 4(i) Page 151].

(iii) $-\frac{y}{x}$. See Exercise 4(J) Ex. 1 (XIV).

Integral Calculus

4. (*b*) (i) $\frac{1}{3}(x^3+3x^2+3x)\log x - x - \frac{x^2}{2} - \frac{x^3}{9} + c$.

(ii) $\sin^{-1}\frac{2x+1}{\sqrt{5}}+c$ [See Exercise II (J) ; Ex. 2(iii)].

(iii) $\sin x - \frac{2\sin^3 x}{3} + \frac{\sin^5 x}{5} + c$.

5. (*a*) $\frac{1}{2}$. (*b*) (i) $\frac{\pi}{4}$ [Exercise IVA 5]. (ii) $\sin^{-1}\frac{1}{4}$.

(iii) $\log 4 - 1$. (iv) $\pi - 2$.

6. (*a*) $\sin^{-1}x - \sqrt{1-x^2}+c$. See Ex. 3 w. out before Ex. IIK.

(*b*) 52 square units.

7. (i) $y=\frac{1}{3}\frac{x+1}{x-1}$. (ii) $\sin y = \sqrt{3}\sin x$.

(iii) $\sin\frac{y}{x}=cx$. (iv) $e^{x-y}=x+c$.

Mechanics

8. (*a*) Incorrect [See Statics cor. 5, § 2·3].
(*b*) Correct. See Cor. 1, § 7·5 (c) Statics.
(*c*) Correct [See Dynamics Short Answer Type questions]
(*d*) Correct [See Dynamics, § 6·6].
9. (*a*) $u=10$ cms./sec. $f=2$ cms./sec^2.
(*b*) See § 8·2 Dynamics.
10. (*a*) Ex. 7 worked out (second part) Chapter VII Dynamics.
(*b*) 2 : 981. (*c*) 1 cm.
11. (*a*) See § 3·5. (*b*) 120° [See Statics Short Answer Type question No. 31. (*c*) See Q. 14, Exercise 4 Statics.

12. (*a*) See § 6·6 (*b*) At distance $\frac{r}{210}$ from the centre of the disc. [See Ex. 2. w. out Page 147].

1982

Group—A

(Answer any *two* questions)

1. (*a*) A function $f(x)$ is defined as follows :

$f(x)=3+2x$, when $x \leqslant 0$

$=3-2x$, when $x>0$.

Draw the graph of $f(x)$ and examine whether $f(x)$ is continuous at $x=0$.

(*b*) If $f(x)=e^{px+q}$, (p, q constants), show that $f(a)\,f(b)\,f(c) =f(a+b+c)e^{2q}$.

(*c*) Evaluate any one of the following limits :

(i) $\underset{x \to 0}{\text{Lt}} \dfrac{\sin 3x}{\sin 2x}$ (ii) $\underset{x \to 1}{\text{Lt}} \dfrac{x^2-3x+2}{x^3-4x+3}$

2. (*a*) Define the derivative of a function $f(x)$ and find, from the definition, the derivative of $\tan x$.

(*b*) Find $\dfrac{dy}{dx}$ (any *one*) :

(i) $y=\dfrac{x \sin x+\cos x}{x \cos x-\sin x}$ (ii) $y=\log_e(x+\sqrt{x^2+a^2})$.

3. (*a*) If $\sin y=x \sin (a+y)$, prove that $\dfrac{dy}{dx}=\dfrac{\sin^2(a+y)}{\sin a}$.

(Here $a \neq n\pi$, $n=0, 1, 2, \cdots$)

(*b*) If $y=\tan^{-1} \dfrac{2t}{1-t^2}$ and $x=\sin^{-1}\dfrac{2t}{1+t^2}$, show that $\dfrac{dy}{dx}=1$.

(*c*) If $y=a \cos \log x+b \sin \log x$, where a, b are constants, show that $x^2 \dfrac{d^2y}{dx^2}+x\dfrac{dy}{dx}+y=0$.

Group B

(Answer Question No. 7 and any *two* from the rest)

4. (*a*) Show that $\displaystyle\int\frac{f'(x)}{f(x)}\,dx=\log|f(x)|+c$

(c is an arbitrary constant)

(*b*) Evaluate any *two* of the following :

(i) $\displaystyle\int\frac{x^3}{x^2+4}\,dx$ (ii) $\displaystyle\int\frac{\sin^2 x}{\cos^4 x}\,dx$

(iii) $\displaystyle\int\frac{dx}{\sqrt{2ax-x^2}}$ (iv) $\displaystyle\int\frac{e^x}{x}(x \log x+1)\,dx$.

5. (*a*) State the Fundamental Theorem of the Integral Calculus.

(*b*) Evaluate any *two* of the following :

(i) $\int_0^2 \frac{x^2}{\sqrt{1+x^2}}dx$ (ii) $\int_0^{\frac{\pi}{2}} \sin^2 x \cos^2 x \, dx$

(iii) $\int_0^1 \frac{dx}{\sqrt{x+1}+\sqrt{x}}$ (iv) $\int_0^1 x \tan^{-1} x \, dx$

6. (*a*) Evaluate $\int_0^1 e^{-x} dx$ from the definition of definite integral.

(*b*) Shade the area enclosed by the two parabolas $y^2=4x$ and $x^2=4y$ and find, by integration, the area of the shaded region

7. Solved (any *two*) :

(i) $x(y^2+1)dx-y(x^2+1)dy=0$ (ii) $\frac{dy}{dx}=\sin(x+y)$.

(iii) $(x^2+y^2)\,dx-2xy\;dy=0$; given that $y=0$ when $x=1$.

(iv) $\tan x\,dy-\tan y\,dx=0$; given that $y=\frac{\pi}{2}$, when $x=\frac{\pi}{4}$.

Group—C

(Answer Question No 8 and any *two* from the rest)

8. Correct or justify, with arguments, any *two* of the following :

(*a*) The resultant of two equal forces acting at a point at an angle of 120° with one another is equal to each of the forces.

(*b*) If three forces acting upon a rigid body can be represented in magnitude, direction and line of action by the three sides of a triangle, taken in order, then the forces will be in equilibrium.

(*c*) For a given velocity of projection, the horizontal range of a projectile is maximum when the angle of projection is 45°.

(*d*) The pressure of a body resting on a horizontal plane moving vertically upwards with a uniform velocity is greater than the weight of the body.

9. (*a*) Prove, with usual notations the formula $s=ut+\frac{1}{2}ft^2$.

Show that the average velocity of a particle moving along a straight line with a uniform acceleration during any interval of time is equal to the mean of the initial and the final velocities of the particle in that interval.

(*b*) The velocity of a particle moving along a straight line is v ft/sec. when the particle is at a distance of x ft. from a fixed point on the line. If the relation between v and x at any instant is given by $v^2=x^2+2x+3$, what is the acceleration of the particle when it is at a distance of 2 ft. from the fixed point ?

(*c*) A particle thrown vertically upwards from the ground takes 5 seconds to reach a height h and in 3 seconds more it reaches the ground again. Find h, if $g=32$ ft/sec^2.

10. (*a*) Prove that the path of a projectile in vacuo is a parabola.

(*b*) A particle executing simple harmonic motion has the velocities 12 ft/sec. and 13 ft/sec. when its distance from the centre are 5 ft. and 4 ft. respectively. Show that the periodic time of the motion is $T=\frac{6\pi}{5}$ sec.

11. (*a*) Enunciate and prove the theorem of the Triangle of Forces.

(*b*) O is the circum-centre of the triangle ABC. Three forces P, Q, R acting along OA, OB, OC are in equilibrium.

Prove that P : Q : R = sin 2A : sin 2B : sin 2C.

(c) Two equal forces act at a point. If the resultant of the forces, when the angle between them is 2α be twice as great as when the angle between them is 2β, prove that $\cos\alpha=2\cos\beta$.

12. (*a*) Find the magnitude and the line of action of the resultant of two like parallel forces 5P and 7P acting upon a rigid body.

(*b*) Forces of magnitudes 1, 2, 3, 4 and $2\sqrt{2}$ act respectively, along the sides AB, BC, CD, DA and the diagonal AC of the square ABCD of side a. Prove that the forces form a couple

(*c*) Equal weights are placed at the vertices of a uniform triangular lamina. Show that the centre of gravity of the weights coincides with the centre of gravity of the triangle.

1983

Group—A (Answer any two questions)

1. (*a*) A function $f(x)$ is defined as follows :

$$f(x)=x+1, \text{ when } x\leq 1 \; = 3-ax^2, \text{ when } x>1.$$

For what value of a will $f(x)$ be continuous ? With this value of a draw the graph of $f(x)$.

(*b*) If $f(x)$ has a derivative $f'(a)$ at $x=a$, show that

$$\underset{x\to a}{\text{Lt}} \frac{x f(a)-a f(x)}{x-a}=f(a)-a f'(a).$$

(c) Evaluate any one of the following limits :

(i) $\underset{x\to 0}{\text{Lt}} \dfrac{e^{\sin x}-1}{x}$ (ii) $\underset{x\to\pi}{\text{Lt}} \dfrac{\sin x}{\pi - x}$.

2. (a) Find from the definition the derivative of x^3+2x.

(b) Answer any *two* of the following :

(i) If $y=e^t \cos t$, $x=e^t \sin t$, verify that $\dfrac{dy}{dx}\cdot\dfrac{dx}{dy}=1$.

(ii) If $y=\tan^{-1}\sqrt{\dfrac{1-\cos x}{1+\cos x}}$, find $\dfrac{dx}{dy}$.

(iii) If $y=(\sin x)^{\log_e x}$, find $\dfrac{dy}{dx}$.

3. (a) If $2x^2+5xy+3y^2=1$, find $\dfrac{dy}{dx}$,

(b) Show that x^3-3x^2+3x, increases as x increases.

(c) If $x=\sin t$, $y=\sin kt$, show that

$$(1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}+k^2y=0. \quad (k=\text{constant}).$$

Group—B

(Answer Question No. 7 and any *two* from the rest.)

4. (a) State for what value of n, the formula

$\int x^n dx=\dfrac{x^{n+1}}{n+1}+c$, c constant, is not true. Find the value of the integral for that particular value of n.

(b) Evaluate *any two* of the following :

(i) $\displaystyle\int\frac{dx}{\sqrt{2+x-x^2}}$; (ii) $\displaystyle\int\frac{dx}{1-\sin x}$;

(iii) $\displaystyle\int\frac{e^{2x}}{e^x+1}\,dx$; (iv) $\displaystyle\int\frac{1+2x^2}{x^2(1+x^2)}dx$

5. Evaluate any *two* of the following :

(i) $\displaystyle\underset{n\to\infty}{\text{Lt}}\left[\frac{1}{n+1}+\frac{1}{n+2}+\frac{1}{n+3}+\cdots+\frac{1}{n+n}\right]$.

(ii) $\displaystyle\int_0^a\frac{x\,dx}{\sqrt{a^2-x^2}}$ (iii) $\displaystyle\int_1^2 x^2e^x dx$

(iv) $\displaystyle\int_0^{\frac{\pi}{2}}\frac{\sin x\cos x}{2\cos^2 x+3\sin^2 x}\,dx$

6. (*a*) State the formula for integration by parts and evaluate $\int \log x \, dx$ with its help.

(*b*) Evaluate $\int_1^2 5x^2 \, dx$ from the definition of a definite integral as the limit of a sum.

(*c*) Find by integration the area of the figure bounded by $y^2=2x+1$ and $x-y-1=0$.

7. Solve (any *two*) :

(i) $\log \frac{dy}{dx}=4x-2y-2$, given that $y=1$, when $x=1$.

(ii) $(2x-2y+5)dy=(x-y+3)dx$. (iii) $\frac{dy}{dx}=\frac{y}{x}+\tan \frac{y}{x}$.

(iv) $\frac{dy}{dx}=y \sec x$, given that $y=1$, when $x=\frac{\pi}{6}$.

Group C

(Answer Question No. 8 and any *two* from the rest)

8. Correct or justify with arguments any *two* of the following :

(i) The centre of a uniform triangular lamina is the same as that of three equal particles placed at the middle points of its three sides.

(ii) A man carries a load at the end of a stick one metre long which is placed on his shoulder. In order that the pressure on his shoulder may be least he should place his hand at the middle point of the stick.

(iii) The velocity at a height h of a particle projected vertically upwards from the ground will be the same in magnitude as that due to fall from rest from maximum height H to the same height.

(iv) The pressure of a mass resting on a lift will increase if the speed of the lift decreases during downward journey.

9. (*a*) Using the usual symbols prove the formula $v^2=u^2+2fs$ for a particle moving with uniform acceleration along a straight line.

(*b*) If a, b, c be the spaces described in the p^{th}, q^{th} and r^{th} seconds by a particle starting with a given velocity and moving with uniforms acceleration in a straight line, show that

$$a(q-r)+b(r-p)+c(p-q)=0$$

10. (*a*) State Newton's Second Law of Motion and prove the formula $P=mf$.

(*b*) In a Simple Harmonic Motion the distances of a particle from the centre of its path are, respectively, 4 cm. 6 cm. and 8 cm. in the t^{th}, $(t+1)^{th}$ and $(t+2)^{th}$ second of its motion ; find the time of a complete oscillation.

11. (*a*) State and prove Lami's Theorem.

(*b*) Two like parallel forces P and Q are acting on a rigid body and the distance between their points of application is x. If two equal and unlike parallel forces S are added to P and Q, show that the new resultant is at a distance $\frac{xS}{P+Q}$ from the old.

12. (*a*) ABCD is uniform lamina in the form of a trapezium whose parallel sides AB and CD are of lengths a and b. Prove that the distance of the centre of gravity of the trapezium from the side AB is $\frac{h}{3}\cdot\frac{a+2b}{a+b}$. ($h$ is the height of the trapezium.)

(*b*) Define a couple and its moment.

Three forces are represented in magnitude, direction and position by the sides of a triangle taken in order. Show that the forces form a couple, whose moment you are to find. What happens if the above forces act at a point ?

1984

Group A

Answer any *two* questions

1. (*a*) Draw the graph of $f(x)=\frac{|x|}{x}$ for $x\neq 0$ and $f(0)=0$.

From the graph examine whether $f(x)$ is continuous at $x=0$.

(*b*) Find from the definition the derivative of $\sin x^2$,

(*c*) Find the limit of *one* of the following :

(i) $\underset{x\to a}{\text{Lt}} \frac{\sqrt{x-b}-\sqrt{a-b}}{x^2-a^2}$, $a>b$; (ii) $\underset{x\to 0}{\text{Lt}} \frac{\sin x(1-\cos x)}{x^3}$.

2. (*a*) Find $\frac{dy}{dx}$ of the following (Answer any *two*) :

(i) $y=e^x \sin x \cos^2 x$; (ii) $y=x^{\sin x}$

(iii) $y=\cos^{-1} \frac{1-x}{1+x}$.

(b) If $f(x)=\dfrac{6-4x}{1+2x+2x^2}$, find $f'(x)$. Also determine the values of x for which $f'(x)=0$.

3. (a) If $f(x)=4x^3+6x^2-24x+1$, show that $f(x)$ decreases in the interval $(-2, 1)$.

(b) Given that $\cos y = x \cos (a+y)$, prove that $\dfrac{dy}{dx}=\dfrac{\cos^2(a+y)}{\sin a}$, where $a(\neq 0)$ is a constant.

(c) If $F(x)=f(x)\,\phi(x)$ and $f'(x)\,\phi'(x)=a$, (a, constant), show that $\dfrac{F''}{F}=\dfrac{f''}{f}+\dfrac{\phi''}{\phi}+\dfrac{2a}{f\phi}$, where $'' \equiv \dfrac{d^2}{dx^2}$ and $' \equiv \dfrac{d}{dx}$, $F(x)\neq 0$.

Group B

Answer Question No. 7 and any *two* from the rest

4. (a) Prove that $\int a^x dx = \dfrac{a^x}{\log_e a}+c$, where $a>0$ and c is a constant : Is the formula valid when $a=1$?

(b) Evaluate any *two* of the following :

(i) $\displaystyle\int \frac{dx}{x^2\sqrt{x+1}}$; (ii) $\displaystyle\int \frac{dx}{\cos 2x+3\sin^2 x}$;

(iii) $\displaystyle\int \frac{\sin x}{\sin (x-a)}\,dx$, a constant ; (iv) $\displaystyle\int e^{-x^2} x^3\,dx$.

5. (a) State the fundamental theorem of Integral Calculus.

(b) Evaluate any *two* of the following :

(i) $\displaystyle\int_0^{\pi} x \cos^2 x\,dx$; (ii) $\displaystyle\int_0^{2} \frac{dx}{\sqrt{x+3}-\sqrt{x+1}}$;

(iii) $\displaystyle\int_0^{e} (\log_e x)^3\,dx$; (iv) $\displaystyle\int_0^{\pi/2} \cos^4 x \sin^3 x\,dx$.

6. (a) From the definition of definite integral as the limit of a sum evaluate $\displaystyle\int_9^{1} 2e^x\,dx$.

(b) Shade the portion of the area above the x-axis bounded by the parabola $y^2=4x$ and the circle $(x-4)=4\cos\theta$, $y=4\sin\theta$ and obtain this area by integration.

7. Solve any *two* of the following :

(*a*) $y(1+x)dx+x(1+y)dy=0$;

(*b*) $\sqrt{1-y^2}\,dx+y^2\sqrt{1-x^2}\,dy=0$;

(*c*) $y-x\frac{dy}{dx}=2\left(1+x^2\frac{dy}{dx}\right)$, given that $y=1$, when $x=1$;

(*d*) $\frac{dy}{dx}=(x+y)^2$, given that $y=1$, when $x=0$.

Group C

Answer Question No. 8 and any *two* from the rest

8. Correct or justify with arguments any *two* of the following :

(*a*) The coplanar forces P, Q and R acting at a point are in equilibrium. If each of the forces is increased by the same amount, then the new forces will also remain in equilibrium.

(*b*) Two forces P and 2P act on a particle. If the angle between the forces is 120°, then their resultant makes a right angle with the force P.

(*c*) For vartical motiom under gravity the time taken by a particle to reach the maximum height from the point of projection is greater than the time of fall from the maximum height to the point of projection.

(*d*) Newton's Second Law of motion gives a method of measuring a force.

9. (*a*) Without assuming any formula prove that for a particle moving with uniform acceleration along a straight line $s=ut+\frac{1}{2}ft^2$, where the symbols have their usual meanings.

(*b*) A particle projected vertically upwards returns to its point of projection after 7 seconds. Find the greatest height attained by the particle, its velocity of projection and its height from the point of projection after four seconds. [Take $g=32$ ft./sec^2]

10. (*a*) Find the path of a particle projected in vacuo at an angle 60° with the horizontal with initial velocity V.

[The motion is supposed to be in a vertical plane. The acceleration due to gravity is 32 ft./sec^2.]

(*b*) In a Simple Harmonic motion the velocities of a particle are 9 cm. sec. and 15 cm./sec. when the distances from the centre of the path are 10 cm. and 6 cm. respectively. Find its time-period.

11. (*a*) State and prove the triangle law of forces.

Examine the correctness or otherwise of the statement : The converse of the polygon of forces is true.

(*b*) in the triangle ABC, D, E and F are the middle points of the sides BC, CA and AB respectively. Prove that three forces acting at a point and represented by $\overline{AD}$, $\overline{BE}$ and $\overline{CF}$ maintain equilibrium.

12. (*a*) Two unlike parallel forces of magnitudes 3P and 2P act at points A and B of a rigid body. Find the magnitude of the resultant and its point of application.

(*b*) ABCD is a uniform square lamina whose centre is O. E and F are the middle points of the sides AB and BC respectively. If the triangle BEF be cut off, find the distance of the centre of gravity of the figure AEFCD from the centre O.

1985

Group A

Answer any *two* questions

1. (*a*) A function $f(x)$ is defined as :

$$f(x)=1+2x \text{ when } x\leqslant 1$$
$$=3-x \text{ when } x>1.$$

Draw the graph of the function and examine whether $f(x)$ is continuous at $x=1$.

(*b*) Find the derivative of sec $3x$ with respect to x from definition.

(*c*) Evaluate any *one* of the following :

(i) $\lim\limits_{x\to 4} \dfrac{\sqrt{x}-2}{x-4}$ (ii) $\lim\limits_{x\to 0} \dfrac{e^{x^2}-1}{\sin^2 x}$

2. (*a*) Find $\dfrac{dy}{dx}$ in the simplest form (any *two*) :

(i) $\tan^2 y=\dfrac{1+\cos 2x}{1-\cos 2x}$

(ii) $x^y=y^x$ (iii) $y=\log_e [x+\sqrt{x^2+a^2}]$

(*b*) If $x=a(\theta+\sin\theta)$, $y=a(1+\cos\theta)$, express $\dfrac{dy}{dx}$ in simplest form and show that $\dfrac{dy}{dx}=-1$ when $\theta=\dfrac{\pi}{2}$.

3. (*a*) Find points on the curve $y=x+\frac{1}{x}$ where $\frac{dy}{dx}=0$ and give geometrical significance of $\frac{dy}{dx}=0$.

(*b*) Rate of change of radius of a circle is $\frac{1}{\pi}$. Find the rate of change of (i) circumferential length and (ii) area of the circle at the instant when the radius is 2 units.

(*c*) If $y=(\sin^{-1}x)^2$, then find the value of

$$(1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}+4.$$

Group B

Answer question No. 7 and any *two* from rest.

4. (*a*) State formula for integration of product of two functions u and v and apply it to find $\int \log_e x\, dx$.

(*b*) Evaluate (any *two*) of the following :

(i) $\int \frac{dx}{\sin^2 x \cos^2 x}$ (ii) $\int x^4(\log_e x)^2 dx$

(iii) $\int \frac{\sqrt{\tan x}}{\cos^4 x} dx$ (iv) $\int \frac{dx}{1+e^x}$

5. (*a*) Perform the following definite integrals (any *two*) :

(i) $\int_1^e \frac{dx}{x(1+\log_e x)^2}$ (ii) $\int_0^a \sin^{-1}\frac{2x}{1+x^2} dx$

(iii) $\int_0^{\frac{\pi}{2}} \sin^4 x\, dx$ (iv) $\int_8^{15} \frac{dx}{(x-3)\sqrt{x+1}}$

(*b*) Using the formula $\int_0^a f(x)\, dx=\int_0^a f(a-x)\, dx$,

prove that $\int_0^{\frac{\pi}{2}} \log \tan x\, dx=0$.

6 (*a*) State definite integration as the limit of sum and apply it to evaluate $\int_1^4 3x^2 dx$.

(*b*) Shade the area bounded by $y^2=8x$ and $y=x$ along positive direction of x-axis and use integration to find the area of that part.

7. Solve (any *two*):

(*a*) $x \cos^2 y\, dx - y \cos^2 x\, dy = 0$

(*b*) $(x^2+y^2)dy - xy\, dx = 0$

(*c*) $\dfrac{dy}{dx} = \dfrac{x-y+1}{2x-2y+3}$

(*d*) $\dfrac{dy}{dx} = \dfrac{1+y^2}{xy}$ when $x=1$, $y=0$.

Group C

Ans ver question No. 8 and any *two* of the rest.

8. Correct of justify with arguments any *two* of the following statements :

(*a*) A particle executing simple harmonic motion always moves with uniform velocity.

(*b*) Three like parallel forces P, P, P act at the vertices of a triangle. Their resultant will act at the centroid of the triangle.

(*c*) The equation of motion of a particle moving along a straight line, given by $x = 16t + 5t^2$, shows that the particle moves with uniform acceleration.

(*d*) Three forces, in a plane acting at a point, represented in magnitude, direction by three sides of a triangle taken in order, are equivalent to a couple.

9. (*a*) State "parallelogram law of forces". Find the magnitude and direction of resultant of two forces R_1, R_2 acting at a point inclined to an angle α with each other. Hence find greatest and least value of the resultant.

(*b*) A transversal meets the lines of action of two forces P and Q (acting at a point) and their resultant R at points A, B and C respectively. Prove that

$$\frac{P}{OA} + \frac{Q}{OB} = \frac{R}{OC}.$$

10. (*a*) State and prove Lami's theorem

(*b*) The resultant of three like parallel forces P, Q, R acting at the vertices of the triangle ABC passes through the Incentre of the triangle. Prove that $\dfrac{P}{\sin 2A} = \dfrac{Q}{\sin 2B} = \dfrac{R}{\sin 2C}$

11. (*a*) Without assuming any formula, prove that $v^2 = u^2 + 2fs$ for a particle moving along a straight line with uniform acceleration (symbols have their usual significance).

(*b*) A stone, dropped from the top of a tower (freely under gravity) covers $\frac{9}{25}$th of the height of the tower in last second of its motion. Find the height of the tower.

12. (*a*) A particle is projected in vacuum with a velocity of projection u and angle of projection α. Find the equation to its path. Hence show that

Horizontal Range, $R = \frac{u^2}{g} \sin 2\alpha$;

Time of flight, $T = \frac{2u \sin \alpha}{g}$.

(*b*) A train starts from Howrah and stops at Kharagpur. Its velocity uniformly increases and reaches to v. The velocity then decreases uniformly. Time taken from Howrah to Kharagpur by the train is t. Prove that the distance between the stations is $\frac{1}{2} vt$.